Solution Manual for *Gravitation (MTW)*

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Chapter 1

GEOMETRODYNAMICS IN BRIEF

1.1 CURVATURE OF A CYLINDER

Show that the Gaussian curvature R of the surface of a cylinder is zero by showing that geodesics on that surface (unroll!) suffer no geodesic deviation. Give an independent argument for the same conclusion by employing the formula $R = 1/\rho_1\rho_2$ where ρ_1 and ρ_2 are the principal radii of curvature at the point in question with respect to the enveloping Euclidean three-dimensional space.

Solution:

The metric of the surface of a cylinder in cylindrical coordinates (ϕ, z) is

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.1.1}$$

The geodesic equation can be gotten by minimizing

$$s = \int \sqrt{r^2 d^2\phi + d^2 z}. \tag{1.1.2}$$

The variation method gives

$$\frac{dz(\phi)}{d\phi} = \text{ constant} \tag{1.1.3}$$

for geodesics. Since the "slope" of the geodesics on the surface is a constant, two initially parallel geodesics will be parallel forever. So the Gaussian curvation of the surface of a cylinder is 0.

The curvature of the curve $\phi =$ costant on the surface is 0. Thus, one of the princpal radii of curvature is infinite. We then have $R = 1/\rho_1\rho_2 = 0$.

1.2 SPRING TIDE VS. NEAP TIDE

Evaluate (1) in conventional units and (2) in geometrized units the magnitude of the Newtonian tide-producing acceleration $R^m{}_{0n0}$ generated at the Earth by (1) the moon ($m_{\text{conv}} = 7.35 \times 10^{25}\text{g}$, $r = 3.84 \times 10^{10}\text{cm}$) and (2) the sun ($m_{\text{conv}} = 1.989 \times 10^{33}\text{g}$, $r = 1.496 \times 10^{13}\text{cm}$). By what factor do you expect spring tides to exceed neap tides?

Solution:

In geometrized units, we have

$$R^m{}_{0n0} = \partial^m \partial_n \phi, \tag{1.2.1}$$

where $\phi = -M/r$. We can figure out that

$$R^m{}_{0n0} = \frac{M(\delta^m_n r^2 - x^m x_n)}{r^5}. \tag{1.2.2}$$

Suppose the coordinates of the earth are $(r, 0, 0)$. We can get

$$R^2{}_{020} = R^3{}_{030} = \frac{M}{r^3}. \tag{1.2.3}$$

As for the moon, we have

$$\frac{M}{r^3} = \frac{G m_{\text{conv}}}{c^2 r^3} = 9.64 \times 10^{-35} \text{cm}^{-2}. \tag{1.2.4}$$

As for the sun, we have

$$\frac{M}{r^3} = 4.41 \times 10^{-35} \text{cm}^{-2}. \tag{1.2.5}$$

Spring tide happens when moon and sun contribute together to the curvature of space-time at earth, while neap tide takes place when they cancels. So, we would expect spring tides to be three times stronger than neaps tides.

The discussion in conventional units is straightforward and will be omitted here.

1.3 KEPLER ENCAPSULATED

A small satellite has a circular frequency $\omega(\text{cm}^{-1})$ in an orbit of radius r about a central object of mass $m(\text{cm})$. From the known value of ω, show that it is possible to determine neither r nor m individually, but only the effective "Kepler density" of the object as averaged over a sphere of the same radius as the orbit. Give the formula for ω^2 in terms of this Kepler density.

Solution:

Since

$$\omega^2 r = \frac{m}{r^2}, \quad m = \frac{4}{3}\pi \rho_{\text{Kepler}} r^3, \tag{1.3.1}$$

we can get

$$\omega^2 = \frac{4}{3}\pi \rho_{\text{Kepler}}. \tag{1.3.2}$$

Chapter 2

FOUNDATIONS OF SPECIAL RELATIVITY

2.1 EXERCISE

Show that equation (2.14) is in accord with the quantum-mechanical properties of a de Broglie wave,

$$\psi = e^{i\phi} = \exp\Big[i(\vec{k}\cdot\vec{x} - \omega t)\Big]. \tag{2.1.1}$$

Solution:

The number of surfaces $\boldsymbol{x}$ pierces in the 4-momentum 1-form $\tilde{\boldsymbol{k}}$ is

$$\phi(\vec{x}, t) - \phi(\mathbf{0}, 0) = \vec{k}\cdot\vec{x} - \omega t = \boldsymbol{k}\cdot\boldsymbol{x}. \tag{2.1.2}$$

So we have

$$\boldsymbol{k}\cdot\boldsymbol{x} = \langle\tilde{\boldsymbol{k}}, \boldsymbol{x}\rangle. \tag{2.1.3}$$

2.2 LOWERING INDEX TO GET THE 1-FORM CORRESPONDING TO A VECTOR

The components u^α of the 1-form $\tilde{\boldsymbol{u}}$ that corresponds to a vector $\boldsymbol{u}$ can be obtained by "lowering an index" with the metric coefficients $\eta_{\alpha\beta}$:

$$u_\alpha = \eta_{\alpha\beta}u^\beta; \text{ i.e., } u_0 = -u^0,\ u_k = u^k. \tag{2.2.1}$$

Solution:

$u_\alpha = \langle\tilde{\boldsymbol{u}}, \boldsymbol{e}_\alpha\rangle = \boldsymbol{u}\cdot\boldsymbol{e}_\alpha = \eta_{\mu\nu}u^\mu \boldsymbol{e}^\mu_\alpha = \eta_{\mu\nu}u^\mu\delta^\nu_\alpha = \eta_{\alpha\beta}u^\beta.$

2.3 RAISING INDEX TO RECOVER THE VECTOR

One can return to the components of $\boldsymbol{u}$ by raising indices,

$$u^\alpha = \eta^{\alpha\beta} u_\beta; \tag{2.3.1}$$

the matrix $||\eta^{\alpha\beta}||$ is defined as the inverse of $||\eta_{\alpha\beta}||$, and happens to equal $||\eta_{\alpha\beta}||$:

$$\eta^{\alpha\beta}\eta_{\beta\gamma} \equiv \delta^\alpha{}_\gamma; \quad \eta^{\alpha\beta} = \eta_{\alpha\beta} \text{ for all } \alpha, \beta. \tag{2.3.2}$$

Solution:

$$\eta^{\alpha\beta} u_\beta = \eta^{\alpha\beta}\eta_{\beta\gamma} u^\gamma = \delta^\alpha{}_\gamma u^\gamma = u^\alpha. \tag{2.3.3}$$

2.4 VARIED ROUTES TO THE SCALAR PRODUCT

The scalar product of $\boldsymbol{u}$ with $\boldsymbol{v}$ can be calculated in any of the following ways:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = u^\alpha v^\beta \eta_{\alpha\beta} = u^\alpha v_\alpha = u_\alpha v_\beta \eta^{\alpha\beta}. \tag{2.4.1}$$

The proposition is straightforward to prove by applying conclusions of the problems above.

2.5 ENERGY AND VELOCITY FROM 4-MOMENTUM

A particle of rest mass m and 4-momentum $\boldsymbol{p}$ is examined by an observer with 4-velocity $\boldsymbol{u}$. Show that just as (a) the energy he measures is

$$E = -\boldsymbol{p} \cdot \boldsymbol{u}; \tag{2.5.1}$$

so (b) the rest mass he attributes to the particle is

$$m^2 = -\boldsymbol{p}^2; \tag{2.5.2}$$

(c) the momentum he measures has magnitude

$$|\vec{p}| = [(\boldsymbol{p} \cdot \boldsymbol{u})^2 + (\boldsymbol{p} \cdot \boldsymbol{p})]^{1/2}; \tag{2.5.3}$$

(d) the ordinary velocity $\vec{v}$ he measures has magnitude

$$|\vec{v}| = \frac{|\vec{p}|}{E}, \tag{2.5.4}$$

where $|\vec{p}|$ and E are as given above; and (e) the 4-vector $\boldsymbol{v}$, whose components in the observer's Lorentz frame are

$$v^0 = 0, \quad v^j = (dx^j/dt)_{\text{for particle}} = \text{ordinary velocity}, \tag{2.5.5}$$

is given by

$$\boldsymbol{v} = \frac{\boldsymbol{p} + (\boldsymbol{p} \cdot \boldsymbol{u})\boldsymbol{u}}{-\boldsymbol{p} \cdot \boldsymbol{u}}. \tag{2.5.6}$$

Solution:

(a) In the frame of observer, we have $\boldsymbol{u} = (1, 0, 0, 0)$ and $p^0 = E$. Thus

$$E = -\boldsymbol{p} \cdot \boldsymbol{u} \tag{2.5.7}$$

holds in observer's frame. Since the two sides of the equation are all geometric objects which is invariant in different reference frames, we have proved that

$$E = -\boldsymbol{p} \cdot \boldsymbol{u}. \tag{2.5.8}$$

(b) In observer's frame, we have $\boldsymbol{p} = (E, \vec{p})$. So the rest mass he attributes to the particle is

$$m^2 = E^2 - |\vec{p}|^2 = -\boldsymbol{p}^2. \tag{2.5.9}$$

(c)

$$|\vec{p}| = (E^2 - m^2)^{1/2} = [(\boldsymbol{p} \cdot \boldsymbol{u})^2 + (\boldsymbol{p} \cdot \boldsymbol{p})]^{1/2}. \tag{2.5.10}$$

(d) In observer's frame, we have $E = \gamma m$ and $\vec{p} = \gamma m \vec{v}$, where $\gamma = (1 - |\vec{v}|^2)^{-1/2}$. Thus

$$\vec{v} = \frac{\vec{p}}{E}, \quad |\vec{v}| = \frac{|\vec{p}|}{E}. \tag{2.5.11}$$

(e) In observer's frame, we have

$$p^0 + (\boldsymbol{p} \cdot \boldsymbol{u})u^0 = E - E = 0, \tag{2.5.12}$$

and

$$\frac{\vec{p} + (\boldsymbol{p} \cdot \boldsymbol{u})\vec{u}}{-\boldsymbol{p} \cdot \boldsymbol{u}} = \frac{\vec{p}}{E} = \vec{v}. \tag{2.5.13}$$

Thus

$$\boldsymbol{v} = \frac{\boldsymbol{p} + (\boldsymbol{p} \cdot \boldsymbol{u})\boldsymbol{u}}{-\boldsymbol{p} \cdot \boldsymbol{u}}. \tag{2.5.14}$$

holds in observer's frame. Since the two sides of the equation are all geometric objects which is invariant in different reference frames, it holds in all frames.

2.6 TEMPERATURE GRADIENT

To each event $\mathcal{L}$ inside the sun one attributes a temperature $T(\mathcal{L})$, the temperature measured by a thermometer at rest in the hot gas there. Then $T(\mathcal{L})$ is a function; no coordinates are required for its definition and discussion. A cosmic ray from outer space flies through the sun with 4-velocity $\boldsymbol{u}$. Show that, as measured by the cosmic ray's clock, the time derivative of temperature in its vicinity is

$$dT/d\tau = \partial_{\boldsymbol{u}} T = \langle \boldsymbol{d}T, \boldsymbol{u} \rangle. \tag{2.6.1}$$

In a local Lorentz frame inside the sun, this equation can be written

$$\frac{dT}{d\tau} = u^\alpha \frac{\partial T}{\partial x^\alpha} = \frac{1}{\sqrt{1 - v^2}} \frac{\partial T}{\partial t} + \frac{v^j}{\sqrt{1 - v^2}} \frac{\partial T}{\partial x^j}. \tag{2.6.2}$$

Why is this result reasonable?

Solution:

The equation (2.15) of the MTW tells

$$T(\mathcal{L}) = T(\mathcal{L}_0) + \langle \boldsymbol{d}T, \mathcal{L} - \mathcal{L}_0 \rangle + (\text{nonlinear terms}). \tag{2.6.3}$$

Suppose the world line of the observer is $P(\tau)$, the we have

$$\mathcal{L}(\tau) - \mathcal{L}(\tau_0) = (\tau - \tau_0)\boldsymbol{u} + (\text{nonlinear terms}). \tag{2.6.4}$$

In the limit $\tau \to \tau_0$, we have

$$\frac{dT}{d\tau} = \langle \boldsymbol{d}T, \boldsymbol{u} \rangle. \tag{2.6.5}$$

In a local Lorentz frame inside the sun, we have

$$\frac{dT}{d\tau} = (\boldsymbol{d}T)_\alpha u^\alpha = u^\alpha \frac{\partial T}{\partial x^\alpha} = \frac{1}{\sqrt{1-v^2}}\frac{\partial T}{\partial t} + \frac{v^j}{\sqrt{1-v^2}}\frac{\partial T}{\partial x^j}. \tag{2.6.6}$$

2.7 BOOST IN AN ARBITRARY DIRECTION

An especially useful Lorentz transformation has the matrix components

$$\begin{aligned}
&\Lambda^{0'}{}_0 = \gamma \equiv \frac{1}{\sqrt{1-\beta^2}},\\
&\Lambda^{0'}{}_j = \Lambda^{j'}{}_0 = -\beta\gamma n^j,\\
&\Lambda^{j'}{}_k = \Lambda^{k'}_j = (\gamma - 1)n^j n^k + \delta^{jk},\\
&\Lambda^{\mu}{}_{\nu'} = (\text{same as } \Lambda^{\nu'}{}_{\mu} \text{ but with} \beta \text{ repalced by } -\beta),
\end{aligned} \tag{2.7.1}$$

where β, n^1, n^2 , and n^3 are parameters, and $\boldsymbol{n}^2 = n_1^2 + n_2^2 + n_3^3 = 1$. Show (a) that this does satisfy the condition $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$ equired of a Lorentz transformation (see Box 2.4); (b) that the primed frame moves with ordinary velocity $\beta\boldsymbol{n}$ as seen in the unprimed frame; (c) that the unprimed frame moves with ordinary velocity $-\beta\boldsymbol{n}$ as seen in the primed frame; and (d) that for motion in the z direction, the transformation matrices reduce to the familiar form

$$\Lambda^{\nu'}{}_{\mu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \quad \Lambda^{\mu}{}_{\nu'} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{2.7.2}$$

Solution:

(a)

$$\Lambda^{\alpha'}{}_0\eta_{\alpha'\beta'}\Lambda^{\beta'}{}_0 = \Lambda^{0'}{}_0\eta_{0'0'}\Lambda^{0'}{}_0 + \Lambda^{j'}{}_0\eta_{j'k'}\Lambda^{k'}{}_0 = -\gamma^2 + \beta^2\gamma^2 n^j n^j = -1 \tag{2.7.3}$$

$$\begin{aligned}
\Lambda^{\alpha'}{}_j\eta_{\alpha'\beta'}\Lambda^{\beta'}{}_k &= \Lambda^{0'}{}_j\eta_{0'0'}\Lambda^{0'}{}_k + \Lambda^{j'}{}_j\eta_{j'k'}\Lambda^{k'}{}_k\\
&= -\beta^2\gamma^2 n^j n^k + [(\gamma-1)n^j n^{i'} + \delta^{ji'}][(\gamma-1)n^k n^{i'} + \delta^{ki'}]\\
&= [-\beta^2\gamma^2 + (\gamma-1)^2 + 2(\gamma-1)]n^j n^k + \delta^{kj}\\
&= \delta_{jk}
\end{aligned} \tag{2.7.4}$$

$$\begin{aligned}
\Lambda^{\alpha'}{}_0\eta_{\alpha'\beta'}\Lambda^{\beta'}{}_j &= \Lambda^{0'}{}_0\eta_{0'0'}\Lambda^{0'}{}_j + \Lambda^{j'}{}_0\eta_{j'k'}\Lambda^{k'}{}_j\\
&= \beta\gamma^2 n^j - \beta\gamma n^{i'}[(\gamma-1)n^{i'}n^j + \delta^{ji'}]\\
&= 0
\end{aligned} \tag{2.7.5}$$

So, we have prove that $\Lambda^{\alpha'}{}_0\eta_{\alpha'\beta'}\Lambda^{\beta'}{}_0 = \eta_{\alpha\beta}$, i.e. $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.

(b) $x^{\mu'} = (t', 0, 0, 0)$ is the world line of one observer who is static at the origin of primed frame. The coordinates of his world line in unprimed frame is

$$x^{\mu} = \Lambda^{\mu}{}_{\nu'} x^{\nu'} = \gamma t'(1, \beta n^j). \tag{2.7.6}$$

So the velocity of the observer as seen by unprimed frame is

$$v^j = \frac{x^j}{t} = \frac{\gamma t' \beta n^j}{\gamma t'} = \beta n^j. \tag{2.7.7}$$

(c) $x^{\mu} = (t, 0, 0, 0)$ is the world line of one observer who is static at the origin of uprimed frame. The coordinates of his world line in primed frame is

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} = \gamma t(1, -\beta n^j). \tag{2.7.8}$$

So the velocity of the observer as seen by primed frame is

$$v'^j = \frac{x'^j}{t'} = \frac{-\gamma t \beta n^j}{\gamma t} = -\beta n^j. \tag{2.7.9}$$

(d) For motion in the z direction, we have $n^j = (0, 0, 1)$. Substitute it into equation (29), it is easy to figure out

$$\Lambda^{\nu'}{}_{\mu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \quad \Lambda^{\mu}{}_{\nu'} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{2.7.10}$$

Chapter 3

THE ELECTROMAGNETIC FIELD

3.1 EXERCISE

Derive equations (3.5) and (3.7) for the components of Faraday by comparing (3.4) with (3.2a,b), and by using definition (3.6).

Solution:

By comparing

$$\frac{dp^0}{d\tau} = e\boldsymbol{E}\cdot\boldsymbol{u} \tag{3.1.1}$$

and

$$\frac{dp^0}{d\tau} = eF^0{}_i\,u^i, \tag{3.1.2}$$

we have

$$F^0{}_i = E^i. \tag{3.1.3}$$

By comparing

$$\frac{d\boldsymbol{p}}{d\tau} = e(u^0\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) \tag{3.1.4}$$

and

$$\frac{dp^i}{d\tau} = eF^i{}_0\,u^0 + eF^i{}_j\,u^j, \tag{3.1.5}$$

we have

$$F^i{}_j = \epsilon_{ijk}B^k. \tag{3.1.6}$$

Now it is straightforward to get equation (3.5) and (3.7).

3.2 TRANSFORMATION LAW FOR COMPONENTS OF A TENSOR

From the transformation laws for components of vectors and 1-forms, derive the transformation law (3.14).

Solution:

$\boldsymbol{S}(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v})$ is invariant under coordinate transformation. In the original frame, we have

$$\boldsymbol{S}(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v}) = S^{\alpha\beta}{}_{\gamma}\sigma_\alpha\rho_\beta v^\gamma. \tag{3.2.1}$$

In the primed frame, we have

$$\boldsymbol{S}(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v}) = S^{\mu'\nu'}{}_{\lambda'}\sigma_{\mu'}\rho_{\nu'}v^{\lambda'}. \tag{3.2.2}$$

From the transformation laws for components of vectors and 1-forms, we have

$$\sigma_\alpha = (\Lambda^{-1})_\alpha{}^{\mu'}\sigma_{\mu'} = \Lambda^{\mu'}{}_\alpha\sigma_{\mu'}, \quad \rho_\beta = (\Lambda^{-1})_\beta{}^{\nu'}\rho_{\nu'} = \Lambda^{\nu'}{}_\beta\sigma_{\nu'}, \quad v^\gamma = (\Lambda^{-1})^\gamma{}_{\lambda'}v^{\lambda'} = \Lambda_{\lambda'}{}^\gamma\sigma_{\lambda'}. \tag{3.2.3}$$

Now, we can get

$$S^{\mu'\nu'}{}_{\lambda'} = \Lambda^{\mu'}{}_\alpha\Lambda^{\nu'}{}_\beta\Lambda_{\lambda'}{}^\gamma S^{\alpha\beta}{}_\gamma \tag{3.2.4}$$

3.3 RAISING AND LOWERING INDICES

Derive equations (3.16) from equation (3.15') plus the law $n_\alpha = \eta_{\alpha\beta}n^\beta$ for getting the components of the 1-form $\tilde{\boldsymbol{n}}$ from the components of its corresponding vector $\boldsymbol{n}$.

Solution:

$$S^\alpha{}_{\beta\gamma}\sigma_\alpha n^\beta v^\gamma = S^{\alpha\beta}{}_\gamma\sigma_\alpha n_\beta v^\gamma = S^{\alpha\mu}{}_\gamma\sigma_\alpha\eta_{\mu\beta}n^\beta v^\gamma. \tag{3.3.1}$$

Thus, we have

$$S^\alpha{}_{\beta\gamma} = \eta_{\mu\beta}S^{\alpha\mu}{}_\gamma. \tag{3.3.2}$$

And we can also get

$$S^{\alpha\mu}{}_\gamma = \eta^{\mu\nu}\eta_{\nu\beta}S^{\alpha\beta}{}_\gamma = \eta^{\mu\nu}S^\alpha{}_{\nu\gamma} = \eta^{\mu\beta}S^\alpha{}_{\beta\gamma}. \tag{3.3.3}$$

3.4 TENSOR PRODUCT

Given any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, one defines the second-rank tensor $\boldsymbol{u}\otimes\boldsymbol{v}$ ("tensor product of $\boldsymbol{u}$ with $\boldsymbol{v}$") to be a machine, with two input slots, whose output is the number

$$(\boldsymbol{u}\otimes\boldsymbol{v})(\boldsymbol{\sigma},\boldsymbol{\lambda}) = \langle\boldsymbol{u},\boldsymbol{\sigma}\rangle\langle\boldsymbol{v},\boldsymbol{\lambda}\rangle \tag{3.4.1}$$

when 1-forms $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ are inserted. Show that the components of $\boldsymbol{T} = \boldsymbol{u}\otimes\boldsymbol{v}$ are the products of the components of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$T^{\alpha\beta} = u^\alpha v^\beta, \quad T_\alpha{}^\beta = u_\alpha v^\beta, \quad T_{\alpha\beta} = u_\alpha v_\beta. \tag{3.4.2}$$

Extend the definition to several vectors and forms,

$$(\boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{\beta}\otimes\boldsymbol{w})(\boldsymbol{\sigma},\boldsymbol{\lambda},\boldsymbol{n},\boldsymbol{\xi}) = \langle\boldsymbol{u},\boldsymbol{\sigma}\rangle\langle\boldsymbol{v},\boldsymbol{\lambda}\rangle\langle\boldsymbol{\beta},\boldsymbol{n}\rangle\langle\boldsymbol{\xi},\boldsymbol{w}\rangle, \tag{3.4.3}$$

and show that the product rule for components still holds:

$$\boldsymbol{S} = \boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{\beta}\otimes\boldsymbol{w} \text{ has components } S^{\mu\nu}{}_\lambda{}^\xi = u^\mu v^\nu\beta_\lambda w^\xi. \tag{3.4.4}$$

Solution:

$$\begin{aligned} T^{\alpha\beta} &= \boldsymbol{T}(\boldsymbol{\omega}^\alpha,\boldsymbol{\omega}^\beta) = \langle\boldsymbol{u},\boldsymbol{\omega}^\alpha\rangle\langle\boldsymbol{v},\boldsymbol{\omega}^\beta\rangle = T^{\alpha\beta};\\ T_\alpha{}^\beta &= \eta_{\alpha\mu}T^{\mu\beta} = \eta_{\alpha\mu}u^\mu v^\beta = u_\alpha v^\beta;\\ T_{\alpha\beta} &= \eta_{\beta\mu}T_\alpha{}^\mu = \eta_{\beta\mu}u_\alpha v^\mu = u_\alpha v_\beta. \end{aligned} \tag{3.4.5}$$

$$S^{\mu\nu}{}_{\lambda}{}^{\xi} = \boldsymbol{S}(\boldsymbol{\omega}^{\mu}, \boldsymbol{\omega}^{\nu}, \boldsymbol{e}_{\lambda}, \boldsymbol{\omega}^{\xi}) = \langle \boldsymbol{u}, \boldsymbol{\omega}^{\mu}\rangle\langle \boldsymbol{v}, \boldsymbol{\omega}^{\nu}\rangle\langle \boldsymbol{\beta}, \boldsymbol{e}_{\lambda}\rangle\langle \boldsymbol{\omega}^{\xi}, \boldsymbol{w}\rangle = u^{\mu}v^{\nu}\beta_{\lambda}w^{\xi}. \tag{3.4.6}$$

3.5 BASIS TENSORS

Show that a tensor $\boldsymbol{M}$ with components $M^{\alpha\beta}{}_{\gamma}{}^{\delta}$ in a given Lorentz frame can be reconstructed from its components and from the basis 1-forms and vectors of that frame as follows:

$$\boldsymbol{M} = M^{\alpha\beta}{}_{\gamma}{}^{\delta}\boldsymbol{e}_{\alpha}\otimes\boldsymbol{e}_{\beta}\otimes\boldsymbol{\omega}^{\gamma}\otimes\boldsymbol{e}_{\delta}. \tag{3.5.1}$$

Solution:

Because

$$\begin{aligned} &M^{\mu\nu}{}_{\rho}{}^{\sigma}\boldsymbol{e}_{\mu}\otimes\boldsymbol{e}_{\nu}\otimes\boldsymbol{\omega}^{\rho}\otimes\boldsymbol{e}_{\sigma}(\boldsymbol{\omega}^{\alpha},\boldsymbol{\omega}^{\beta},\boldsymbol{e}_{\gamma},\boldsymbol{\omega}^{\delta}) \\ &= M^{\mu\nu}{}_{\rho}{}^{\sigma}\langle\boldsymbol{\omega}^{\alpha},\boldsymbol{e}_{\mu}\rangle\langle\boldsymbol{\omega}^{\beta},\boldsymbol{e}_{\nu}\rangle\langle\boldsymbol{\omega}^{\rho},\boldsymbol{e}_{\gamma}\rangle\langle\boldsymbol{\omega}^{\delta},\boldsymbol{e}_{\sigma}\rangle \\ &= M^{\mu\nu}{}_{\rho}{}^{\sigma}\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\delta^{\rho}{}_{\gamma}\delta^{\delta}{}_{\sigma} \\ &= M^{\alpha\beta}{}_{\gamma}{}^{\delta}, \end{aligned} \tag{3.5.2}$$

$$\boldsymbol{M}(\boldsymbol{\omega}^{\alpha},\boldsymbol{\omega}^{\beta},\boldsymbol{\omega}^{\gamma},\boldsymbol{\omega}^{\delta}) = M^{\alpha\beta}{}_{\gamma}{}^{\delta}, \tag{3.5.3}$$

we have

$$\boldsymbol{M} = M^{\alpha\beta}{}_{\gamma}{}^{\delta}\boldsymbol{e}_{\alpha}\otimes\boldsymbol{e}_{\beta}\otimes\boldsymbol{\omega}^{\gamma}\otimes\boldsymbol{e}_{\delta}. \tag{3.5.4}$$

3.6 Faraday MACHINERY AT WORK

An observer with 4-velocity $\boldsymbol{u}$ picks out three directions in spacetime that are orthogonal and purely spatial (no time part) as seen in his frame. Let $\boldsymbol{e}_{\hat{1}}$, $\boldsymbol{e}_{\hat{2}}$, $\boldsymbol{e}_{\hat{3}}$ be unit vectors in those directions and let them be oriented in a righthanded way ($\boldsymbol{e}_{\hat{1}}\cdot\boldsymbol{e}_{\hat{2}}\times\boldsymbol{e}_{\hat{3}} = +1$ in three-dimensional language). Why do the following relations hold?

$$\boldsymbol{e}_{\hat{j}}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{e}_{\hat{j}}\cdot\boldsymbol{e}_{\hat{k}} = \delta_{jk}. \tag{3.6.1}$$

What vectors are to be inserted in the two slots of the electromagnetic field tensor Faraday if one wants to get out the electric field along $\boldsymbol{e}_{\hat{j}}$; as measured by this observer? What vectors must be inserted to get the magnetic field he measures along $\boldsymbol{e}_{\hat{j}}$?

Solution:

As $\boldsymbol{e}_{\hat{j}}$ are orthogonal and purely spatial as seen by the observer with four velocity $\boldsymbol{u}$, we have the following relations:

$$\boldsymbol{e}_{\hat{j}}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{e}_{\hat{j}}\cdot\boldsymbol{e}_{\hat{k}} = \delta_{jk}. \tag{3.6.2}$$

The electric field along $\boldsymbol{e}_{\hat{j}}$ as measured by this observer is

$$\boldsymbol{F}(\boldsymbol{e}_{\hat{j}},\boldsymbol{u}). \tag{3.6.3}$$

The magnetic field he measures along $\boldsymbol{e}_{\hat{j}}$ is

$$\boldsymbol{F}(\boldsymbol{e}_{\hat{i}},\boldsymbol{e}_{\hat{k}}) \tag{3.6.4}$$

where $\boldsymbol{e}_{\hat{j}}\cdot\boldsymbol{e}_{\hat{i}}\times\boldsymbol{e}_{\hat{k}} = +1$.

3.7 MAXWELL'S EQUATIONS

Show, by explicit examination of components, that the geometric laws

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \quad F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\alpha}, \tag{3.7.1}$$

do reduce to Maxwell's equations (3.26), (3.31), (3.34), (3.35), as claimed above.

Solution:

If two among the indexes α, β and γ are the same, it is easy to find

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.7.2}$$

would hold automatically due to the antisymmetry of the Faraday tensor. Now we assume all indexes are different. We have

$$F_{0i,j} + F_{ij,0} + F_{j0,i} = -E_{i,j} + \epsilon_{ijk}\dot{B}_k + E_{j,i} = \epsilon_{ijk}\left[\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right]_k = 0, \tag{3.7.3}$$

and

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = \epsilon_{ijl}B_{l,k} + \epsilon_{jkl}B_{l,i} + \epsilon_{kil}B_{l,j} = B_{k,k} + B_{i,i} + B_{j,j} = 3\nabla \cdot \boldsymbol{B} = 0. \tag{3.7.4}$$

They reduce to the (3.26) and (3.31).

As for the second equation, we have

$$F^{0i}{}_{,i} = E_{i,i} = \nabla \cdot \boldsymbol{E} = 4\pi J^0 \tag{3.7.5}$$

and

$$F^{i0}{}_{,0} + F^{ij}{}_{,j} = -\dot{E}_i + \epsilon_{ijk}B_{k,j} = \left[\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t}\right]_i = 4\pi \boldsymbol{J}_i. \tag{3.7.6}$$

They reduce to the (3.34) and (3.35).

3.8 CONTRACTION IS FRAME-INDEPENDENT

Show that contraction, as defined in equation (3.40), does not depend on which Lorentz frame $\boldsymbol{e}_a$ and $\boldsymbol{\omega}^a$ are taken from. Also show that equation (3.40) implies

$$\boldsymbol{M}(\boldsymbol{u}, \boldsymbol{v}) = R_{\alpha\mu}{}^{\alpha}{}_{\nu}u^{\mu}v^{\nu}. \tag{3.8.1}$$

Solution:

Suppose in another frame, the base vector and 1-form are $\boldsymbol{f}$ and $\boldsymbol{\xi}$. Then we have

$$\boldsymbol{f}_{\alpha} = \Lambda_{\alpha}{}^{\beta}\boldsymbol{e}_{\beta}, \quad \boldsymbol{\xi}^{\alpha} = \Lambda^{\alpha}{}_{\beta}\boldsymbol{\omega}^{\beta}. \tag{3.8.2}$$

Thus,

$$\boldsymbol{R}(\boldsymbol{f}_{\alpha}, \boldsymbol{u}, \boldsymbol{\xi}^{\alpha}, \boldsymbol{v}) = \Lambda_{\alpha}{}^{\beta}\Lambda^{\alpha}{}_{\gamma}\boldsymbol{R}(\boldsymbol{e}_{\beta}, \boldsymbol{u}, \boldsymbol{\omega}^{\gamma}, \boldsymbol{v}) = \delta^{\beta}_{\gamma}\boldsymbol{R}(\boldsymbol{e}_{\beta}, \boldsymbol{u}, \boldsymbol{\omega}^{\gamma}, \boldsymbol{v}) = \boldsymbol{R}(\boldsymbol{e}_{\beta}, \boldsymbol{u}, \boldsymbol{\omega}^{\beta}, \boldsymbol{v}). \tag{3.8.3}$$

So contraction is frame independent. And

$$\boldsymbol{M}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{R}(\boldsymbol{e}_{\alpha}, \boldsymbol{u}, \boldsymbol{\xi}^{\alpha}, \boldsymbol{v}) = R_{\beta\mu}{}^{\gamma}{}_{\nu}(\boldsymbol{e}_{\alpha})^{\beta}u^{\mu}(\boldsymbol{\omega}^{\alpha})_{\gamma}v^{\nu} = R_{\alpha\mu}{}^{\alpha}{}_{\nu}u^{\mu}v^{\nu}. \tag{3.8.4}$$

3.9 DIFFERENTIATION

(a) Justify the formula

$$d(u_\mu v^\nu)/d\tau = (d(u_\mu)/d\tau)v^\nu + u_\mu(d(v^\nu)/d\tau). \tag{3.9.1}$$

by considering the special case $\mu = 0$, $\nu = 1$.

(b) Explain why

$$(T^{\alpha\beta}v_\beta)_{,\mu} = T^{\alpha\beta}{}_{,\mu}\, v_\beta + T^{\alpha\beta}v_{\beta,\mu}. \tag{3.9.2}$$

Solution:

(a)

$$d(u_0v^1)/d\tau = d(u_0)/d\tau v^1 + u_0 d(v^1)/d\tau. \tag{3.9.3}$$

(b) The contraction of a tensor is a kind of summation as for the components. Since the order of differention and summation can be exchaged. So we have the equation 3.9.2.

3.10 MORE DIFFERENTIATION

(a) Justify the formula,

$$d(u_\mu u^\nu)/d\tau = 2u_\mu(du^\mu/d\tau) \tag{3.10.1}$$

by writing out the summation $u_\mu u^\mu \equiv \eta_{\mu\nu}u^\mu u^\nu$ explicitly. (b) Let δ indicate a variation or small change, and justify the formula

$$\delta(F^{\alpha\beta}F_{\alpha\beta}) = 2F_{\alpha\beta}\delta(F^{\alpha\beta}). \tag{3.10.2}$$

(c) Compute $(F^{\alpha\beta}F_{\alpha\beta})_{,\mu} = ?$

Solution:

(a)

$$\begin{aligned} d(u_\mu u^\nu)/d\tau &= d(\eta_{\mu\nu}u^\mu u^\nu)/d\tau \\ &= \eta_{\mu\nu}u^\mu(du^\nu/d\tau) + \eta_{\mu\nu}u^\nu(du^\mu/d\tau) \\ &= u_\nu(du^\nu/d\tau) + u_\mu(du^\mu/d\tau) \\ &= 2u_\mu(du^\mu/d\tau). \end{aligned} \tag{3.10.3}$$

(b)

$$\begin{aligned} \delta(F^{\alpha\beta}F_{\alpha\beta}) &= \delta(\eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}F^{\mu\nu}) \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}\delta(F^{\mu\nu}) + \eta_{\mu\alpha}\eta_{\nu\beta}F^{\mu\nu}\delta(F^{\alpha\beta}) \\ &= F_{\mu\nu}\delta(F^{\mu\nu}) + F_{\alpha\beta}\delta(F^{\alpha\beta}) \\ &= 2F_{\alpha\beta}\delta(F^{\alpha\beta}). \end{aligned} \tag{3.10.4}$$

(c)

$$(F^{\alpha\beta}F_{\alpha\beta})_{,\mu} = 2F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu}\,. \tag{3.10.5}$$

3.11 SYMMETRIES

Let $A_{\mu\nu}$ be an anbsymmetric tensor so that $A_{\mu\nu} = -A_{\nu\mu}$; and let $S^{\mu\nu}$ be a symmetrIc tensor so that $S^{\mu\nu} = S^{\nu\mu}$.

(a) Justify the equations $A_{\mu\nu}S^{\mu\nu} = 0$ in two ways: first, by writing out the sum explIcitly (all sixteen terms) and showing how the terms in the sum cancel in pairs; second, by giving an argument to justify each equals sign in the following string:

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\alpha\beta}S^{\alpha\beta} = -A_{\mu\nu}S^{\mu\nu} = 0. \tag{3.11.1}$$

(b) Establish the following two identities for any arbitrary tensor $V_{\mu\nu}$:

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} - V^{\nu\mu})A_{\mu\nu}; \quad V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} + V^{\nu\mu})S_{\mu\nu}. \tag{3.11.2}$$

Solution:

(a) i.

$$\begin{aligned} A_{\mu\nu}S^{\mu\nu} &= A_{00}S^{00} + A_{11}S^{11} + A_{22}S^{22} + A_{33}S^{33} \\ &+ (A_{01}S^{01} + A_{10}S^{10}) + (A_{02}S^{02} + A_{20}S^{20}) \\ &+ (A_{03}S^{03} + A_{30}S^{30}) + (A_{12}S^{12} + A_{21}S^{21}) \\ &+ (A_{13}S^{13} + A_{31}S^{31}) + (A_{23}S^{23} + A_{32}S^{32}) \\ &= 0. \end{aligned} \tag{3.11.3}$$

ii. The first equals sign is due to the antisymmetry of $A_{\mu\nu}$. The second equals sign is due to the symmetry of $S_{\mu\nu}$. The third and fourth equals sign is a transformation of dumb indexes. The last equals sign is due to the fact that $A_{\mu\nu}S^{\mu\nu} = -A_{\mu\nu}S^{\mu\nu}$.

(b) Since

$$V^{\mu\nu}A_{\mu\nu} = -V^{\mu\nu}A_{\nu\mu} = -V^{\nu\mu}A_{\mu\nu} \tag{3.11.4}$$

and

$$V^{\mu\nu}S_{\mu\nu} = V^{\mu\nu}S_{\nu\mu} = V^{\nu\mu}S_{\mu\nu}, \tag{3.11.5}$$

we have

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} - V^{\nu\mu})A_{\mu\nu}; \quad V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} + V^{\nu\mu})S_{\mu\nu}. \tag{3.11.6}$$

3.12 SYMMETRIZATION AND ANTISYMMETRIZATION

To "symmetrize" a tensor, one averages it with all of its transposes. The components of the new, symmetrized tensor are distinguished by round brackets:

$$\begin{aligned} V_{(\mu\nu)} &\equiv \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu}); \\ V_{(\mu\nu\lambda)} &\equiv \frac{1}{3!}(V_{\mu\nu\lambda} + V_{\nu\lambda\mu} + V_{\lambda\mu\nu} + V_{\nu\mu\lambda} + V_{\lambda\nu\mu} + V_{\mu\lambda\nu}). \end{aligned} \tag{3.12.1}$$

One "antisymmetrizes" a tensor (square brackets) similarly:

$$\begin{aligned} V_{[\mu\nu]} &\equiv \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu}); \\ V_{[\mu\nu\lambda]} &\equiv \frac{1}{3!}(V_{\mu\nu\lambda} + V_{\nu\lambda\mu} + V_{\lambda\mu\nu} - V_{\nu\mu\lambda} - V_{\lambda\nu\mu} - V_{\mu\lambda\nu}). \end{aligned} \tag{3.12.2}$$

(a) Show that such symmetrized and antisymmetrized tensors are, indeed, symmetric and antisymmetric under interchange of the vectors inserted into their slots:

$$V_{(\alpha\beta\gamma)}u^\alpha v^\beta w^\gamma = +V_{(\alpha\beta\gamma)}v^\alpha u^\beta w^\gamma = \cdots$$
$$V_{[\alpha\beta\gamma]}u^\alpha v^\beta w^\gamma = -V_{(\alpha\beta\gamma)}v^\alpha u^\beta w^\gamma = \cdots \tag{3.12.3}$$

(b) Show that a second-rank tensor can be reconstructed from its symmetric and antisymmetric parts,

$$V_{\mu\nu} = V_{(\mu\nu)} + V_{[\mu\nu]}, \tag{3.12.4}$$

but that a third-rank tensor cannot; $V_{(\alpha\beta\gamma)}$ and $V_{[\alpha\beta\gamma]}$ contain together "less information" than $V_{\alpha\beta\gamma}$. "Young diagrams" (see, e.g., Messiah [1961], appendix D) describe other symmetries, more subtle than these two, which contain the missing information.

(c) Show that the electromagnetic field tensor satisfies

$$F_{(\alpha\beta)} = 0; \quad F_{[\alpha\beta]} = 0. \tag{3.12.5}$$

(d) Show that Maxwell's "magnetic" equations

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.12.6}$$

can be rewritten in the form

$$F_{[\alpha\beta,\gamma]} = 0. \tag{3.12.7}$$

Solution:

(a)

$$V_{(\alpha\beta\gamma)}u^\alpha v^\beta w^\gamma = V_{(\beta\alpha\gamma)}v^\beta u^\alpha w^\gamma = V_{(\alpha\beta\gamma)}u^\alpha v^\beta w^\gamma,$$
$$V_{[\alpha\beta\gamma]}u^\alpha v^\beta w^\gamma = -V_{[\beta\alpha\gamma]}v^\beta u^\alpha w^\gamma = -V_{(\alpha\beta\gamma)}v^\alpha u^\beta w^\gamma. \tag{3.12.8}$$

(b)

$$V_{(\mu\nu)} + V_{[\mu\nu]} = \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu}) + \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu}) = V_{\mu\nu}. \tag{3.12.9}$$

The degree of freedom of $V_{\alpha\beta\gamma}$, $V_{(\alpha\beta\gamma)}$, $V_{[\alpha\beta\gamma]}$ is d^3, $C_d^3 + C_d^2 + C_d^1$, and C_d^3 respectively. d is the dimension of the space. We have

$$d^3 - (2C_d^3 + C_d^2 + C_d^1) = \frac{(4d+7)(d-1)d}{6} > 0 \tag{3.12.10}$$

if $d > 1$. So a third-rank tensor cannot be reconstructed from its symmetric and antisymmetric parts.

(c) Since $F_{\alpha\beta} + F_{\beta\alpha} = 0$, it is easy to get

$$F_{(\alpha\beta)} = 0; \quad F_{[\alpha\beta]} = 0. \tag{3.12.11}$$

(d) Since $F_{\alpha\beta} + F_{\beta\alpha} = 0$, we have

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = -(F_{\beta\alpha,\gamma} + F_{\gamma\beta,\alpha} + F_{\alpha\gamma,\beta}). \tag{3.12.12}$$

Thus,

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{3.12.13}$$

is equal to

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} - (F_{\beta\alpha,\gamma} + F_{\gamma\beta,\alpha} + F_{\alpha\gamma,\beta}) = 6F_{[\alpha\beta,\gamma]} = 0. \tag{3.12.14}$$

3.13 LEVI-CIVITA TENSOR

The "Levi-Civita tensor" $\boldsymbol{\epsilon}$ in spacetime is a fourth-rank, completely antisymmetric tensor:

$$\boldsymbol{\epsilon}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \text{ changes sign when any two of the vectors are interchanged.} \tag{3.13.1}$$

Choose an arbitrary but specIfic Lorentz frame, with $\boldsymbol{e}_0$ pointing toward the future and with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ a righthanded set of spatial basis vectors. The covariant components of $\boldsymbol{\epsilon}$ in this frame are

$$\epsilon_{0123} = \boldsymbol{\epsilon}(\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3) = +1. \tag{3.13.2}$$

[Note: In an n-dimensional space, $\boldsymbol{\epsilon}$ is the analogous completely antisymmetric rank-n tensor. Its components are

$$\epsilon_{1,\cdots,n} = \boldsymbol{\epsilon}(\boldsymbol{e}_1,\cdots,\boldsymbol{e}_n) = +1, \tag{3.13.3}$$

when computed on a "positively oriented," orthonormal basis $\boldsymbol{e}_1,\cdots,\boldsymbol{e}_n$.]

(a) Use the antisymmetry to show that

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} &= 0 \text{ unless } \alpha,\beta,\gamma,\delta \text{ are all different,} \\ \epsilon_{\pi 0\pi 1\pi 2\pi 3} &= \begin{cases} +1 \text{ for even permutations of } 0,1,2,3 \text{ and} \\ -1 \text{ for odd permutations.} \end{cases} \end{aligned} \tag{3.13.4}$$

(b) Show that

$$\epsilon^{\pi 0\pi 1\pi 2\pi 3} = -\epsilon_{\pi 0\pi 1\pi 2\pi 3}. \tag{3.13.5}$$

(c) By means of a Lorentz transformation show that $\epsilon^{\bar\alpha\bar\beta\bar\gamma\bar\delta}$ and $\epsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}$ have these same values in any other Lorentz frame with $\boldsymbol{e}_0$ pointing toward the future and with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ a righthanded set.

(d) What are the components of $\boldsymbol{\epsilon}$ in a Lorentz frame with past-pointing $\boldsymbol{e}_0$? with lefthanded $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$?

(e) From the Levi-Civita tensor, one can construct several "permutation tensors." In index notation:

$$\begin{aligned} \delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} &\equiv -\epsilon^{\alpha\beta\gamma\rho}\epsilon_{\mu\nu\lambda\rho}; \\ \delta^{\alpha\beta}{}_{\mu\nu} &\equiv \frac{1}{2}\delta^{\alpha\beta\lambda}{}_{\mu\nu\lambda} = -\frac{1}{2}\epsilon^{\alpha\beta\lambda\rho}\epsilon_{\mu\nu\lambda\rho}; \\ \delta^{\alpha}{}_{\mu} &\equiv \frac{1}{3}\delta^{\alpha\beta}{}_{\mu\beta} = \frac{1}{6}\delta^{\alpha\beta\lambda}{}_{\mu\beta\lambda} = -\frac{1}{6}\epsilon^{\alpha\beta\lambda\rho}\epsilon_{\mu\beta\lambda\rho}. \end{aligned} \tag{3.13.6}$$

Show that:

$$\begin{aligned} \delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} &= \begin{cases} +1 \text{ if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda, \\ -1 \text{ if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda, \\ 0 \text{ otherwise;} \end{cases} \\ \delta^{\alpha\beta}{}_{\mu\nu} &= \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu} \\ &= \begin{cases} +1 \text{ if } \alpha\beta \text{ is an even permutation of } \mu\nu, \\ -1 \text{ if } \alpha\beta \text{ is an odd permutation of } \mu\nu, \\ 0 \text{ otherwise;} \end{cases} \\ \delta^{\alpha}{}_{\mu} &= \begin{cases} +1 \text{ if } \alpha = \mu, \\ 0 \text{ otherwise.} \end{cases} \end{aligned} \tag{3.13.7}$$

Solution:

(a) If $\alpha,\beta,\gamma,\delta$ are not all different, say $\alpha = \beta$, we then have

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\beta\alpha\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta}. \tag{3.13.8}$$

Thus $\epsilon_{\alpha\beta\gamma\delta} = 0$ unless $\alpha, \beta, \gamma, \delta$ are all different. If π is an even permutation, we have

$$\epsilon_{\pi 0 \pi 1 \pi 2 \pi 3} = (-1)^{\pi} \epsilon_{0123} = 1. \tag{3.13.9}$$

similarly, for odd permutations of $0, 1, 2, 3$, we have

$$\epsilon_{\pi 0 \pi 1 \pi 2 \pi 3} = (-1)^{\pi} \epsilon_{0123} = -1. \tag{3.13.10}$$

(b)

$$\epsilon^{\pi 0 \pi 1 \pi 2 \pi 3} = g^{\pi 0 \alpha} g^{\pi 1 \beta} g^{\pi 2 \gamma} g^{\pi 3 \delta} \epsilon_{\alpha\beta\gamma\delta} = \begin{cases} \det g = -1 \text{ if } \pi \text{ is an even permutation} \\ -\det g = 1 \text{ if } \pi \text{ is an odd permutation} \end{cases} \tag{3.13.11}$$

Therefore,

$$\epsilon^{\pi 0 \pi 1 \pi 2 \pi 3} = -\epsilon_{\pi 0 \pi 1 \pi 2 \pi 3}. \tag{3.13.12}$$

(c) Firstly, we have

$$\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = \Lambda^{\bar{\alpha}}{}_{\alpha} \Lambda^{\bar{\beta}}{}_{\beta} \Lambda^{\bar{\gamma}}{}_{\gamma} \Lambda^{\bar{\delta}}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta}. \tag{3.13.13}$$

If $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are not all different, say $\bar{\alpha} = \bar{\beta} = \bar{0}$, we have

$$\epsilon^{\bar{0}\bar{0}\bar{\gamma}\bar{\delta}} = \Lambda^{\bar{0}}{}_{\alpha} \Lambda^{\bar{0}}{}_{\beta} \Lambda^{\bar{\gamma}}{}_{\gamma} \Lambda^{\bar{\delta}}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = -\Lambda^{\bar{0}}{}_{\beta} \Lambda^{\bar{0}}{}_{\alpha} \Lambda^{\bar{\gamma}}{}_{\gamma} \Lambda^{\bar{\delta}}{}_{\delta} \epsilon^{\beta\alpha\gamma\delta} = -\Lambda^{\bar{0}}{}_{\alpha} \Lambda^{\bar{0}}{}_{\beta} \Lambda^{\bar{\gamma}}{}_{\gamma} \Lambda^{\bar{\delta}}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = -\epsilon^{\bar{0}\bar{0}\bar{\gamma}\bar{\delta}}. \tag{3.13.14}$$

Thus $\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$ is 0 unless $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are all different. If $\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}$ is a permutation of $\bar{0}\bar{1}\bar{2}\bar{3}$, we have

$$\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = \Lambda^{\pi\bar{0}}{}_{\alpha} \Lambda^{\pi\bar{1}}{}_{\beta} \Lambda^{\pi\bar{2}}{}_{\gamma} \Lambda^{\pi\bar{3}}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = -(-1)^{\pi} \Lambda^{\bar{0}}{}_{\alpha} \Lambda^{\bar{1}}{}_{\beta} \Lambda^{\bar{2}}{}_{\gamma} \Lambda^{\bar{3}}{}_{\delta} \epsilon_{\alpha\beta\gamma\delta} = -(-1)^{\pi} \det \Lambda. \tag{3.13.15}$$

Secondly, from the defination of Lorentz transformation, we have $\Lambda^{\mathrm{T}} \eta \Lambda = \eta$. Take the determinant of the both side of the equation, we have

$$\det \Lambda \det \eta \det \Lambda = \det \eta. \tag{3.13.16}$$

Thus $\det \Lambda = \pm 1$. If $\boldsymbol{e}_0$ is pointing toward the future with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are a righthanded set. The transformation can be made up of successive infinitesimal boost or space rotation $I + \delta\omega$. Because the determinant of $I + \delta\omega$ cannot be -1. We can conclude that $\det \Lambda = +1$. Now, we have

$$\begin{aligned} \epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} &= 0 \text{ unless } \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \text{ are all different,} \\ \epsilon^{\pi\bar{0}\pi\bar{1}\pi\bar{2}\pi\bar{3}} &= \begin{cases} -1 \text{ for even permutations of } \bar{0}, \bar{1}, \bar{2}, \bar{3} \text{ and} \\ +1 \text{ for odd permutations.} \end{cases} \end{aligned} \tag{3.13.17}$$

So $\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$ have these same values in any other Lorentz frame with $\boldsymbol{e}_0$ pointing toward the future and with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ a righthanded set.

(d) In a Lorentz frame with past-pointing $\boldsymbol{e}_0$ or lefthanded $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, we have $\det \Lambda = -1$. So the components of $\boldsymbol{\epsilon}$ are the opposite number of those in the original frame.

(e) Firstly we focus on

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} \equiv -\epsilon^{\alpha\beta\gamma\rho} \epsilon_{\mu\nu\lambda\rho}. \tag{3.13.18}$$

If α, β, γ or μ, ν, λ are not all different, $\epsilon^{\alpha\beta\gamma\rho}$ or $\epsilon_{\mu\nu\lambda\rho}$ will be zero, and so $\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda}$ will be zero. If $\alpha\beta\gamma$ is not a permutation of $\mu\nu\lambda$, α, β, γ and μ, ν, λ are all different, then ρ cannot be different from α, β, γ and μ, ν, λ simutaneously. So $\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda}$ will be zero. If $\alpha\beta\gamma$ is an even permutation of $\mu\nu\lambda$, then $\alpha\beta\gamma\rho$ is also an even permutation of $\mu\nu\lambda\rho$ if $\alpha\beta\gamma\rho$ ($\mu\nu\lambda\rho$) are all different (in this case, ρ has only one possible value). So

$$-\epsilon^{\alpha\beta\gamma\rho} \epsilon_{\mu\nu\lambda\rho} = -(\pm 1)(\mp 1) = 1. \tag{3.13.19}$$

If $\alpha\beta\gamma$ is an odd permutation of $\mu\nu\lambda$, then $\alpha\beta\gamma\rho$ is also an odd permutation of $\mu\nu\lambda\rho$ if $\alpha\beta\gamma\rho$ ($\mu\nu\lambda\rho$) are all different. So

$$-\epsilon^{\alpha\beta\gamma\rho} \epsilon_{\mu\nu\lambda\rho} = -(\pm 1)(\pm 1) = -1. \tag{3.13.20}$$

As a result, we can get

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = \begin{cases} +1 \text{ if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda, \\ -1 \text{ if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda, \\ 0 \text{ otherwise.} \end{cases} \tag{3.13.21}$$

Secondly, let us foucs on

$$\delta^{\alpha\beta}{}_{\mu\nu} \equiv \frac{1}{2}\delta^{\alpha\beta\lambda}{}_{\mu\nu\lambda}. \tag{3.13.22}$$

If $\alpha\beta$ is not a permutation of $\mu\nu$, $\alpha\beta\lambda$ will not be a permutation of $\mu\nu\lambda$. So $\delta^{\alpha\beta}{}_{\mu\nu}$ will be zero. If $\alpha\beta$ is an even permutation of $\mu\nu$, then $\alpha\beta\lambda$ will be an even permutation of $\mu\nu\lambda$ if α, β, λ and μ, ν, λ are all different. In this case, there are two possible values for λ and

$$\delta^{\alpha\beta\lambda}{}_{\mu\nu\lambda} = 2. \tag{3.13.23}$$

If $\alpha\beta$ is an odd permutation of $\mu\nu$, we have

$$\delta^{\alpha\beta\lambda}{}_{\mu\nu\lambda} = -2. \tag{3.13.24}$$

As a result, we can get

$$\delta^{\alpha\beta}{}_{\mu\nu} = \begin{cases} +1 \text{ if } \alpha\beta \text{ is an even permutation of } \mu\nu, \\ -1 \text{ if } \alpha\beta \text{ is an odd permutation of } \mu\nu, \\ 0 \text{ otherwise.} \end{cases} \tag{3.13.25}$$

$\delta^{\alpha\beta}{}_{\mu\nu} = \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}$ is easy to verify according to the equation above.

Finally, let us move to

$$\delta^{\alpha}{}_{\mu} \equiv \frac{1}{3}\delta^{\alpha\beta}{}_{\mu\beta}. \tag{3.13.26}$$

If $\alpha \neq \mu$, $\alpha\beta$ can not be a permutation of $\mu\beta$, and so $\delta^{\alpha}{}_{\mu}$ will be 0. If $\alpha = \mu$, $\alpha\beta$ will be an even permutation of $\mu\beta$ if α, β and μ, β are all different. In this case, there are three possible values for β and

$$\delta^{\alpha\beta}{}_{\mu\beta} = 3. \tag{3.13.27}$$

As a result, we can get

$$\delta^{\alpha}{}_{\mu} = \begin{cases} +1 \text{ if } \alpha = \mu, \\ 0 \text{ otherwise.} \end{cases} \tag{3.13.28}$$

3.14 DUALS

From any vector $\boldsymbol{J}$, any second-rank antisymmetric tensor $\boldsymbol{F}$ and any third-rank antisymmetric tensor $\boldsymbol{B}$, one can construct new tensors defined by

$${}^*J_{\alpha\beta\gamma} = J^{\mu}\epsilon_{\mu\alpha\beta\gamma}, \quad {}^*F_{\alpha\beta} = \frac{1}{2}F^{\mu\nu}\epsilon_{\mu\nu\alpha\beta}, \quad {}^*B_{\alpha} = \frac{1}{3!}B^{\lambda\mu\nu}\epsilon_{\lambda\mu\nu\alpha} \tag{3.14.1}$$

One calls ${}^*\boldsymbol{J}$ the "dual" of $\boldsymbol{J}$, ${}^*\boldsymbol{F}$ the dual of $\boldsymbol{F}$, and ${}^*\boldsymbol{B}$ the dual of $\boldsymbol{B}$. [A previous and entirely distinct use of the word "dual" (section 2.7) called a set of basis one-forms $\boldsymbol{\omega}^{\alpha}$ dual to a set of basIs vectors $\boldsymbol{e}_{\alpha}$ if $\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = \delta^{\alpha}{}_{\beta}$. Fortunately there are no grounds for confusion between the two types of duality. One relates sets of vectors to sets of 1-forms. The other relates antisymmetric tensors of rank p to antisymmetric tensors of rank $4 - p$.]

(a) Show that

$$^{**}\boldsymbol{J} = \boldsymbol{J}, \quad ^{**}\boldsymbol{F} = -\boldsymbol{F}, \quad ^{**}\boldsymbol{B} = \boldsymbol{B}. \tag{3.14.2}$$

so (aside from sign) one can recover any completely antisymmetric tensor $\boldsymbol{H}$ from its dual $^*\boldsymbol{H}$ by taking the dual once again, $^{**}\boldsymbol{J}$. This shows that $\boldsymbol{H}$ and $^*\boldsymbol{H}$ contain precisely the same information.

(b) Make explicit this fact of same-information-content by writing out the components $^*A^{\alpha\beta\gamma}$ in terms of A^α, also $^*F^{\alpha\beta}$ in terms of $F^{\alpha\beta}$, also $^*B^\alpha$ in terms of $B^{\alpha\beta\gamma}$.

Solution:

(a)

$$\begin{aligned}
^{**}J^\alpha &= \frac{1}{3!}{}^*J_{\lambda\mu\nu}\epsilon^{\lambda\mu\nu\alpha} = \frac{1}{3!}J^\beta\epsilon_{\beta\lambda\mu\nu}\epsilon^{\lambda\mu\nu\alpha} = -\frac{1}{6}J^\beta\epsilon_{\beta\lambda\mu\nu}\epsilon^{\alpha\lambda\mu\nu} = J^\beta\delta^\alpha{}_\beta = J^\alpha;\\
^{**}F^{\alpha\beta} &= \frac{1}{2}{}^*F_{\mu\nu}\,\epsilon^{\mu\nu\alpha\beta} = \frac{1}{4}F^{\rho\sigma}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\mu\nu\alpha\beta} = \frac{1}{4}F^{\rho\sigma}\epsilon_{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\mu\nu} = -\frac{1}{2}F^{\rho\sigma}\delta^{\alpha\beta}{}_{\rho\sigma} = -F^{\alpha\beta};\\
^{**}B^{\alpha\beta\gamma} &= {}^*B_\mu\epsilon^{\mu\alpha\beta\gamma} = \frac{1}{3}B^{\rho\sigma\lambda}\epsilon_{\rho\sigma\lambda\mu}\epsilon^{\mu\alpha\beta\gamma} = -\frac{1}{3!}B^{\rho\sigma\lambda}\epsilon_{\rho\sigma\lambda\mu}\epsilon^{\alpha\beta\gamma\mu} = \frac{1}{3!}B^{\rho\sigma\lambda}\delta^{\alpha\beta\gamma}{}_{\rho\sigma\lambda} = B^{\alpha\beta\gamma}.
\end{aligned} \tag{3.14.3}$$

(b)

$$^*A^{012} = A^3, \quad ^*A^{013} = -A^2, \quad ^*A^{023} = A^1, \quad ^*A^{123} = A^0. \tag{3.14.4}$$

$$^*F^{01} = -F^{23}, \quad ^*F^{02} = F^{13}, \quad ^*F^{03} = -F^{12}, \quad ^*F^{12} = F^{03}, \quad ^*F^{13} = -F^{02}, \quad ^*F^{23} = F^{01}. \tag{3.14.5}$$

$$^*B^0 = B^{123}, \quad ^*B^1 = B^{023}, \quad ^*B^2 = -B^{013}, \quad ^*B^3 = B^{012}. \tag{3.14.6}$$

3.15 GEOMETRIC VERSIONS OF MAXWELL EQUATIONS

Show that, if $\boldsymbol{F}$ is the electromagnetic field tensor, then $\boldsymbol{\nabla}\cdot{}^*\boldsymbol{F} = 0$ is a geometric frame-independent version of the Maxwell equations

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{3.15.1}$$

Similarly show that $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 4\pi\boldsymbol{J}$ (divergence on second slot of $\boldsymbol{F}$) is a geometric version of $F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$.

Solution:

From the defination of star operator, we have

$$[\boldsymbol{\nabla}\cdot{}^*\boldsymbol{F}]^\alpha = {}^*F^{\alpha\beta}{}_{,\beta} = \frac{1}{2}F_{\mu\nu,\beta}\epsilon^{\mu\nu\alpha\beta}\frac{1}{2}F_{\mu\nu,\beta}\epsilon^{\alpha\mu\nu\beta}. \tag{3.15.2}$$

So $\boldsymbol{\nabla}\cdot{}^*\boldsymbol{F} = 0$ is equal to the equation

$$F_{[\alpha\beta,\gamma]} = 0. \tag{3.15.3}$$

According to the conclusion of exercise 13(d), we can get the first pair of Maxwell equations,

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{3.15.4}$$

Similarly, we have

$$[\boldsymbol{\nabla}\cdot\boldsymbol{F}]^\alpha = F^{\alpha\beta}{}_{,\beta}\,. \tag{3.15.5}$$

Thus it is easy to see that $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 4\pi\boldsymbol{J}$ is a geometric version of $F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$.

3.16 CHARGE CONSERVATION

From Maxwell's equations $F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\alpha}$, derive the "equation of charge conservation"

$$J^{\alpha}{}_{,\alpha} = 0. \tag{3.16.1}$$

Show that this equation does, indeed, correspond to conservation of charge. It will be studied further in Chapter 5.

Solution:

$$J^{\alpha}{}_{,\alpha} = \frac{1}{4\pi} F^{\alpha\beta}{}_{,\beta\alpha} = \frac{1}{4\pi}[F^{\alpha\beta}{}_{,\beta\alpha} + F^{\beta\alpha}{}_{,\alpha\beta}] = \frac{1}{4\pi}[F^{\alpha\beta}{}_{,\beta\alpha}\, F^{\alpha\beta}{}_{,\alpha\beta}] = 0. \tag{3.16.2}$$

3.17 VECTOR POTENTIAL

The vector potential $\boldsymbol{A}$ of electromagnetic theory generates the electromagnetic field tensor via the geometric equation

$$\boldsymbol{F} = -(\text{antisymmetric part of } \boldsymbol{\nabla A}), \tag{3.17.1}$$

i.e.,

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{3.17.2}$$

(a) Show that the electric and magnetic fields in a specific Lorentz frame are given by

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \boldsymbol{\nabla} A^0. \tag{3.17.3}$$

(b) Show that $\boldsymbol{F}$ will satisfy Maxwell's equations if and only if $\boldsymbol{A}$ satisfies

$$A^{\alpha,\mu}{}_{,\mu} - A^{\mu}{}_{,\mu}{}^{,\alpha} = -4\pi J^{\alpha}. \tag{3.17.4}$$

(c) Show that "gauge transformations"

$$\boldsymbol{A}_{\text{NEW}} = \boldsymbol{A}_{\text{OLD}} + \boldsymbol{d}\phi, \quad \phi = \text{ arbitrary function}, \tag{3.17.5}$$

leave $\boldsymbol{F}$ unaffected.

(d) Show that one can adjust the gauge so that

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{A} &= 0 \quad (\text{"Lorentz gauge"}), \\ \Box \boldsymbol{A} &= -4\pi \boldsymbol{J}. \end{aligned} \tag{3.17.6}$$

Here $\Box$ is the wave operator ("d' Alembertian "):

$$\Box \boldsymbol{A} = A^{\alpha,\mu}{}_{,\mu} \boldsymbol{e}_{\alpha}. \tag{3.17.7}$$

Solution:

(a)

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}[\partial_j A_k - \partial_k A_j] = \epsilon^{ijk}\partial_j A_k,$$
$$E^i = F^{0i} = -\partial_t A^i - \partial_i A^0. \tag{3.17.8}$$

(b)

$$-F^{\alpha\mu}{}_{,\mu} = -(A^{\mu,\alpha} - A^{\alpha,\mu})_\mu = (A^{\alpha,\mu}{}_{,\mu} - A^{\mu}{}_{,\mu}{}^{,\alpha}) = -4\pi J^\alpha. \tag{3.17.9}$$

(c)

$$F^{\mu\nu}_{\rm NEW} = A^{\nu,\mu}_{\rm NEW} - A^{\mu,\nu}_{\rm NEW} = (A^{\nu,\mu}_{\rm OLD} + \phi^{,\nu\mu}) - (A^{\mu,\nu}_{\rm OLD} + \phi^{,\mu\nu}) = A^{\nu,\mu}_{\rm OLD} - A^{\mu,\nu}_{\rm OLD} = F^{\mu\nu}_{\rm OLD}. \tag{3.17.10}$$

(d) If we want

$$\partial_\mu A^\mu_{\rm NEW} = \partial_\mu A^\mu_{\rm OLD} + \partial_\mu\partial^\mu\phi = 0, \tag{3.17.11}$$

we should choose ϕ to satisfy the equation

$$\partial_\mu\partial^\mu\phi = -\partial_\mu A^\mu_{\rm OLD}. \tag{3.17.12}$$

In this case, we have

$$A^{\alpha,\mu}{}_{,\mu} - A^{\mu}{}_{,\mu}{}^{,\alpha} = A^{\alpha,\mu}{}_{,\mu} = -4\pi J^\alpha, \tag{3.17.13}$$

i.e.

$$\Box\boldsymbol{A} = -4\pi\boldsymbol{J}. \tag{3.17.14}$$

3.18 DIVERGENCE OF ELECTROMAGNETIC STRESS-ENERGY TENSOR

From an electromagnetic field tensor $\boldsymbol{F}$, one constructs a second-rank, symmetric tensor $\boldsymbol{T}$ as follows:

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{3.18.1}$$

As an exercise in index gymnastics:

(a) Show that $\boldsymbol{\nabla}\cdot\boldsymbol{T}$ has components

$$T^{\mu\nu}{}_{,\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta}{}^{,\mu}F^{\alpha\beta}\right]. \tag{3.18.2}$$

(b) Manipulate this expression into the form

$$T_\mu{}^{\nu}{}_{,\nu} = \frac{1}{4\pi}\left[-F_{\mu\alpha}F^{\alpha\nu}{}_{,\nu} - \frac{1}{2}F^{\alpha\beta}(F_{\alpha\beta,\mu} + F_{\mu\alpha,\beta} + F_{\beta\mu,\alpha})\right]; \tag{3.18.3}$$

note that the first term of (123) arises directly from the second term of (122).

(c) Use Maxwell's equations to conclude that

$$T^{\mu\nu}{}_{,\nu} = -F^{\mu\alpha}J_\alpha. \tag{3.18.4}$$

Solution:

(a)

$$\begin{aligned} T^{\mu\nu}{}_{,\nu} &= \frac{1}{4\pi}\left(F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta,\nu}F^{\alpha\beta} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}{}_{,\nu}\right) \\ &= \frac{1}{4\pi}\left(F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{4}F_{\alpha\beta}{}^{,\mu}F^{\alpha\beta} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta,\mu}\right) \\ &= \frac{1}{4\pi}\left(F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{4}F_{\alpha\beta}{}^{,\mu}F^{\alpha\beta} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}{}^{,\mu}\right) \\ &= \frac{1}{4\pi}\left(F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta}{}^{,\mu}F^{\alpha\beta}\right). \end{aligned} \tag{3.18.5}$$

(b)

$$\begin{aligned} T_{\mu}{}^{\nu}{}_{,\nu} &= \frac{1}{4\pi}\left(F_{\mu}{}^{\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta,\mu}F^{\alpha\beta}\right) \\ &= \frac{1}{4\pi}\left(F_{\mu\alpha,\nu}F^{\nu\alpha} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta,\mu}F^{\alpha\beta}\right) \\ &= \frac{1}{4\pi}\left(-F_{\mu\alpha,\beta}F^{\alpha\beta} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta,\mu}F^{\alpha\beta}\right) \\ &= \frac{1}{4\pi}\left(-\frac{1}{2}F_{\mu\alpha,\beta}F^{\alpha\beta} - \frac{1}{2}F_{\alpha\mu,\beta}F^{\beta\alpha} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta,\mu}F^{\alpha\beta}\right) \\ &= \frac{1}{4\pi}\left(-\frac{1}{2}F_{\mu\alpha,\beta}F^{\alpha\beta} - \frac{1}{2}F_{\beta\mu,\alpha}F^{\alpha\beta} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta,\mu}F^{\alpha\beta}\right) \\ &= \frac{1}{4\pi}\left[-F_{\mu\alpha}F^{\alpha\nu}{}_{,\nu} - \frac{1}{2}F^{\alpha\beta}(F_{\alpha\beta,\mu} + F_{\mu\alpha,\beta} + F_{\beta\mu,\alpha})\right]. \end{aligned} \tag{3.18.6}$$

(c) We have the Maxwell equations:

$$F^{\alpha\nu}{}_{,\nu} = 4\pi J^{\alpha}, \quad F_{\alpha\beta,\mu} + F_{\mu\alpha,\beta} + F_{\beta\mu,\alpha} = 0. \tag{3.18.7}$$

Thus

$$T_{\mu}{}^{\nu}{}_{,\nu} = -F_{\mu\alpha}J^{\alpha}, \tag{3.18.8}$$

i.e.

$$T^{\mu\nu}{}_{,\nu} = -F^{\mu\alpha}J_{\alpha}. \tag{3.18.9}$$

Chapter 4

ELECTROMAGNETISM AND DIFFERENTIAL FORMS

4.1 GENERIC LOCAL ELECTROMAGNETIC FIELD EXPRESSED IN SIMPLEST FORM

In the laboratory Lorentz frame, the electric field is $\boldsymbol{E}$, the magnetic field $\boldsymbol{B}$. Special cases are: (1) pure electrIc field ($\boldsymbol{B} = 0$); (2) pure magnetic field ($\boldsymbol{E} = 0$); and (3) "radiation field" or "null field" ($\boldsymbol{E}$ and $\boldsymbol{B}$ equal in magnitude and perpendicular in direction). All cases other than (1), (2), and (3) are "generic". In the generic case, calculate the Poynting density of flow of energy $\boldsymbol{E} \times \boldsymbol{B}/4\pi$ and the density of energy $(\boldsymbol{E}^2 + \boldsymbol{B}^2)/8\pi$. Define the direction of a unit vector $\boldsymbol{n}$ and the magnitude of a velocity parameter α by the ratio of energy flow to energy density:

$$\boldsymbol{n} \tanh 2\alpha = \frac{2\boldsymbol{E} \times \boldsymbol{B}}{\boldsymbol{E}^2 + \boldsymbol{B}^2}. \tag{4.1.1}$$

View the same electromagnetic field in a rocket frame moving in the direction of $\boldsymbol{n}$ with the velocity parameter α (not 2α; factor 2 comes in because energy flow and energy density are components, not of a vector, but of a tensor). By employing the formulas for a Lorentz transformation (equation 3.23), or otherwise, show that the energy flux vanishes in the rocket frame, with the consequence that $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ are parallel. No one can prevent the z-axis from being put in the direction common to $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$. Show that with thiS choice of direction, $\boldsymbol{Faraday}$ becomes

$$\boldsymbol{F} = \bar{E}_z \boldsymbol{d}\bar{z} \wedge \boldsymbol{d}\bar{t} + \bar{B}_z \boldsymbol{d}\bar{x} \wedge \boldsymbol{d}\bar{y} \tag{4.1.2}$$

(only two wedge products needed to represent the generic local field; "canonical representation"; valid in one frame, valid in any frame).

Solution:

By employing the formulas for a Lorentz transformation, we have

$$\begin{aligned} \bar{\boldsymbol{E}} &= \gamma(\boldsymbol{E}_\perp + \boldsymbol{\beta} \times \boldsymbol{B}) + \boldsymbol{E}_\parallel \\ &= \cosh\alpha[\boldsymbol{E} - (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{n} + \tanh\alpha\, \boldsymbol{n} \times \boldsymbol{B}] + (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{n} \\ &= \cosh\alpha[\boldsymbol{E} + \tanh\alpha\, \boldsymbol{n} \times \boldsymbol{B}] \\ &= \cosh\alpha\, \boldsymbol{E} + \sinh\alpha\, \boldsymbol{n} \times \boldsymbol{B}. \end{aligned} \tag{4.1.3}$$

$$\begin{aligned}\bar{\boldsymbol{B}} &= \gamma(\boldsymbol{B}_\perp - \boldsymbol{\beta}\times\boldsymbol{E}) + \boldsymbol{B}_\parallel \\ &= \cosh\alpha[\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{n})\boldsymbol{n} - \tanh\alpha\,\boldsymbol{n}\times\boldsymbol{E}] + (\boldsymbol{B}\cdot\boldsymbol{n})\boldsymbol{n} \\ &= \cosh\alpha[\boldsymbol{B} - \tanh\alpha\,\boldsymbol{n}\times\boldsymbol{E}] \\ &= \cosh\alpha\,\boldsymbol{B} - \sinh\alpha\,\boldsymbol{n}\times\boldsymbol{E}. \end{aligned} \tag{4.1.4}$$

The cross product of electric field and magnetic field in the rocket frame is

$$\begin{aligned}\bar{\boldsymbol{E}}\times\bar{\boldsymbol{B}} &= (\cosh\alpha\,\boldsymbol{E} + \sinh\alpha\,\boldsymbol{n}\times\boldsymbol{B})\times(\cosh\alpha\,\boldsymbol{B} - \sinh\alpha\,\boldsymbol{n}\times\boldsymbol{E}) \\ &= \cosh^2\alpha\,\boldsymbol{E}\times\boldsymbol{B} + \sinh\alpha\cosh\alpha[(\boldsymbol{n}\times\boldsymbol{B})\times\boldsymbol{B} + (\boldsymbol{n}\times\boldsymbol{E})\times\boldsymbol{E}] - \sinh^2\alpha(\boldsymbol{n}\times\boldsymbol{B})\times(\boldsymbol{n}\times\boldsymbol{E}).\end{aligned} \tag{4.1.5}$$

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$$\begin{aligned}(\boldsymbol{n}\times\boldsymbol{B})\times\boldsymbol{B} &= (\boldsymbol{B}\cdot\boldsymbol{n})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{n} = -\boldsymbol{B}^2\boldsymbol{n}; \\ (\boldsymbol{n}\times\boldsymbol{E})\times\boldsymbol{E} &= (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{E} - (\boldsymbol{E}\cdot\boldsymbol{E})\boldsymbol{n} = -\boldsymbol{E}^2\boldsymbol{n}; \\ (\boldsymbol{n}\times\boldsymbol{B})\times(\boldsymbol{n}\times\boldsymbol{E}) &= [(\boldsymbol{n}\times\boldsymbol{B})\cdot\boldsymbol{E}]\boldsymbol{n} - [(\boldsymbol{n}\times\boldsymbol{B})\cdot\boldsymbol{n}]\boldsymbol{E} \\ &= [(\boldsymbol{B}\times\boldsymbol{E})\cdot\boldsymbol{n}]\boldsymbol{n} \\ &= -\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}\tanh 2\alpha\,\boldsymbol{n}.\end{aligned} \tag{4.1.6}$$

Finally we can get

$$\begin{aligned}\bar{\boldsymbol{E}}\times\bar{\boldsymbol{B}} &= \frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}\cosh^2\alpha\tanh 2\alpha\,\boldsymbol{n} - (\boldsymbol{E}^2+\boldsymbol{B}^2)\sinh\alpha\cosh\alpha\,\boldsymbol{n} + \frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}\sinh^2\alpha\tanh 2\alpha\,\boldsymbol{n} \\ &= \frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}(\cosh^2\alpha\tanh 2\alpha + \sinh^2\alpha\tanh 2\alpha - 2\sinh\alpha\cosh\alpha)\boldsymbol{n} \\ &= \frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2}(\cosh 2\alpha\tanh 2\alpha - \sinh 2\alpha)\boldsymbol{n} \\ &= 0.\end{aligned} \tag{4.1.7}$$

Thus $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ are parallel in the rocket frame. If we put z-axis in the direction common to $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$, we have

$$F_{30} = \bar{F}_{03} = \bar{E}_z; \quad F_{12} = -F_{21} = \bar{B}_z, \tag{4.1.8}$$

and all other components vanish. So

$$\boldsymbol{F} = \bar{E}_z\boldsymbol{d}\bar{z}\wedge\boldsymbol{d}\bar{t} + \bar{B}_z\boldsymbol{d}\bar{x}\wedge\boldsymbol{d}\bar{y}. \tag{4.1.9}$$

4.2 FREEDOM OF CHOICE OF 1-FORMS IN CANONICAL REPRESENTATION OF GENERIC LOCAL FIELD

Deal with a region so small that the variation of the field from place to place can be neglected. Write Faraday in canonical representation in the form

$$\boldsymbol{F} = \boldsymbol{d}p_I\wedge\boldsymbol{d}q^I + \boldsymbol{d}p_{II}\wedge\boldsymbol{d}q^{II}, \tag{4.2.1}$$

where p_A ($A = I$ or II) and q^A are scalar functions of position in spacetime. Define a "canonical transformation" to new scalar functions of position p_A and q_A by way of the "equation of transformation"

$$p_A\boldsymbol{d}q^A = \boldsymbol{d}S + p_{\bar{A}}\boldsymbol{d}q^{\bar{A}}, \tag{4.2.2}$$

where the "generating function" S of the transformation is an arbitrary function of the q^A and the $q^{\bar{A}}$:

$$\boldsymbol{d}S = (\partial S/\partial q^A)\boldsymbol{d}q^A + (\partial S/\partial q^{\bar{A}})\boldsymbol{d}q^{\bar{A}}. \tag{4.2.3}$$

(a) Derive expressions for the two p_A's and the two $p_{\bar{A}}$'s in terms of S by equating coefficients of dq^I, dq^{II}, $dq^{\bar{I}}$, $dq^{\bar{I}\bar{I}}$ individually on the two sides of the equation of transformation.

(b) Use these expressions for the p_A's and $p_{\bar{A}}$'s to show that $\boldsymbol{F} = \boldsymbol{d}p_A \wedge \boldsymbol{d}q^A$ and $\bar{\boldsymbol{F}} = \boldsymbol{d}p_{\bar{A}} \wedge \boldsymbol{d}q^{\bar{A}}$, ostensibly different, are actually expressions for one and the same 2-form in terms of alternative sets of 1-forms.

Solution:

(a) From the definition of S, we have

$$p_A \boldsymbol{d}q^A = \boldsymbol{d}S + p_{\bar{A}} \boldsymbol{d}q^{\bar{A}} = (\partial S/\partial q^A)\boldsymbol{d}q^A + (\partial S/\partial q^{\bar{A}})\boldsymbol{d}q^{\bar{A}} + p_{\bar{A}} \boldsymbol{d}q^{\bar{A}}. \tag{4.2.4}$$

Thus,

$$p_A = \partial S/\partial q^A; \quad p_{\bar{A}} = -\partial S/\partial q^{\bar{A}}. \tag{4.2.5}$$

i.e.

$$\begin{aligned} p_I &= \partial S/\partial q^I; \quad p_{\bar{I}} = -\partial S/\partial q^{\bar{I}} \\ p_{II} &= \partial S/\partial q^{II}; \quad p_{\bar{I}\bar{I}} = -\partial S/\partial q^{\bar{I}\bar{I}}. \end{aligned} \tag{4.2.6}$$

(b)

$$\boldsymbol{F} = \boldsymbol{d}p_A \wedge \boldsymbol{d}q^A = \boldsymbol{d}\left(\frac{\partial S}{\partial q^A}\right) \wedge dq^A = \frac{\partial^2 S}{\partial q^A \partial q^{\bar{A}}} dq^{\bar{A}} \wedge dq^A. \tag{4.2.7}$$

$$\bar{\boldsymbol{F}} = \boldsymbol{d}p_{\bar{A}} \wedge \boldsymbol{d}q^{\bar{A}} = -\boldsymbol{d}\left(\frac{\partial S}{\partial q^{\bar{A}}}\right) \wedge dq^{\bar{A}} = -\frac{\partial^2 S}{\partial q^A \partial q^{\bar{A}}} dq^A \wedge dq^{\bar{A}}. \tag{4.2.8}$$

So we have $\boldsymbol{F} = \bar{\boldsymbol{F}}$.

4.3 A CLOSED OR CURL-FREE 1-FORM IS A GRADIENT

Given a 1-form a such that $\boldsymbol{d}\boldsymbol{\sigma} = 0$, show that $\boldsymbol{\sigma}$ can be expressed in the form $\boldsymbol{\sigma} = \boldsymbol{d}f$, where f is some scalar. The 1-form $\boldsymbol{\sigma}$ is said to be "curl-free," a narrower category of 1-form than the "rotation-free" 1-form of the next exercise (expressible as $\boldsymbol{\sigma} = h\boldsymbol{d}f$), and it in turn is narrower (see Figure 4.7) than the category of "1-forms with rotation" (not expressible in the form $\boldsymbol{\sigma} = h\boldsymbol{d}f$). When the 1-form $\boldsymbol{\sigma}$ is expressed in terms of basis 1-forms $\boldsymbol{d}x^\alpha$, multiplied by corresponding components σ_α, show that "curl-free" implies $\sigma_{[\alpha,\beta]} = 0$.

Solution:

Suppose $\boldsymbol{\sigma} = \sigma_\alpha dx^\alpha$. Then $\boldsymbol{d}\boldsymbol{\sigma} = 0$ implies that

$$\boldsymbol{d}\boldsymbol{\sigma} = \sigma_{\alpha,\beta} dx^\beta \wedge dx^\alpha = 0, \tag{4.3.1}$$

i.e.

$$\sigma_{[\alpha,\beta]} = 0. \tag{4.3.2}$$

From the theory of PDE, we can find a solution to the equations

$$\sigma_\alpha = \frac{\partial f}{\partial x^\alpha} \tag{4.3.3}$$

if and only if

$$\sigma_{\alpha,\beta} = \sigma_{\beta,\alpha}. \tag{4.3.4}$$

Thus given a 1-form a such that $\boldsymbol{d}\boldsymbol{\sigma} = 0$, $\boldsymbol{\sigma}$ can be expressed in the form $\boldsymbol{\sigma} = \boldsymbol{d}f$, where f is some scalar.

4.4 CANONICAL EXPRESSION FOR A ROTATION-FREE 1-FORM

In three dimensions a rigid body turning with angular velocity ω about the z-axis has components of velocity $\nu_y = \omega x$, and $\nu_x = -\omega y$. The quantity curl $\boldsymbol{\nu} = \nabla \times \boldsymbol{\nu}$ has z-component equal to 2ω, and all other components equal to zero. Thus the scalar product of $\boldsymbol{\nu}$ and curl $\boldsymbol{\nu}$ vanishes:

$$\nu_{[i,j}\nu_{k]} = 0. \tag{4.4.1}$$

The same concept generalizes to four dimensions,

$$\nu_{[\alpha,\beta}\nu_{\gamma]} = 0, \tag{4.4.2}$$

and lends itself to expression in coordinate-free language, as the requirement that a certain 3-form must vanish:

$$\boldsymbol{d\nu} \wedge \boldsymbol{\nu} = 0. \tag{4.4.3}$$

Any 1-form $\boldsymbol{\nu}$ satisfying this condition is said to be "rotation-free." Show that a 1-form is rotation-free if and only if it can be written in the form

$$\boldsymbol{\nu} = h\boldsymbol{d}f, \tag{4.4.4}$$

where h and f are scalar functions of position (the "Frobenius theorem").

Solution:

If $\boldsymbol{\nu} = h\boldsymbol{d}f$, we have

$$\boldsymbol{d\nu} \wedge \boldsymbol{\nu} = h\boldsymbol{d}h \wedge \boldsymbol{d}f \wedge \boldsymbol{d}f + h^2\boldsymbol{d}^2 f \wedge \boldsymbol{d}f = 0. \tag{4.4.5}$$

The condition $\boldsymbol{\nu} \wedge \boldsymbol{d\nu} = 0$ on $\boldsymbol{\nu}$ implies that its kernel

$$V = \{\boldsymbol{u} \in TM : \boldsymbol{\nu}(\boldsymbol{u}) = 0\} \tag{4.4.6}$$

is an involutive distribution, meaning it is closed under the Lie bracket. To see this, let X, Y be smooth vector fields in V (i.e. such that $\boldsymbol{\nu}(X) = \boldsymbol{\nu}(Y) = 0$) and compute

$$0 = \boldsymbol{\nu} \wedge \boldsymbol{d\nu}(X, Y, Z) = \boldsymbol{\nu}(Z) \wedge \boldsymbol{d\nu}(X, Y) = \boldsymbol{\nu}(Z) \wedge \boldsymbol{\nu}([X, Y]); \tag{4.4.7}$$

so choosing Z such that $\boldsymbol{\nu}(Z)$ vanishes only where $\boldsymbol{\nu}$ does we conclude that $\boldsymbol{\nu}([X, Y]) = 0$.

The Frobenius theorem tells us that an involutive distribution is tangent to a foliation by submanifolds. The idea of the proof is to construct a basis of the distribution consisting of pairwise commuting vector fields, so that the flows of these vector fields will provide natural adapted coordinates.

Once we know V is tangent to a foliation by submanifolds, the local version of the claim follows: in adapted coordinates x^i such that the surfaces $x^0 =$ constants are tangent to V, we know that $\boldsymbol{\nu}(\partial/\partial x^i)$ for $i \neq 0$; so $\boldsymbol{\nu}$ must have component expansion $\boldsymbol{\nu} = \nu_0 dx^0$. Thus $h = \nu_0$, $f = x_0$ does the job.

This proof is provided by **Anthony Carapetis**. You can find the original answer on the **StackExchange**.

4.5 FORMS ENDOWED WITH POLAR SINGULARITIES

List the principal results on how such forms are representable, such as

$$\boldsymbol{\Phi}_1 = \frac{\boldsymbol{d}S}{S} \wedge \boldsymbol{\Psi}_1 + \boldsymbol{\theta}_1, \tag{4.5.1}$$

and the conditions under which each applies [for the meaning and answer to this exercise, see Lascoux (1968)].

Solution:

This exercise is a proposition in the paper *Perturbation Theory in Quantum Field Theory and Homology* by Jean Lascoux (1968). The paper is included in the book *Battelle Rencontres, 1967 Lectures in Mathematics and Physics*, edited by Cecile M. DeWitt and John A. Wheeler. The complete statement of the proposition is as follows.

Let X be a nonsingular complex analytic manifold dim $X = l$. Introduce local coordinates $x = (x_1, x_2, \cdots, x_l)$ for the complex structure, but for the real C^∞-differentiable structure, we shall use $x = (x_1, \bar{x}_1, \cdots, x_l, \bar{x}_l)$. We shall note, for all complex manifolds of codimension 1, their local irreducible equation as follows: S, in the neighborhood of $y \in S$, is given by $s(x, y) = 0$. Let $(S_i)i = 1, \cdots, m$ be such that $ds_i \neq 0$ for $x = y$ and we shall say that $(S_1, \cdots, S_m)$ are in general position at $y \in S_I = \bigcup_{i \in I} S_i$ if the ds_i are linearly independent.

By a regular function, one means a function which is C^∞-differentiable over X. A differential form $\phi(x)$ is regular if it has regular coefficients. If $d\phi = 0$, ϕ is closed.

ϕ has a polar singularity of order p along S if $\phi(x)$ is regular on $X - S$ and if there exists a number p such that $s(u, x)^p \phi(x)$ is regular on X. Now, let us define the residue form. Let $d\phi = 0$ on $X - S$, let ϕ have a polar singularity of order p along S. Then

$$\phi = \frac{ds}{S} \wedge \psi + \theta \tag{4.5.2}$$

where the restriction of the differential form ψ on S, $\psi|_S$ is closed.

The proof of the proposition is too technical and out of my scope. So please refer to the original paper of Lascoux for more information.

4.6 THE FIELD OF THE OSCILLATING DIPOLE

Verify that the expressions given for the electromagnetic field of an oscillating dipole in equations (4.23) and (4.24) satisfy $\boldsymbol{dF} = 0$ everywhere and $\boldsymbol{d^*F}$ everywhere except at the origin.

Solution:

$$\begin{aligned}
\boldsymbol{dF} = p_1 e^{i\omega r - i\omega t} \Bigg[& -2\sin\theta \left(\frac{1}{r^3} - \frac{i\omega}{r^2} \right) \boldsymbol{d\theta} \wedge \boldsymbol{dr} \wedge \boldsymbol{dt} \\
& + \sin\theta \left(-\frac{2}{r^3} + \frac{2i\omega}{r^2} + \frac{\omega^2}{r} - i\omega^3 \right) \boldsymbol{dr} \wedge \boldsymbol{d\theta} \wedge \boldsymbol{dt} \\
& + \sin\theta \left(-\frac{\omega^2}{r} + i\omega^3 \right) \boldsymbol{dt} \wedge \boldsymbol{dr} \wedge \boldsymbol{d\theta} \Bigg] \\
= 0. &
\end{aligned} \tag{4.6.1}$$

$$\begin{aligned}
\boldsymbol{d^*F} = p_1 e^{i\omega r - i\omega t} \Bigg[& \sin^2\theta \left(\frac{i\omega}{r^2} + \frac{\omega^2}{r} - i\omega^3 \right) \boldsymbol{dr} \wedge \boldsymbol{dt} \wedge \boldsymbol{d\phi} + 2\sin\theta\cos\theta \left(\frac{-i\omega}{r} - \omega^2 \right) \boldsymbol{d\theta} \wedge \boldsymbol{dt} \wedge \boldsymbol{d\phi} \\
& + 2\sin\theta\cos\theta \left(-\frac{1}{r^2} + \frac{i\omega}{r} + \omega^2 \right) \boldsymbol{dr} \wedge \boldsymbol{d\theta} \wedge \boldsymbol{d\phi} + 2\sin\theta\cos\theta \left(-\frac{i\omega}{r} - \omega^2 \right) \boldsymbol{dt} \wedge \boldsymbol{d\theta} \wedge \boldsymbol{d\phi} \\
& + \sin^2\theta \left(-\frac{i\omega}{r^2} - \frac{\omega^2}{r} + i\omega^3 \right) \boldsymbol{dt} \wedge \boldsymbol{d\phi} \wedge \boldsymbol{dr} + 2\sin\theta\cos\theta \left(\frac{1}{r^2} - \frac{i\omega}{r} - \omega^2 \right) \boldsymbol{d\theta} \wedge \boldsymbol{d\phi} \wedge \boldsymbol{dr} \Bigg] \\
= 0. &
\end{aligned} \tag{4.6.2}$$

4.7 THE 2-FORM MACHINERY TRANSLATED INTO TENSOR MACHINERY

This exercise is stated at the end of the legend caption of Figure 4.1.

Solution:

For general $\boldsymbol{F}$, we have

$$\boldsymbol{F} = \sum_{i<j} F_{ij} \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j. \tag{4.7.1}$$

Here F_{ij} is the number of tubes which is normal to the surface made by x^i and x^j per unit area. For an infinitesimal surface $\boldsymbol{u} \wedge \boldsymbol{v}$. The area of its projection to the surface made by x^i and x^j is $u^i v^j - u^j v^i$. So the number of tubes intersecting with the infinitesimal surface $\boldsymbol{u} \wedge \boldsymbol{v}$ is

$$\sum_{i<j} F_{ij}(u^i v^j - u^j v^i) = \sum F_{ij} u^i v^j. = \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{v}). \tag{4.7.2}$$

4.8 PANCAKING THE COULOMB FIELD

Figure 4.5 shows a spacelike slice, $t =$ const, through the $\boldsymbol{Maxwell}$ of a point-charge at rest. By the following pictorial steps, verify that the electric-field lines get compressed into the transverse direction when viewed from a moving Lorentz frame: (1) Draw a picture of an equatorial slice ($\theta = \pi/2$; t, r, ϕ, variable) through $\boldsymbol{Maxwell} = {}^*\boldsymbol{F}$. (2) Draw various spacelike slices, corresponding to constant time in various Lorentz frames, through the resultant geometric structure. (3) Interpret the intersection of $\boldsymbol{Maxwell} = {}^*\boldsymbol{F}$ with each Lorentz slice in the manner of Figure 4.3.

I am really not good at proving a mathematical statements by drawing these weird pictures.

4.9 COMPUTATION OF SURFACE INTEGRALS

In Box 4.1 the definition

$$\int \boldsymbol{\alpha} = \int \cdots \int \left\langle \boldsymbol{\alpha}, \frac{\partial \mathcal{P}}{\partial \lambda^1} \wedge \cdots \wedge \frac{\partial \mathcal{P}}{\partial \lambda^p} \right\rangle d\lambda^1 \cdots d\lambda^p \tag{4.9.1}$$

is given for the integral of a p-form $\boldsymbol{\alpha}$ over a p-surface $\mathcal{P}(\lambda^1, \cdots, \lambda^p)$ in n-dimensional space. From this show that the following computational rule (also given in Box 4.1) works: (1) substitute the equation for the surface,

$$x^k = x^k(\lambda^1, \cdots, \lambda^p), \tag{4.9.2}$$

into $\boldsymbol{\alpha}$ and collect terms in the form

$$\boldsymbol{\alpha} = a(\lambda^1, \cdots, \lambda^p) \boldsymbol{d}\lambda^1 \wedge \cdots \wedge \boldsymbol{d}\lambda^p; \tag{4.9.3}$$

(2) integrate

$$\int \boldsymbol{\alpha} = \int \cdots \int a(\lambda^1, \cdots, \lambda^p) d\lambda^1 \cdots d\lambda^p \tag{4.9.4}$$

using the elementary definition of integration.

Solution:

We have

$$\boldsymbol{\alpha} = \alpha_{i_1\cdots i_p}\boldsymbol{d}x^{i_1}\wedge\cdots\wedge\boldsymbol{d}x^{i_p} \tag{4.9.5}$$

and

$$\frac{\partial\mathcal{P}}{\partial\lambda^1}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial\lambda^p} = \frac{\partial x^{k_1}}{\partial\lambda^1}\cdots\frac{\partial x^{k_p}}{\partial\lambda^p}\frac{\partial\mathcal{P}}{\partial x^{k_1}}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial x^{k_p}} \tag{4.9.6}$$

Recall that

$$\left\langle \boldsymbol{d}x^{i_1}\wedge\cdots\wedge\boldsymbol{d}x^{i_p}, \frac{\partial\mathcal{P}}{\partial x^{k_1}}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial x^{k_p}}\right\rangle = \delta^{i_1\cdots i_p}_{k_1\cdots k_p}. \tag{4.9.7}$$

We can get

$$\left\langle \boldsymbol{\alpha}, \frac{\partial\mathcal{P}}{\partial\lambda^1}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial\lambda^p}\right\rangle = \alpha_{i_1\cdots i_p}\frac{\partial x^{k_1}}{\partial\lambda^1}\cdots\frac{\partial x^{k_p}}{\partial\lambda^p}\delta^{i_1\cdots i_p}_{k_1\cdots k_p} \tag{4.9.8}$$

Note the antisymmetry of indexes of $\alpha_{i_1\cdots i_p}$, we have

$$\left\langle \boldsymbol{\alpha}, \frac{\partial\mathcal{P}}{\partial\lambda^1}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial\lambda^p}\right\rangle = p!\alpha_{k_1\cdots k_p}\frac{\partial x^{k_1}}{\partial\lambda^1}\cdots\frac{\partial x^{k_p}}{\partial\lambda^p}. \tag{4.9.9}$$

If we substitute the equation for the surface into $\boldsymbol{\alpha}$, we have

$$\boldsymbol{\alpha} = \alpha_{k_1\cdots k_p}\frac{\partial x^{k_1}}{\partial\lambda^{i_1}}\cdots\frac{\partial x^{k_p}}{\partial\lambda^{i_p}}\boldsymbol{d}\lambda^{i_1}\wedge\cdots\wedge\boldsymbol{d}\lambda^{i_p}. \tag{4.9.10}$$

If we collect terms in the form

$$\boldsymbol{\alpha} = a(\lambda^1,\cdots,\lambda^p)\boldsymbol{d}\lambda^1\wedge\cdots\wedge\boldsymbol{d}\lambda^p, \tag{4.9.11}$$

we have

$$a(\lambda^1,\cdots,\lambda^p) = p!\alpha_{k_1\cdots k_p}\frac{\partial x^{k_1}}{\partial\lambda^1}\cdots\frac{\partial x^{k_p}}{\partial\lambda^p}. \tag{4.9.12}$$

Thus,

$$a(\lambda^1,\cdots,\lambda^p) = \left\langle \boldsymbol{\alpha}, \frac{\partial\mathcal{P}}{\partial\lambda^1}\wedge\cdots\wedge\frac{\partial\mathcal{P}}{\partial\lambda^p}\right\rangle. \tag{4.9.13}$$

4.10 WHITAKER'S CALUMOID, OR, THE LIFE OF A LOOP

Take a closed loop, bounding a 2-dimensional surface $\mathcal{S}$. It entraps a certain flux of ***Faraday*** $\Phi_F = \int_{\mathcal{S}}\boldsymbol{F}$ ("magnetic tubes") and a certain flux of ***Maxwell*** $\Phi_M = \int_{\mathcal{S}}{}^*\boldsymbol{F}$ ("electric tubes").

(a) Show that the fluxes Φ_F and Φ_M depend only on the choice of loop, and not on the choice of the surface $\mathcal{S}$ bounded by the loop, if and only if $\boldsymbol{dF} = \boldsymbol{d}^*\boldsymbol{F} = 0$ (no magnetic charge; no electric charge). Hint: use generalized Stokes theorem, Boxes 4.1 and 4.6.

(b) Move the loop in space and time so that it continues to entrap the same two fluxes. Move it forward a little more here, a little less there, so that it continues to do so. In this way trace out a 2-dimensional surface ("calumoid"; see E. T. Whitaker 1904) $\mathcal{P} = \mathcal{P}(a,b)$; $x^\mu = x^\mu(a,b)$. Show that the elementary bivector in this surface, $\boldsymbol{\Sigma} = \partial\mathcal{P}/\partial a\wedge\partial\mathcal{P}/\partial b$ satisfies $\langle\boldsymbol{F},\boldsymbol{\Sigma}\rangle = 0$ and $\langle{}^*\boldsymbol{F},\boldsymbol{\Sigma}\rangle = 0$.

(c) Show that these differential equations for $x^\mu(a,b)$ can possess a solution, with given initial condition $x^\mu = x^\mu(a,0)$ for the initial location of the loop, if $\boldsymbol{dF} = 0$ and $\boldsymbol{d}^*\boldsymbol{F} = 0$ (no magnetic charge, no electric charge).

(d) Consider a static, uniform electric field $\boldsymbol{F} = -E_x\boldsymbol{d}t\wedge\boldsymbol{d}x$. Solve the equations, $\langle\boldsymbol{F},\boldsymbol{\Sigma}\rangle = 0$ and $\langle{}^*\boldsymbol{F},\boldsymbol{\Sigma}\rangle = 0$ to find the equation $\mathcal{P}(a,b)$ for the most general calumoid. Exhibit two special cases: (i) a calumoid that lies entirely in a hypersurface of constant time [loop moves at infinite velocity; analogous

to super-light velocity of point of crossing for two blades of a pair of scissors]; (ii) a calumoid whose loop remains forever at rest in the t, x, y, z Lorentz frame.

Solution:

(a) Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are two surfaces bounded by the loop. Then we have

$$\Phi_1 - \Phi_2 = \int_{\mathcal{S}_1} \boldsymbol{F} - \int_{\mathcal{S}_2} \boldsymbol{F} = \oint_{\mathcal{S}_1+(-\mathcal{S}_2)} \boldsymbol{F} = \int_V \boldsymbol{dF}, \tag{4.10.1}$$

where V is enclosed by $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\boldsymbol{dF} = 0$, we have $\Phi_1 = \Phi_2$, i.e. Φ_F depend only on the choice of loop, and not on the choice of the surface bounded by the loop. By the arbitraryness of V, the necessity of proposition is obvious.

(b) $\mathcal{S}_1$ and $\mathcal{S}_2$ are the surfaces bounded by the two loops respectively. $\mathcal{S}_1$, $\mathcal{S}_2$ and calumoid make up of a closed surface. Because $\mathcal{S}_1$ and $\mathcal{S}_2$ entrap the same two fluxes and the net flux crossed the closed surface must vanish, we know the flux entraped by the calumoid must be zero. By the arbitraryness of the movement of the loop, we have $\boldsymbol{\Sigma} = \partial\mathcal{P}/\partial a \wedge \partial\mathcal{P}/\partial b$ satisfies $\langle \boldsymbol{F}, \boldsymbol{\Sigma}\rangle = 0$ and $\langle {}^*\boldsymbol{F}, \boldsymbol{\Sigma}\rangle = 0$ everywhere on the calumoid.

(c) Since

$$\boldsymbol{F} = \frac{1}{2} F_{\mu\nu} \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu, \quad \boldsymbol{\Sigma} = \frac{\partial x^\rho}{\partial a}\frac{\partial x^\sigma}{\partial b}\frac{\partial \mathcal{P}}{\partial x^\rho} \wedge \frac{\partial \mathcal{P}}{\partial x^\sigma}, \tag{4.10.2}$$

we have

$$\langle \boldsymbol{F}, \boldsymbol{\Sigma}\rangle = F_{\mu\nu}\frac{\partial x^\mu}{\partial a}\frac{\partial x^\nu}{\partial b}. \tag{4.10.3}$$

Similarly, we have

$$\langle {}^*\boldsymbol{F}, \boldsymbol{\Sigma}\rangle = {}^*F_{\mu\nu}\frac{\partial x^\mu}{\partial a}\frac{\partial x^\nu}{\partial b} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}\frac{\partial x^\mu}{\partial a}\frac{\partial x^\nu}{\partial b}. \tag{4.10.4}$$

I do not know how to prove the existence of the solution yet.

(d) Because $\boldsymbol{F} = -E_x \boldsymbol{dt} \wedge \boldsymbol{dx}$, we have ${}^*\boldsymbol{F} = E_x \boldsymbol{dy} \wedge \boldsymbol{dz}$. Equations above can be simplified to

$$E_x\left(\frac{\partial x^0}{\partial a}\frac{\partial x^1}{\partial b} - \frac{\partial x^1}{\partial a}\frac{\partial x^0}{\partial b}\right) = 0 \tag{4.10.5}$$

and

$$E_x\left(\frac{\partial x^2}{\partial a}\frac{\partial x^3}{\partial b} - \frac{\partial x^3}{\partial a}\frac{\partial x^2}{\partial b}\right) = 0. \tag{4.10.6}$$

It is easy to see a general solution to the equations are

$$t = t(a),\ x = x(a),\ y = y(b),\ z = z(b). \tag{4.10.7}$$

We now exhibit two special cases:

(i) a calumoid that lies entirely in a hypersurface of constant time. In this case, $\frac{\partial x^0}{\partial a}\frac{\partial x^1}{\partial b} - \frac{\partial x^1}{\partial g}\frac{\partial x^0}{\partial b} = 0$ holds automatically. So the value of x^1 is arbitrary. The second equation then indicates x^2 and x^3 are correlated and so form a loop in $y-z$ plane. So the calumoid is formed by translating an arbitrary loop in $y-z$ plane in the direction of x.

(ii) a calumoid whose loop remains forever at rest in the Lorentz frame. In this case, we know x^1 is not coupled with x^0 so the first equation indicates that x^1 must be constant. By similar analysis, we know the calumoid is formed by translating an arbitrary loop in $y-z$ plane in the direction of t.

4.11 DIFFERENTIAL FORMS AND HAMilTONIAN MECHANICS

Consider a dynamic system endowed with two degrees of freedom. For the definition of this system as a Hamiltonian system (special case: here the Hamiltonian is independent of time), one needs (1) a definition of canonical variables (see Box 4.5) and (2) a knowledge of the Hamiltonian H as a function of the coordinates q^1, q^2 and the canonically conjugate momenta p_1, p_2 to derive the laws of mechanics, consider the five-dimensional space of p_1, p_2, q^1, q^2 and t, and a curve in this space leading from starting values of the five coordinates (subscript A) to final values (subscript B), and the value

$$I = \int_A^B p_1 \boldsymbol{d}q^1 + p_2 \boldsymbol{d}q^2 - H(p,q)\boldsymbol{d}t = \int_A^B \boldsymbol{\omega} \tag{4.11.1}$$

of the integral I taken along this path. The difference of the integral for two "neighboring" paths enclosing a two-dimensional region $\mathcal{S}$, according to the theorem of Stokes (Boxes 4.1 and 4.6), is

$$\delta I = \int_{\mathcal{S}} \boldsymbol{d\omega}. \tag{4.11.2}$$

The principle of least action (principle of "extremal history") states that the representative point of the system must travel along a route in the five-dimensional manifold (route with tangent vector $d\mathcal{P}/dt$) such that the variation vanishes for this path; i.e.,

$$\boldsymbol{d\omega}(\cdots, d\mathcal{P}/dt) = 0 \tag{4.11.3}$$

(2-form $\boldsymbol{d\omega}$ with a single vector argument supplied, and other slot left unfilled, gives the 1-form in 5-space that must vanish). This fixes only the direction of $d\mathcal{P}/dt$; its magnitude can be normalized by requiring $\langle \boldsymbol{d}t, d\mathcal{P}/dt\rangle = 1$.

(a) Evaluate $\boldsymbol{d\omega}$ from the expression $\boldsymbol{\omega} = p_j \boldsymbol{d}q^j - H\boldsymbol{d}t$.

(b) Set $d\mathcal{P}/dt = \dot{q}^j(\partial\mathcal{P}/\partial q^j) + \dot{p}_j(\partial\mathcal{P}/\partial p_j) + \dot{t}(\partial\mathcal{P}/\partial t)$, and expand $\boldsymbol{d\omega}(\cdots, d\mathcal{P}/dt) = 0$ in terms of the basis $\{\boldsymbol{d}p_j, \boldsymbol{d}q^k, \boldsymbol{d}t\}$

(c) Show that this five-dimensional equation can be written in the 4-dimensional phase space of $\{q^j, p_k\}$ as

$$\boldsymbol{\Theta}(\cdots, d\mathcal{P}/dt) = \boldsymbol{d}H, \tag{4.11.4}$$

where $\boldsymbol{\Theta}$ is the 2-form defined in Box 4.5.

(d) Show that the components of $\boldsymbol{\Theta}(\cdots, d\mathcal{P}/dt) = \boldsymbol{d}H$ in the $\{q^j, p_k\}$ coordinate system are the familiar Hamilton equations. Note that this conclusion depends only on the form assumed for $\boldsymbol{\Theta}$, so that one also obtains the standard Hamilton equations In any other phase-space coordinates

$$\{\bar{q}^j, \bar{p}_k\}$$

("canonical variables") for which

$$\boldsymbol{\Theta} = \boldsymbol{d}\bar{p}_1 \wedge \boldsymbol{d}\bar{q}^1 + \boldsymbol{d}\bar{p}_2 \wedge \boldsymbol{d}\bar{q}^2. \tag{4.11.5}$$

Solution:

(a)

$$\boldsymbol{d\omega} = \boldsymbol{d}p_j \wedge \boldsymbol{d}q^j - \boldsymbol{d}H \wedge \boldsymbol{d}t = \boldsymbol{d}p_j \wedge \boldsymbol{d}q^j - \frac{\partial H}{\partial q^j}\boldsymbol{d}q^j \wedge \boldsymbol{d}t - \frac{\partial H}{\partial p_j}\boldsymbol{d}p_j \wedge \boldsymbol{d}t. \tag{4.11.6}$$

(b) We have

$$\boldsymbol{d\omega}(\cdots, d\mathcal{P}/dt) = \left(\dot{q}^j - \dot{t}\frac{\partial H}{\partial p_j}\right)\boldsymbol{d}p_j - \left(\dot{p}^j + \dot{t}\frac{\partial H}{\partial q^j}\right)\boldsymbol{d}q^j + \left(\dot{q}^j\frac{\partial H}{\partial q^j} + \dot{p}_j\frac{\partial H}{\partial p_j}\right)\boldsymbol{d}t. \tag{4.11.7}$$

If $\boldsymbol{d\omega}(\cdots, d\mathcal{P}/dt) = 0$, we can get

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q^j}. \tag{4.11.8}$$

Note $\dot{q}^j \partial H/\partial q^j + \dot{p}_j \partial H/\partial p_j = 0$ will hold automatically if the equations above are satisfied.

(c) In four dimensional space, we have

$$\frac{d\mathcal{P}}{dt} = \dot{q}^j \frac{\partial \mathcal{P}}{\partial q^j} + \dot{p}_j \frac{\partial \mathcal{P}}{\partial p_j}, \quad \boldsymbol{\Theta} = \boldsymbol{d}p_j \wedge \boldsymbol{d}q^j, \quad \boldsymbol{d}H = \frac{\partial H}{\partial q^j}\boldsymbol{d}q^j + \frac{\partial H}{\partial p_j}\boldsymbol{d}p_j. \tag{4.11.9}$$

Now $\boldsymbol{\Theta}(\cdots, d\mathcal{P}/dt) = \boldsymbol{d}H$ can be expanded to

$$\left(\dot{q}^j - \frac{\partial H}{\partial p_j}\right)\boldsymbol{d}p_j - \left(\dot{p}^j + \frac{\partial H}{\partial q^j}\right)\boldsymbol{d}q^j = 0. \tag{4.11.10}$$

Now we can see $\boldsymbol{\Theta}(\cdots, d\mathcal{P}/dt) = \boldsymbol{d}H$ is equivalent to $\boldsymbol{d\omega}(\cdots, d\mathcal{P}/dt) = 0$.

(d) From the discussion in (b) and (c), it is obvious that $\boldsymbol{\Theta}(\cdots, d\mathcal{P}/dt) = \boldsymbol{d}H$ in the $\{q^j, p_k\}$ coordinate system are the familiar Hamilton equations.

4.12 SYMMETRY OPERATIONS AS TENSORS

We define the meaning of square and round brackets enclosing a set of indices as follows:

$$V_{(\alpha_1\cdots\alpha_p)} \equiv \frac{1}{p!}\sum V_{\alpha_{\pi_1}\cdots\alpha_{\pi_p}}, \quad V_{[\alpha_1\cdots\alpha_p]} \equiv \frac{1}{p!}\sum (-1)^\pi V_{\alpha_{\pi_1}\cdots\alpha_{\pi_p}}. \tag{4.12.1}$$

Here the sum is taken over all permutations π of the numbers $1, 2, \cdots, p$, and $(-1)^\pi$ is $+1$ or -1 depending on whether the permutation is even or odd. The quantity V may have other indices, not shown here, besides the set of p indices $\alpha_1, \alpha_2, \cdots, \alpha_p$, but only this set of indices is affected by the operations described here. The numbers $\pi_1, \pi_2, \cdots, \pi_p$ are the numbers $1, 2, \cdots, p$ rearranged according to the permutation π. (Cases $p = 2, 3$ were treated in exercise 3.12.) We therefore have machinery to convert any rank-p tensor with components $V_{(\alpha_1\cdots\alpha_p)}$ into a new tensor with components

$$[\boldsymbol{Alt}(\boldsymbol{V})]_{\mu_1\cdots\mu_p} = V_{[\mu_1\cdots\mu_p]}. \tag{4.12.2}$$

Since this machinery ***Alt*** is linear, it can be viewed as a tensor which, given suitable arguments $\boldsymbol{u}$, $\boldsymbol{v}$, ...,$\boldsymbol{w}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, ..., $\boldsymbol{\gamma}$ produces a number

$$u^\mu v^\nu \cdots w^\lambda \alpha_{[\mu}\beta_\nu \cdots \gamma_{\lambda]}. \tag{4.12.3}$$

(a) Show that the components of this tensor are

$$\boldsymbol{Alt}_{\beta_1\cdots\beta_p}{}^{\alpha_1\cdots\alpha_p} = (p!)^{-1}\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}, \tag{4.12.4}$$

where

$$\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} = \begin{cases} +1 \text{ if } (\alpha_1\cdots\alpha_p) \text{ is an even permutation of } (\beta_1\cdots\beta_p), \\ -1 \text{ if } (\alpha_1\cdots\alpha_p) \text{ is an odd permutation of } (\beta_1\cdots\beta_p), \\ 0 \text{ if (i) any two of the } \alpha\text{'s are the same,} \\ 0 \text{ if (ii) any two of the } \beta\text{'s are the same,} \\ 0 \text{ if (iii) the } \alpha\text{'s and } \beta\text{'s are different sets of integers.} \end{cases} \tag{4.12.5}$$

Note that the demonstration, and therefore these component values, are correct in any frame.

(b) Show for any "alternating" (i.e., "completely antisymmetric") tensor $A_{\alpha_1\cdots\alpha_p} = A_{[\alpha_1\cdots\alpha_p]}$ that

$$\frac{1}{p!}A_{\alpha_1\cdots\alpha_p}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}} = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_p} A_{\alpha_1\cdots\alpha_p}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}} \equiv A_{|\alpha_1\cdots\alpha_p|}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}. \tag{4.12.6}$$

The final line here introduces the convention that a summation over indIces enclosed between vertical bars includes only terms with those indices in increasing order. Show, consequently or similarly, that

$$\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}\delta^{\mu_1\cdots\mu_q}_{|\beta_1\cdots\beta_q|} = \delta^{\alpha_1\cdots\alpha_p\mu_1\cdots\mu_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}. \tag{4.12.7}$$

(c) Define the exterior ("wedge") product of any two alternating tensors by

$$(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})_{\lambda_1\cdots\lambda_{p+q}} = \delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}_{\lambda_1\cdots\lambda_p\lambda_{p+1}\cdots\lambda_{p+q}}\alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|}; \tag{4.12.8}$$

and similarly

$$(\boldsymbol{U}\wedge\boldsymbol{V})^{\lambda_1\cdots\lambda_{p+q}} = \delta^{\lambda_1\cdots\lambda_p\lambda_{p+1}\cdots\lambda_{p+q}}_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}U^{|\mu_1\cdots\mu_p|}V^{|\nu_1\cdots\nu_q|}. \tag{4.12.9}$$

Show that this implies equation (3.45b). Establish the associative law for this product rule by showing that

$$[(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})\wedge\boldsymbol{\gamma}]_{\sigma_1\cdots\sigma_{p+q+r}} = \delta^{\lambda_1\cdots\lambda_p\mu_1\cdots\mu_q\nu_1\cdots\nu_r}_{\sigma_1\cdots\sigma_{p+q+r}}\alpha_{\lambda_1\cdots\lambda_p}\beta_{\mu_1\cdots\mu_q}\gamma_{\nu_1\cdots\nu_r} = [\boldsymbol{\alpha}\wedge(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})]_{\sigma_1\cdots\sigma_{p+q+r}}; \tag{4.12.10}$$

and show that this reduces to the 3-form version of Equation (3.45c) when $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ all 1-forms.

(d) Derive the following formula for the components of the exterior product of p vectors

$$(\boldsymbol{u}_1\wedge\boldsymbol{u}_2\wedge\cdots\wedge\boldsymbol{u}_p)^{\alpha_1\cdots\alpha_p} = \delta^{\alpha_1\cdots\alpha_p}_{\mu\cdots\nu}(u_1)^\mu\cdots(u_p)^\nu = p!u_1^{[\alpha_1}u_2^{\alpha_2}\cdots u_p^{\alpha_p]} = \delta^{\alpha_1\cdots\alpha_p}_{1,\cdots,p}\det\big[(u_\mu)^\lambda\big]. \tag{4.12.11}$$

Solution:

(a)

$$\begin{aligned}
\boldsymbol{Alt}_{\beta_1\cdots\beta_p}{}^{\alpha_1\cdots\alpha_p} &= (\boldsymbol{e}_{\beta_1})^\mu(\boldsymbol{e}_{\beta_2})^\nu\cdots(\boldsymbol{e}_{\beta_p})^\lambda(\boldsymbol{\omega}^{\alpha_1})_{[\mu}(\boldsymbol{\omega}^{\alpha_2})_\nu\cdots(\boldsymbol{\omega}^{\alpha_p})_{\lambda]}\\
&= \frac{1}{p!}\sum_\pi(-1)^\pi(\boldsymbol{e}_{\beta_1})^\mu(\boldsymbol{e}_{\beta_2})^\nu\cdots(\boldsymbol{e}_{\beta_p})^\lambda(\boldsymbol{\omega}^{\alpha_1})_{\pi(\mu)}(\boldsymbol{\omega}^{\alpha_2})_{\pi(\nu)}\cdots(\boldsymbol{\omega}^{\alpha_p})_{\pi(\lambda)}\\
&= \frac{1}{p!}\sum_\pi(-1)^\pi(\boldsymbol{e}_{\beta_1})^\mu(\boldsymbol{e}_{\beta_2})^\nu\cdots(\boldsymbol{e}_{\beta_p})^\lambda(\boldsymbol{\omega}^{\pi^{-1}(\alpha_1)})_\mu(\boldsymbol{\omega}^{\pi^{-1}(\alpha_2)})_\nu\cdots(\boldsymbol{\omega}^{\pi^{-1}(\alpha_p)})_\lambda\\
&= \frac{1}{p!}\sum_\pi(-1)^\pi\delta^{\pi^{-1}(\alpha_1)}_{\beta_1}\delta^{\pi^{-1}(\alpha_2)}_{\beta_2}\cdots\delta^{\pi^{-1}(\alpha_p)}_{\beta_p}\\
&= \frac{1}{p!}\sum_\pi(-1)^\pi\delta^{\pi(\alpha_1)}_{\beta_1}\delta^{\pi(\alpha_2)}_{\beta_2}\cdots\delta^{\pi(\alpha_p)}_{\beta_p}\\
&= \frac{1}{p!}\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}.
\end{aligned} \tag{4.12.12}$$

(b)

$$\begin{aligned}
\frac{1}{p!}A_{\alpha_1\cdots\alpha_p}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}} &= \frac{1}{p!}\sum_{\alpha_1<\alpha_2<\cdots<\alpha_p}\sum_\pi A_{\pi(\alpha_1)\cdots\pi(\alpha_p)}\delta^{\pi(\alpha_1)\cdots\pi(\alpha_p)\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}\\
&= \frac{1}{p!}\sum_{\alpha_1<\alpha_2<\cdots<\alpha_p}\sum_\pi(-1)^\pi A_{\alpha_1\cdots\alpha_p}(-1)^\pi\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}\\
&= \sum_{\alpha_1<\alpha_2<\cdots<\alpha_p}A_{\alpha_1\cdots\alpha_p}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}.
\end{aligned} \tag{4.12.13}$$

$$\begin{aligned}
\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}\delta^{\mu_1\cdots\mu_q}_{|\beta_1\cdots\beta_q|} &= \sum_{\beta_1<\cdots<\beta_p}\delta^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}\delta^{\mu_1\cdots\mu_q}_{\beta_1\cdots\beta_q}\\
&= \delta^{\alpha_1\cdots\alpha_p\pi(\mu_1)\cdots\pi(\mu_q)}_{\gamma_1\cdots\cdots\gamma_{p+q}}(-1)^\pi\\
&= \delta^{\alpha_1\cdots\alpha_p\mu_1\cdots\mu_q}_{\gamma_1\cdots\cdots\gamma_{p+q}}
\end{aligned} \tag{4.12.14}$$

(c) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are 1-forms, we have

$$(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})_{\lambda_1\lambda_2}=\delta^{\mu\nu}_{\lambda_1\lambda_2}\alpha_\mu\beta_\nu=\alpha_{\lambda_1}\beta_{\lambda_2}-\alpha_{\lambda_2}\beta_{\lambda_1}; \tag{4.12.15}$$

So we have

$$\boldsymbol{\alpha}\wedge\boldsymbol{\beta}=\boldsymbol{\alpha}\otimes\boldsymbol{\beta}-\boldsymbol{\beta}\otimes\boldsymbol{\alpha}. \tag{4.12.16}$$

Similarly,

$$\boldsymbol{U}\wedge\boldsymbol{V}=\boldsymbol{U}\otimes\boldsymbol{V}-\boldsymbol{V}\otimes\boldsymbol{U}. \tag{4.12.17}$$

Then we move on to establish the associative law for product rule.

$$\begin{aligned}[(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})\wedge\boldsymbol{\gamma}]_{\sigma_1\cdots\sigma_{p+q+r}}&=\delta^{\rho_1\cdots\rho_{p+q}\lambda_1\cdots\lambda_r}_{\sigma_1\cdots\sigma_{p+q+r}}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})_{|\rho_1\cdots\rho_{p+q}|}\gamma_{|\lambda_1\cdots\lambda_r|}\\&=\delta^{\rho_1\cdots\rho_{p+q}\lambda_1\cdots\lambda_r}_{\sigma_1\cdots\sigma_{p+q+r}}\delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}_{|\rho_1\cdots\rho_{p+q}|}\alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|}\gamma_{|\lambda_1\cdots\lambda_r|}\\&=\delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_q\lambda_1\cdots\lambda_r}_{\sigma_1\cdots\sigma_{p+q+r}}\alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|}\gamma_{|\lambda_1\cdots\lambda_r|};\end{aligned} \tag{4.12.18}$$

$$\begin{aligned}[\boldsymbol{\alpha}\wedge(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})]_{\sigma_1\cdots\sigma_{p+q+r}}&=\delta^{\mu_1\cdots\mu_p\rho_1\cdots\rho_{q+r}}_{\sigma_1\cdots\sigma_{p+q+r}}\alpha_{|\mu_1\cdots\mu_p|}(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})_{|\rho_1\cdots\rho_{p+q}|}\\&=\delta^{\mu_1\cdots\mu_p\rho_1\cdots\rho_{q+r}}_{\sigma_1\cdots\sigma_{p+q+r}}\alpha_{|\mu_1\cdots\mu_p|}\delta^{\nu_1\cdots\nu_q\lambda_1\cdots\lambda_r}_{|\rho_1\cdots\rho_{p+q}|}\beta_{|\nu_1\cdots\nu_q|}\gamma_{|\lambda_1\cdots\lambda_r|}\\&=\delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_q\lambda_1\cdots\lambda_r}_{\sigma_1\cdots\sigma_{p+q+r}}\alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|}\gamma_{|\lambda_1\cdots\lambda_r|}.\end{aligned} \tag{4.12.19}$$

So we have

$$(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})\wedge\boldsymbol{\gamma}=\boldsymbol{\alpha}\wedge(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}). \tag{4.12.20}$$

If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are all 1-forms, then we have

$$\begin{aligned}[\boldsymbol{\alpha}\wedge(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})]_{\sigma_1\sigma_2\sigma_3}&=\delta^{\mu\nu\lambda}_{\sigma_1\sigma_2\sigma_3}\alpha_\mu\beta_\nu\gamma_\lambda\\&=\alpha_{\sigma_1}\beta_{\sigma_2}\gamma_{\sigma_3}+\alpha_{\sigma_2}\beta_{\sigma_3}\gamma_{\sigma_1}+\alpha_{\sigma_3}\beta_{\sigma_1}\gamma_{\sigma_2}\\&\quad-\alpha_{\sigma_2}\beta_{\sigma_1}\gamma_{\sigma_3}-\alpha_{\sigma_3}\beta_{\sigma_2}\gamma_{\sigma_1}-\alpha_{\sigma_1}\beta_{\sigma_3}\gamma_{\sigma_2},\end{aligned} \tag{4.12.21}$$

i.e. 3-form version of Equation (3.45c) holds.

(d) Suppose

$$(\boldsymbol{u}_1\wedge\boldsymbol{u}_2\wedge\cdots\wedge\boldsymbol{u}_k)^{\alpha_1\cdots\alpha_k}=\delta^{\alpha_1\cdots\alpha_k}_{\mu\cdots\nu}(u_1)^\mu\cdots(u_k)^\nu, \tag{4.12.22}$$

then

$$\begin{aligned}(\boldsymbol{u}_1\wedge\boldsymbol{u}_2\wedge\cdots\wedge\boldsymbol{u}_k\wedge\boldsymbol{u}_{k+1})^{\alpha_1\cdots\alpha_k\alpha_{k+1}}&=\delta^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_k\gamma}(\boldsymbol{u}_1\wedge\boldsymbol{u}_2\wedge\cdots\wedge\boldsymbol{u}_k)^{|\beta_1\cdots\beta_k|}(u_{k+1})^\gamma\\&=\delta^{\alpha_1\cdots\alpha_{k+1}}_{\beta_1\cdots\beta_k\gamma}\delta^{|\beta_1\cdots\beta_k|}_{\mu\cdots\nu}(u_1)^\mu\cdots(u_k)^\nu(u_{k+1})^\gamma\\&=\delta^{\alpha_1\cdots\alpha_k\alpha_{k+1}}_{\mu\cdots\nu\gamma}(u_1)^\mu\cdots(u_k)^\nu(u_{k+1})^\gamma.\end{aligned} \tag{4.12.23}$$

Part (c) tells us the equation holds when $k=2$. Thanks to finite induction, we know the equation holds for any $p\geq 2$. Then we have

$$\begin{aligned}(\boldsymbol{u}_1\wedge\boldsymbol{u}_2\wedge\cdots\wedge\boldsymbol{u}_p)^{\alpha_1\cdots\alpha_p}&=\delta^{\alpha_1\cdots\alpha_p}_{\mu\cdots\nu}(u_1)^\mu\cdots(u_p)^\nu\\&=\sum_\pi(-1)^\pi(u_1)^{\pi(\alpha_1)}\cdots(u_p)^{\pi(\alpha_p)}\\&=p!u_1^{[\alpha_1}u_2^{\alpha_2}\cdots u_p^{\alpha_p]}\\&=\delta^{\alpha_1\cdots\alpha_p}_{1,\cdots,p}\sum_\pi(-1)^\pi(u_1)^{\pi(1)}\cdots(u_p)^{\pi(p)}\\&=\delta^{\alpha_1\cdots\alpha_p}_{1,\cdots,p}\det\big[(u_\mu)^\lambda\big].\end{aligned} \tag{4.12.24}$$

Chapter 5

STRESS-ENERGY TENSOR AND CONSERVATION LAWS

5.1 EXERCISE

Show that expression (5.22), evaluated in a Lorentz coordinate frame, gives

$$T^{00} = \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi},\ T^{0j} = T^{j0} = \frac{(\boldsymbol{E}\times\boldsymbol{B})^j}{4\pi},\ T^{jk} = \frac{1}{4\pi}\left[-(E^jE^k + B^jB^k) + \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\delta^{jk}\right]. \quad (5.1.1)$$

Show that the stress tensor does describe a tension $(\boldsymbol{E}^2 + \boldsymbol{B}^2)/8\pi$ along the field lines and a pressure $(\boldsymbol{E}^2 + \boldsymbol{B}^2)/8\pi$ perpendicular to the field lines, as stated in the text.

Solution:

$$4\pi T^{00} = F^{0i}F^0{}_i + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \boldsymbol{E}^2 + \frac{1}{4}(2\boldsymbol{B}^2 - 2\boldsymbol{E}^2) = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2); \quad (5.1.2)$$

$$4\pi T^{0j} = 4\pi T^{j0} = F^{0i}F^j{}_i = E^i\epsilon_{jik}B^k = (\boldsymbol{E}\times\boldsymbol{B})^j; \quad (5.1.3)$$

$$\begin{aligned} T^{jk} &= F^{j0}F^k{}_0 + F^{ji}F^k{}_i - \frac{1}{4}\delta^{jk}F_{\alpha\beta}F^{\alpha\beta} \\ &= -E^jE^k + \epsilon_{jim}\epsilon_{kin}B^mB^n - \frac{1}{4}\delta^{jk}(2\boldsymbol{B}^2 - 2\boldsymbol{E}^2) \\ &= -E^jE^k + \delta^{jk}\boldsymbol{B}^2 - B^jB^k - \frac{1}{4}\delta^{jk}(2\boldsymbol{B}^2 - 2\boldsymbol{E}^2) \\ &= -E^jE^k - B^jB^k + \frac{1}{2}\delta^{jk}(\boldsymbol{B}^2 + \boldsymbol{E}^2). \end{aligned} \quad (5.1.4)$$

5.2 CHARGE CONSERVATION

Exercise 3.16 revealed that the charge-current 4-vector $\boldsymbol{J}$ satisfies the differential conservation law $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$. Write down the corresponding integral conservation law, and interpret it for the four closed surfaces of

Fig. 5.3.

Solution:

Suppose there are two surfaces $\Sigma(t=t_1)$ and $\Sigma(t=t_2)$, and $\boldsymbol{J}$ decays rapidly at space infinity. Denote the space-time between $\Sigma(t=t_1)$ and $\Sigma(t=t_2)$ as $\mathcal{V}$. Then we have stokes theorem

$$\int_{\Sigma(t=t_1)} \boldsymbol{J}\cdot\boldsymbol{dS} - \int_{\Sigma(t=t_2)} \boldsymbol{J}\cdot\boldsymbol{dS} = \int \boldsymbol{\nabla}\cdot\boldsymbol{J}\boldsymbol{dV} = 0, \tag{5.2.1}$$

i.e. $Q(t)=\int_{\Sigma(t)} J^0 \boldsymbol{d\Sigma_0}$ is a constant or conserved charge.

5.3 PARTICLE PRODUCTION

Inside highly evolved, massive stars, the temperature is so high that electron-positron pairs are continually produced and destroyed. Let $\boldsymbol{S}$ be the number-flux vector for electrons and positrons, and denote its divergence by

$$\epsilon = \boldsymbol{\nabla}\cdot\boldsymbol{S}. \tag{5.3.1}$$

Use Gauss's theorem to show that ϵ is the number of particles created (minus the number destroyed) in a unit four-dimensional volume of spacetime.

Solution:

Gauss's theorem indicates that

$$\oint_{\Sigma} \boldsymbol{S}\cdot\boldsymbol{d\Sigma} = \int_{\mathcal{V}} \epsilon \boldsymbol{d\mathcal{V}}. \tag{5.3.2}$$

$\oint_{\Sigma} \boldsymbol{S}\cdot\boldsymbol{d\Sigma}$ is the number of particles escaping from the four-dimensional volume $\mathcal{V}$. So ϵ is the number of particles created in a unit four-dimensional volume of spacetime.

5.4 INERTIAL MASS PER UNIT VOLUME

Consider a stressed medium in motion with ordinary velocity $|\boldsymbol{v}| \ll 1$ with respect to a specific Lorentz frame.

(a) Show by Lorentz transformations that the spatial components of the momentum density are

$$T^{0j} = \sum_k m^{jk} v^k, \tag{5.4.1}$$

where

$$m^{jk} = T^{\bar{0}\bar{0}}\delta^{jk} + T^{\bar{j}\bar{k}} \tag{5.4.2}$$

and $T^{\bar{\mu}\bar{\nu}}$ are the components of the stress-energy tensor in the rest frame of the medium. Throughout the solar system $T^{\bar{0}\bar{0}} \gg |T^{\bar{j}\bar{k}}|$ so one is accustomed to write $T^{0j} = T^{\bar{0}\bar{0}}v^j$, i.e. "(momentum density) = (rest-mass density) $\times$ (velocity)". But inside a neutron star, $T^{\bar{0}\bar{0}}$ may be of the same order of magnitude as $T^{\bar{j}\bar{k}}$, so one must replace "(momentum density) = (rest-mass density) $\times$ (velocity)" by equations 5.4.1 and 5.4.2, at low velocities.

(b) Derive equations 5.4.1 and 5.4.2 from Newtonian considerations plus the equivalence of mass and energy. (Hint: the total mass-energy carried past the observer by a volume V of the medium includes both the rest mass $T^{00}V$ and the work done by forces acting across the volume's faces as they "push" the volume through a distance.)

(c) As a result of relation 5.4.1, the force per unit volume required to produce an acceleration dv^k/dt in a stressed medium, which is at rest with respect to the man who applies the force, is

$$F^j = dT^{0j}/dt = \sum_k m^{jk} dv^k/dt. \tag{5.4.3}$$

This equation suggests that one call m^{jk} the "inertial mass per unit volume" of a stressed medium at rest. In general m^{jk} is a symmetric 3-tensor. What does it become for the special case of a perfect fluid?

(d) Consider an isolated, stressed body at rest and in equilibrium ($T^{\alpha\beta}{}_{,0} = 0$) in the laboratory frame. Show that its total inertial mass, defined by

$$M^{jk} = \int m^{ij}\, dx\, dy\, dz, \tag{5.4.4}$$

is isotropic and equals the rest mass of the body

$$M^{jk} = \delta^{jk} \int T^{00} dxdydz. \tag{5.4.5}$$

Solution:

(a) The infinitesimal boost from rest frame to lab frame is

$$\Lambda = \begin{pmatrix} 1 & \boldsymbol{v} \\ \boldsymbol{v} & 1 \end{pmatrix}. \tag{5.4.6}$$

The stress-energy tensor in the rest frame is

$$\bar{T} = \begin{pmatrix} T^{\bar{0}\bar{0}} & 0 \\ 0 & T^{\bar{j}\bar{k}} \end{pmatrix}. \tag{5.4.7}$$

The stress-energy tensor in the lab frame is

$$T = \Lambda \bar{T} \Lambda^{\mathrm{T}} = \begin{pmatrix} 1 & \boldsymbol{v} \\ \boldsymbol{v} & 1 \end{pmatrix} \begin{pmatrix} T^{\bar{0}\bar{0}} & 0 \\ 0 & T^{\bar{j}\bar{k}} \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{v} \\ \boldsymbol{v} & 1 \end{pmatrix}. \tag{5.4.8}$$

Keep the first order of $\boldsymbol{v}$, we have

$$T^{0j} = T^{\bar{0}\bar{0}} v^j + T^{\bar{j}\bar{k}} v^k. \tag{5.4.9}$$

(b) The hint of the exercise is confusing and I can not figure out what does "the work done by forces acting across the volume's faces as they "push" the volume through a distance" mean. Any help is appreciated.

(c) For the special case of a perfect fluid, we have $T^{\bar{0}\bar{0}} = \rho$ and $T^{\bar{j}\bar{k}} = p\delta^{\bar{j}\bar{k}}$, and so

$$m^{jk} = (p + \rho)\delta^{jk}. \tag{5.4.10}$$

(d) The conservation of energy and momentum indicates that $T^{\alpha\beta}{}_{,\beta} = 0$. If $T^{\alpha\beta}{}_{,0} = 0$, we have $T^{jk}{}_{,k} = 0$. So

$$\int T^{jk} dxdydz = \oint_S T^{jk} dS^k. \tag{5.4.11}$$

The stress of the fluid must vanish at the boundary. So $\int T^{jk} dxdydz = 0$. And we have

$$M^{jk} = \delta^{jk} \int T^{00} dxdydz. \tag{5.4.12}$$

5.5 DETERMINANTS AND JACOBIANS

(a) Write out explicitly the sum defining d^2S_{01} in

$$d^2S_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta}\frac{\partial x^\alpha}{\partial a}\frac{\partial x^\beta}{\partial b}da\,db. \tag{5.5.1}$$

Thereby establish the formula

$$d^2S_{\mu\nu} = \epsilon_{\mu\nu|\alpha\beta|}\frac{\partial(x^\alpha, x^\beta)}{\partial(a,b)}da\,db = \frac{1}{2!}\epsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\alpha, x^\beta)}{\partial(a,b)}da\,db. \tag{5.5.2}$$

The notation used here for Jacobian determinants is

$$\frac{\partial(f,g)}{\partial(a,b)} = \begin{vmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial b} \end{vmatrix}. \tag{5.5.3}$$

(b) By a similar inspection of a specific case, show that

$$d^2\Sigma_\mu \equiv \epsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial a}\frac{\partial x^\beta}{\partial b}\frac{\partial x^\gamma}{\partial c}da\,db\,dc = \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\frac{\partial(x^\alpha, x^\beta, x^\gamma)}{\partial(a,b,c)}da\,db\,dc. \tag{5.5.4}$$

(c) Cite a precise definition of the value of a determinant as a sum of terms (with suitably alternating signs), with each term a product containing one factor from each row and simultaneously one factor from each column. Show that this definition can be stated (in the 4×4 case, with the $p \times p$ case an obvious extension) as

$$\det A \equiv \det ||A^\lambda{}_\rho|| = \epsilon_{\alpha\beta\gamma\delta}A^\alpha{}_0A^\beta{}_1A^\gamma{}_2A^\delta{}_3. \tag{5.5.5}$$

(d) Show that

$$\det A = \frac{1}{4!}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}A^\alpha{}_\mu A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma. \tag{5.5.6}$$

(e) Use properties of the δ-symbol to show that the matrix A^{-1} inverse to A has entries $(A^{-1})^\mu{}_\alpha$ given by

$$(A^{-1})^\mu{}_\alpha(\det A) = \frac{1}{3!}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma. \tag{5.5.7}$$

(f) By an "index-mechanics" computation, from the formula for $\det A$ in part (d) derive the followIng expression for the derivative of the logarithm of the determinant

$$\boldsymbol{d}\ln|\det A| = \mathrm{trace}(A^{-1}\boldsymbol{d}A). \tag{5.5.8}$$

Here $\boldsymbol{d}A$ is the matrix $||\boldsymbol{d}A^\alpha{}_\mu||$ whose entries are 1-forms.

Solution:

(a)

$$d^2S_{0,1} = \epsilon_{01\alpha\beta}\frac{\partial x^\alpha}{\partial a}\frac{\partial x^\beta}{\partial b}da\,db = \left(\frac{\partial x^2}{\partial a}\frac{\partial x^3}{\partial b} - \frac{\partial x^3}{\partial a}\frac{\partial x^2}{\partial b}\right)da\,db = \frac{\partial(x^2,x^3)}{\partial(a,b)}da\,db = \epsilon_{01|\alpha\beta|}\frac{\partial(x^\alpha,x^\beta)}{\partial(a,b)}da\,db. \tag{5.5.9}$$

(b)

$$
\begin{aligned}
d^2\Sigma_0 &= \epsilon_{0\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial a}\frac{\partial x^\beta}{\partial b}\frac{\partial x^\gamma}{\partial c}da\,db\,dc \\
&= \left(\frac{\partial x^1}{\partial a}\frac{\partial x^2}{\partial b}\frac{\partial x^3}{\partial c} - \frac{\partial x^1}{\partial a}\frac{\partial x^3}{\partial b}\frac{\partial x^2}{\partial c} + \frac{\partial x^2}{\partial a}\frac{\partial x^3}{\partial b}\frac{\partial x^1}{\partial c} - \frac{\partial x^2}{\partial a}\frac{\partial x^1}{\partial b}\frac{\partial x^3}{\partial c} + \frac{\partial x^3}{\partial a}\frac{\partial x^1}{\partial b}\frac{\partial x^2}{\partial c} - \frac{\partial x^3}{\partial a}\frac{\partial x^2}{\partial b}\frac{\partial x^1}{\partial c}\right)da\,db\,dc \\
&= \frac{\partial(x^1,x^2,x^3)}{\partial(a,b,c)}da\,db\,dc \\
&= \epsilon_{0|\alpha\beta\gamma|}\frac{\partial(x^\alpha,x^\beta,x^\gamma)}{\partial(a,b,c)}da\,db\,dc \\
&= \frac{1}{3!}\epsilon_{0\alpha\beta\gamma}\frac{\partial(x^\alpha,x^\beta,x^\gamma)}{\partial(a,b,c)}da\,db\,dc
\end{aligned}
\tag{5.5.10}
$$

(c)

$$
\begin{aligned}
\det A &= \sum_\pi (-1)^\pi A^{\pi(0)}{}_0 A^{\pi(1)}{}_1 A^{\pi(2)}{}_2 A^{\pi(3)}{}_3 \\
&= \sum_\pi \epsilon_{\pi(0)\pi(1)\pi(2)\pi(3)} A^{\pi(0)}{}_0 A^{\pi(1)}{}_1 A^{\pi(2)}{}_2 A^{\pi(3)}{}_3 \\
&= \epsilon_{\alpha\beta\gamma\delta} A^\alpha{}_0 A^\beta{}_1 A^\gamma{}_2 A^\delta{}_3.
\end{aligned}
\tag{5.5.11}
$$

(d)

$$
\begin{aligned}
\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\mu A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma &= \sum_\pi \delta^{\pi(0)\pi(1)\pi(2)\pi(3)}_{\alpha\beta\gamma\delta} A^\alpha{}_{\pi(0)} A^\beta{}_{\pi(1)} A^\gamma{}_{\pi(2)} A^\delta{}_{\pi(3)} \\
&= \sum_\pi (-1)^\pi \delta^{0123}_{\alpha\beta\gamma\delta} A^\alpha{}_{\pi(0)} A^\beta{}_{\pi(1)} A^\gamma{}_{\pi(2)} A^\delta{}_{\pi(3)} \\
&= \sum_\pi (-1)^\pi \epsilon_{\alpha\beta\gamma\delta} A^\alpha{}_{\pi(0)} A^\beta{}_{\pi(1)} A^\gamma{}_{\pi(2)} A^\delta{}_{\pi(3)} \\
&= \sum_\pi (-1)^\pi \epsilon_{\alpha\beta\gamma\delta} A^{\pi^{-1}(\alpha)}{}_0 A^{\pi^{-1}(\beta)}{}_1 A^{\pi^{-1}(\gamma)}{}_2 A^{\pi^{-1}(\delta)}{}_3 \\
&= \sum_\pi (-1)^\pi (-1)^{\pi^{-1}} \epsilon_{\pi^{-1}(\alpha)\pi^{-1}(\beta)\pi^{-1}(\gamma)\pi^{-1}(\delta)} A^{\pi^{-1}(\alpha)}{}_0 A^{\pi^{-1}(\beta)}{}_1 A^{\pi^{-1}(\gamma)}{}_2 A^{\pi^{-1}(\delta)}{}_3 \\
&= \sum_\pi \epsilon_{\alpha\beta\gamma\delta} A^\alpha{}_0 A^\beta{}_1 A^\gamma{}_2 A^\delta{}_3 \\
&= 4!\epsilon_{\alpha\beta\gamma\delta} A^\alpha{}_0 A^\beta{}_1 A^\gamma{}_2 A^\delta{}_3 \\
&= 4!\det A.
\end{aligned}
\tag{5.5.12}
$$

(e) Let us examine $\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma$ first. If $\lambda \neq \mu$, since μ, ν, ρ, σ are all different, λ must equal to one of ν, ρ, σ. For example, if $\lambda = \nu$, then $\sum_{\alpha\beta} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu$ must vanish. It is easy to conclude that $\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma = 0$ if $\lambda \neq \mu$. Now suppose $\lambda = \mu$. (d) tells us that $\delta^{\pi(0)\pi(1)\pi(2)\pi(3)}_{\alpha\beta\gamma\delta} A^\alpha{}_{\pi(0)} A^\beta{}_{\pi(1)} A^\gamma{}_{\pi(2)} A^\delta{}_{\pi(3)} = \det A$ regardless of the specific permutation π. So we have $\sum_{\alpha\beta\gamma\delta} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma = \det A$ when $\lambda = \mu$. Once the value of λ and μ is given, there are $3!$ groups of possible values of ν, ρ, σ. Thus $\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma = 3!\det A$ if $\lambda = \mu$. Put it together, we have

$$
\frac{1}{3!}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} A^\alpha{}_\lambda A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma = \delta^\mu_\lambda \det A.
\tag{5.5.13}
$$

If we multiply both side of the equation by $(A^{-1})^\lambda{}_\epsilon$, we have

$$
\frac{1}{3!}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\delta^\alpha{}_\epsilon A^\beta{}_\nu A^\gamma{}_\rho A^\delta{}_\sigma = (A^{-1})^\mu{}_\epsilon \det A,
\tag{5.5.14}
$$

i.e.

$$(A^{-1})^{\mu}{}_{\epsilon}\det A = \frac{1}{3!}\delta^{\mu\nu\rho\sigma}_{\epsilon\beta\gamma\delta}A^{\beta}{}_{\nu}A^{\gamma}{}_{\rho}A^{\delta}{}_{\sigma}. \tag{5.5.15}$$

(f)

$$\begin{aligned}
\boldsymbol{d}\ln|\det A| &= \frac{\boldsymbol{d}\det A}{\det A}\\
&= \frac{\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\boldsymbol{d}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}A^{\gamma}{}_{\rho}A^{\delta}{}_{\sigma}}{4!\det A}\\
&= \frac{\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}(A^{\beta}{}_{\nu}A^{\gamma}{}_{\rho}A^{\delta}{}_{\sigma}\boldsymbol{d}A^{\alpha}{}_{\mu} + A^{\alpha}{}_{\mu}A^{\gamma}{}_{\rho}A^{\delta}{}_{\sigma}\boldsymbol{d}A^{\beta}{}_{\nu} + A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}A^{\delta}{}_{\sigma}\boldsymbol{d}A^{\gamma}{}_{\rho} + A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}A^{\gamma}{}_{\rho}\boldsymbol{d}A^{\delta}{}_{\sigma})}{4!\det A}\\
&= \frac{3!\det A((A^{-1})^{\mu}{}_{\alpha}\boldsymbol{d}A^{\alpha}{}_{\mu} + (A^{-1})^{\nu}{}_{\beta}\boldsymbol{d}A^{\nu}{}_{\beta} + (A^{-1})^{\rho}{}_{\gamma}\boldsymbol{d}A^{\rho}{}_{\gamma} + (A^{-1})^{\sigma}{}_{\delta}\boldsymbol{d}A^{\alpha}{}_{\mu})}{4!\det A}\\
&= (A^{-1})^{\mu}{}_{\alpha}\boldsymbol{d}A^{\alpha}{}_{\mu}\\
&= \mathrm{trace}(A^{-1}\boldsymbol{d}A).
\end{aligned} \tag{5.5.16}$$

5.6 CENTROIDS AND SIZES

Consider an isolated system with stress-energy tensor $T^{\mu\nu}$, total4-momentum P^{α}, magnitude of 4-momentum $M = (-\boldsymbol{P}\cdot\boldsymbol{P})^{-1/2}$, intrinsic angular momentum tensor $S^{\alpha\beta}$, and intrinsic angular momentum vector S^{α}. An observer with 4-velocity u^{α} defines the centroid of the system, at his Lorentz time $x^0 = t$ and in his own Lorentz frame, by

$$X^j_{\boldsymbol{u}}(t) = (1/P^0)\int_{x^0=t} x^j T^{00} d^3x \text{ in Lorentz frame where } \boldsymbol{u} = \partial\mathcal{P}/\partial x^0. \tag{5.6.1}$$

This centroid depends on (i) the particular system being studied, (ii) the 4-velocity $\boldsymbol{u}$ of the observer, and (iii) the time t at which the system is observed.

(a) Show that the centroid moves with a uniform velocity

$$dX^j_{\boldsymbol{u}}(t)/dt = P^j/P^0, \tag{5.6.2}$$

corresponding to the 4-velocity

$$\boldsymbol{U} = \boldsymbol{P}/M. \tag{5.6.3}$$

Note that this "4-velocity of centroid" is independent of the 4-velocity $\boldsymbol{u}$ used in defining the centroid.

(b) The centroid assocIated with the rest frame of the system (i.e., the centroid defined with $\boldsymbol{u} = \boldsymbol{U}$) is called the center of mass; see Box 5.6. Let $\boldsymbol{\xi_u}$ be a vector reaching from any event on the center-of-mass world line to any event on the world line of the centroid associated with 4-velocity $\boldsymbol{u}$; thus the components of $\boldsymbol{\xi_u}$ in any coordinate system are

$$\xi^{\alpha}_{\boldsymbol{u}} = X^{\alpha}_{\boldsymbol{u}} - X^{\alpha}_{\boldsymbol{U}}. \tag{5.6.4}$$

Show that $\boldsymbol{\xi_u}$ satisfies the equation

$$[(\xi^{\alpha}_{\boldsymbol{u}}P^{\beta} - \xi^{\beta}_{\boldsymbol{u}}P^{\alpha}) - S^{\alpha\beta}]u_{\beta} = 0. \tag{5.6.5}$$

[Hint: perform the calculation in a Lorentz frame where $\boldsymbol{u} = \partial\mathcal{P}/\partial x^0$.]

(c) Show that, as seen in the rest-frame of the system at any given moment of time, the above equation reduces to the three-dimensional Euclidean equation

$$\boldsymbol{\xi_u} = -(\boldsymbol{v}\times\boldsymbol{S})/M, \tag{5.6.6}$$

where $\boldsymbol{v} = \boldsymbol{u}/u^0$ is the ordinary velocity of the frame associated with the centroid.

(d) Assume that the energy density measured by any observer anywhere in spacetime is non-negative ($\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u} \geq 0$ for all timelike $\boldsymbol{u}$). In the rest frame of the system, construct the smallest possible cylinder that is parallel to S and that contains the entire system ($T^{\alpha\beta} = 0$ everywhere outside the cylinder). Show that the radius r_0 of this cylinder is limited by

$$r_0 \geq |\boldsymbol{S}|/M. \tag{5.6.7}$$

Thus, a system with given intrinsic angular momentum $\boldsymbol{S}$ and given mass M has a minimum possible size $r_{0\,\min} = |\boldsymbol{S}|/M$ as measured in its rest frame.

Solution:

(a)

$$\begin{aligned} dX^j_{\boldsymbol{u}}(t)/dt &= (1/P^0)\int_{x^0=t} x^j \partial_0 T^{00} d^3x \\ &= (1/P^0)\int_{x^0=t} (\partial_i x^j) T^{0i} d^3x \\ &= (1/P^0)\int_{x^0=t} T^{0j} d^3x \\ &= P^j/P^0. \end{aligned} \tag{5.6.8}$$

(b)The defination of $S^{\alpha\beta}$ can be generalized to an arbitrary reference system as

$$\begin{aligned} S^{\alpha\beta} &\equiv \int d^3x[(x^\alpha - X^\alpha_{\boldsymbol{U}})T^{0\beta} - (x^\beta - X^\beta_{\boldsymbol{U}})T^{0\alpha}] \\ &= \int d^3x(x^\alpha T^{0\beta} - x^\beta T^{0\alpha}) - X^\alpha_{\boldsymbol{U}} P^\beta + X^\beta_{\boldsymbol{U}} P^\alpha. \end{aligned} \tag{5.6.9}$$

As a result, we have

$$(\xi^\alpha_{\boldsymbol{u}} P^\beta - \xi^\beta_{\boldsymbol{u}} P^\alpha) - S^{\alpha\beta} = X^\alpha_{\boldsymbol{u}} P^\beta - X^\beta_{\boldsymbol{u}} P^\alpha - \int d^3x(x^\alpha T^{0\beta} - x^\beta T^{0\alpha}). \tag{5.6.10}$$

We now perform the calculation in a Lorentz frame where $\boldsymbol{u} = \partial\mathcal{P}/\partial x^0$. In this frame, $u_\beta = (-1, 0, 0, 0)$. Thus all we need to prove is $X^i_{\boldsymbol{u}} P^0 - X^0_{\boldsymbol{u}} P^i - \int d^3x(x^i T^{00} - x^0 T^{0i}) = 0$. From the defination of centroid, we know

$$\int d^3x x^i T^{00} = P^0 X^i_{\boldsymbol{u}}. \tag{5.6.11}$$

Time x^0 is a constant in space integration and we have $x^0 = X^0_{\boldsymbol{u}}$ in this frame. So

$$\int d^3x x^0 T^{0i} = P^i X^0_{\boldsymbol{u}}. \tag{5.6.12}$$

Now, the proof is completed. In conclusion, we have

$$[(\xi^\alpha_{\boldsymbol{u}} P^\beta - \xi^\beta_{\boldsymbol{u}} P^\alpha) - S^{\alpha\beta}]u_\beta = 0. \tag{5.6.13}$$

(c) In the rest frame of the system, we have $P^0 = M$, $P^i = 0$ and $S^{0i} = -S^{i0} = 0$. So

$$[(\xi^i_{\boldsymbol{u}} P^\beta - \xi^\beta_{\boldsymbol{u}} P^i) - S^{i\beta}]u_\beta = (\xi^i_{\boldsymbol{u}} M)u_0 - S^{ij}u_j = 0. \tag{5.6.14}$$

Note that $S^{ij}u_j = \epsilon^{ijk}S^k u_j$ and $u_0 = -u^0$. We have

$$\boldsymbol{\xi}_{\boldsymbol{u}} = -(\boldsymbol{v} \times \boldsymbol{S})/M. \tag{5.6.15}$$

(d) Suppose $r_0 < |\boldsymbol{S}|/M$, then there exists a $\boldsymbol{v}$, to make $|(\boldsymbol{v} \times \boldsymbol{S})/M| > r_0$, i.e. $|\boldsymbol{X}_{\boldsymbol{u}} - \boldsymbol{X}_{\boldsymbol{U}}| > r_0$. So the centroid of the system in one frame is outside the cylinder of the matter, which is impossible. So we must have $r_0 \geq |\boldsymbol{S}|/M$.

Chapter 6

ACCELERATED OBSERVERS

6.1 A TRIP TO THE GALACTIC NUCLEUS

Compute the proper time required for the occupants of a rocket Ship to travel the 30000 light-years from the Earth to the center of the Galaxy. Assume that they maintain an acceleration of one "earth gravity" ($10^3\mathrm{cm}/\mathrm{sec}^2$) for half the trip, and then decelerate at one earth gravity for the remaining half.

Solution:

The equation of motion of the rocket is

$$\frac{d\boldsymbol{u}}{d\tau} = \boldsymbol{w}, \tag{6.1.1}$$

where $\boldsymbol{w}$ is four-acceleration. In the rest frame of rocket, we have $\boldsymbol{w} = (0, a, 0, 0)$. So, in the galaxy frame, we have $w^1 = \gamma a$ and so

$$\frac{du^1}{d\tau} = \gamma \frac{d\gamma v}{dt} = \gamma a. \tag{6.1.2}$$

In the first half of the trip, we have

$$\gamma v = at \tag{6.1.3}$$

and so

$$v = \frac{at}{\sqrt{1+a^2t^2}}, \gamma = \sqrt{1+a^2t^2}. \tag{6.1.4}$$

We can solve the differential equation to get

$$ax = \sqrt{1+a^2t^2} - 1. \tag{6.1.5}$$

The proper time of the first half of the trip is

$$\int d\tau = \int \frac{dt}{\gamma} = \int \frac{dx}{at} = \int \frac{dx}{\sqrt{(1+ax)^2 - 1}} = \left.\frac{\ln\left(ax + \sqrt{a^2x^2 + 2ax} + 1\right)}{a}\right|_0^{L/2}. \tag{6.1.6}$$

By the symmetry of the first half and second half of the trip, we can get

$$\tau = \frac{2\ln\left(aL/2 + \sqrt{a^2L^2/4 + aL} + 1\right)}{a}. \tag{6.1.7}$$

Put back the speed of light, we have

$$\tau = \frac{2c\ln\left(aL/2c^2 + \sqrt{a^2L^2/4c^4 + aL/c^2} + 1\right)}{a} \approx 19.7\mathrm{years}. \tag{6.1.8}$$

6.2 ROCKET PAYLOAD

What fraction of the initial mass of the rocket can be payload for the journey considered in exercise 6.1? Assume an ideal rocket that converts rest mass into radiation and ejects all the radiation out the back of the rocket with 100 per cent efficiency and perfect collimation.

Solution:

Suppose the velocity of the spaceship in the galaxy frame is v and the rest mass is m at one moment of the first half of the trip. It converts a little fraction of its rest mass Δm into photons whose momentum is opposite with that of spaceship. And the velocity of the spaceship then becomes $v + \Delta v$. Due to the conservation of energy and momentumm, we have

$$\begin{aligned} 0 &= mv\Delta\gamma + \gamma m\Delta v - \gamma v\Delta m - \Delta p_{\text{photon}}; \\ 0 &= m\Delta\gamma - \gamma\Delta m + \Delta E_{\text{photon}}. \end{aligned} \tag{6.2.1}$$

As for photon, we have $\Delta E_{\text{photon}} = \Delta p_{\text{photon}}$. And we also have $\Delta\gamma = \gamma^3 v\Delta v$. Now we can figure out that

$$\gamma^2 m\Delta v = \Delta m. \tag{6.2.2}$$

At one moment of the second half of the trip, the spaceship will convert a little fraction of its rest mass Δm into photons whose momentum is equidirectional with that of spaceship. And the velocity of the spaceship then becomes $v - \Delta v$. Similar analysis also shows that

$$\gamma^2 m\Delta v = \Delta m. \tag{6.2.3}$$

Suppose the orginal mass of the spaceship is M_0. At the middle point the trip, the velocity of the spaceship is $v_{\max}$ and the mass of the spaceship is M_1. When the spaceship reach the center of the Galaxy, its remaining mass is m_0. We have

$$\int_0^{v_{\max}} \gamma^2 dv = -\int_{M_0}^{M_1} \frac{dm}{m}, \quad \int_{v_{\max}}^{0} \gamma^2 dv = \int_{M_1}^{m_0} \frac{dm}{m}. \tag{6.2.4}$$

From exercise 6.1, we know

$$\gamma_{\max} = \sqrt{1 + a^2 t_{1/2}^2}, \quad \frac{aL}{2} = \sqrt{1 + a^2 t_{1/2}^2} - 1. \tag{6.2.5}$$

We can figure out that

$$\frac{m_0}{M_0} = \frac{1}{\gamma_{\max}^2 (1 + v_{\max})^2} \approx \frac{1}{(2 + aL)^2} \approx 10^{-9}. \tag{6.2.6}$$

6.3 TWIN PARADOX

(a) Show that, of all tlmelike world lines connecting two events ($\mathcal{A}$ and $\mathcal{B}$), the one with the longest lapse of proper time is the unaccelerated one. (Hint: perform the calculation in the inertial frame of the unaccelerated world line.)

(b) One twin chooses to move from $\mathcal{A}$ and $\mathcal{B}$ along the unaccelerated world line. Show that the other twin, by an appropriate choice of accelerations, can get from $\mathcal{A}$ and $\mathcal{B}$ in arbitrarily small proper time.

(c) If the second twin prefers to ride in comfort, with the acceleration he feels never exceeding one earth gravity, g, what is the shortest proper time-lapse he can achieve between $\mathcal{A}$ and $\mathcal{B}$. Express the answer in terms of g and the proper time-lapse $\Delta\tau$ measured by the unaccelerated twin.

(d) Evaluate the answer numerically for several interesting trips.

Solution:

(a) In the frame of unaccelerated world line, the coordinates of $\mathcal{A}$ and $\mathcal{B}$ are $(0,0,0,0)$ and $(T,0,0,0)$ respectively. For an time-like arbitrary world line $x^i(t)$, we have

$$\tau = \int_0^T d\tau = \int_0^T \frac{dt}{\sqrt{1-\dot{x}^i(t)\dot{x}_i(t)}} < \int_0^T dt = T. \tag{6.3.1}$$

Thus, of all tlmelike world lines connecting two events ($\mathcal{A}$ and $\mathcal{B}$), the one with the longest lapse of proper time is the unaccelerated one.

(b) Let us design the trip of accelerated twin as follows. During coordinate time $t \in [0, T/4]$, he moves with uniform acceleration a measured in his own frame. His velocity reaches maximum V at $t = T/4$. During coordinate time $t \in [T/4, T/2]$, he moves with uniform acceleration $-a$. His displacement reaches maximum L at $t = T/4$. During coordinate time $t \in [T/2, 3T/4]$, he moves back with uniform acceleration $-a$. During coordinate time $t \in [3T/4, T]$, he decelerates with uniform acceleration a and come back to the origin at $t = T$. Follow the calculation in exercise 6.1, we have

$$\tau(a,T) = 4\int_0^{T/4} \frac{dt}{\sqrt{1+a^2t^2}} = \frac{4\ln\Big(aT/4 + \sqrt{(aT/4)^2+1}\Big)}{a}. \tag{6.3.2}$$

It is easy to see $\tau \to 0$ as $a \to \infty$.

(c) If we want τ to be small, we should let $1/\sqrt{1-\dot{x}^i(t)\dot{x}_i(t)}$ be as larger as possible, i.e. we should let the velocity of accelerated world line as larger as possible. Given the maximum acceleration g, it is obvious the trip with minimun proper time must be the form stated in (b).So we have

$$\tau(g) = \frac{4\ln\Big(a\Delta\tau/4 + \sqrt{(a\Delta\tau/4)^2+1}\Big)}{a}. \tag{6.3.3}$$

6.4 RADAR SPEED INDICATOR

A radar set measures velocity by emitting a signal at a standard frequency and comparing it with the frequency of the signal reflected back by another object. This redshift measurement is then converted, using the standard special-relativistic formula, into the corresponding velocity, and the radar reads out this velocity. How useful is this radar set as a velocity-measuring instrument for a uniformly accelerated observer?

(a) Consider this problem first for the special case where the object and the radar set are at rest with respect to each other at the instant the radar pulse is reflected. Compute the redshift $1 + z = \omega_e/\omega_0$ that the radar set measures in this case, and the resulting (incorrect) velocity it infers. Simplify by making use of the symmetries of the situation.

(b) Now consider the situation where the object has a non-zero velocity in the momentary rest frame of the observer at the instant it reflects the radar pulse. Compute the ratio of the actual relative velocity to the velocity read out by the radar set.

Solution:

(a) The world line of the object is $t = 0$ in its resframe, while the world line of the observer (radar set) is

$$g(x - L) = \sqrt{1 + g^2t^2} - 1, \tag{6.4.1}$$

where L is the distance between the object and the observer at time $t = 0$. It is easy to see the observer is at rest at $t = 0$. Since the object and the radar set are at rest with respect to each other at the instant the radar pulse is reflected, we know the light is reflected at $t = 0$ and so emitted at $t_1 = -L(2-gL)/(2-2gL)$ and recived at $t_2 = L(2 - gL)/(2 - 2gL)$. According to $\gamma = \sqrt{1 + g^2t^2}$, we have $\gamma(t_1) = \gamma(t_2) = [(1 - gL)^2 + 1]/2(1 - gL)$.

If the light is reflected at $t = \Delta$, we can figure out $t'_1 = -(L - \Delta)(2 - gL + g\Delta)/(2 - 2gL + 2g\Delta)$ and $t'_2 = (L + \Delta)(2 - gL - g\Delta)/(2 - 2gL - 2g\Delta)$. And we now have

$$\tau_1 = \frac{t'_1 - t_1}{\gamma(t_1)} = \frac{\Delta + \frac{1-gL}{g^2L^2-2gL+2}g\Delta^2}{1 - gL + g\Delta} \tag{6.4.2}$$

and

$$\tau_2 = \frac{t'_2 - t_2}{\gamma(t_2)} = \frac{\Delta - \frac{1-gL}{g^2L^2-2gL+2}g\Delta^2}{1 - gL - g\Delta}. \tag{6.4.3}$$

As a result,

$$\frac{\omega_e}{\omega_0} = \frac{\tau_1}{\tau_2} \approx \frac{1 - \frac{g\Delta}{(1-gL)(g^2L^2-2gL+2)}}{1 + \frac{g\Delta}{(1-gL)(g^2L^2-2gL+2)}}. \tag{6.4.4}$$

If the radar set was moving uniformly with velocity v, we would have

$$\frac{\omega_e}{\omega_0} = \frac{1 - v}{1 + v}. \tag{6.4.5}$$

By comparision, we can get

$$v_{\text{err}} = \frac{g\Delta}{(1 - gL)(g^2L^2 - 2gL + 2)}. \tag{6.4.6}$$

(2) This time suppose the light is reflected at $t = T$. So we can figure out the light is emitted at $t_1 = -(L-T)(2-gL+gT)/(2-2gL+2gT)$ and received at $t_2 = (L+T)(2-gL-gT)/(2-2gL-2gT)$. And we have $\gamma(t_1) = [(1-gL+gT)^2+1]/2(1-gL+gT)$ and $\gamma(t_2) = [(1-gL-gT)^2+1]/2(1-gL-gT)$. According to Figure 6.4 of the textbook, the equal time line of the observer which cross the event $x = 0, t = T$ is $t = gTx/(1 - gL) + T$. The velocity of the object moving away from the radar set in the momentary rest frame of the observer at the instant it reflects the radar pulse is $v = gT/(1 - gL)$.

If the light is reflected at $t = T + \Delta$, we can figure out $t'_1 = -(L - T - \Delta)(2 - gL + gT + g\Delta)/(2 - 2gL + 2gT + 2g\Delta)$ and $t'_2 = (L + T + \Delta)(2 - gL - gT - g\Delta)/(2 - 2gL - 2gT - 2g\Delta)$. And we now have

$$\tau_1 = \frac{t'_1 - t_1}{\gamma(t_1)} = \frac{\Delta + \frac{1-gL+gT}{g^2(L-T)^2-2g(L-T)+2}g\Delta^2}{1 - gL + gT + g\Delta} \tag{6.4.7}$$

and

$$\tau_2 = \frac{t'_2 - t_2}{\gamma(t_2)} = \frac{\Delta - \frac{1-gL-gT}{g^2(L+T)^2-2g(L+T)+2}g\Delta^2}{1 - gL - gT - g\Delta}. \tag{6.4.8}$$

As a result,

$$\frac{\omega_e}{\omega_0} = \frac{\tau_1}{\tau_2} \approx \frac{1 - \frac{gT}{1-gL} + O(g\Delta)}{1 + \frac{gT}{1-gL} + O(g\Delta)}. \tag{6.4.9}$$

The coefficients of $g\Delta$ is very tedious and will not be listed here, then the velocity the object the observer measured is

$$v' = \frac{gT}{1 - gL} + O(g\Delta), \tag{6.4.10}$$

i.e.

$$\frac{v'}{v} \sim 1 + O(g\Delta). \tag{6.4.11}$$

6.5 RADAR DISTANCE INDICATOR

Use radar as a distance-measuring device. The radar set measures its proper time τ between the instant at which it emits a pulse and the later instant when it receives the reflected pulse. It then performs the simple computation $L_0 = \tau/2$ and supplies as output the "distance" L_0. How accurate is the output reading of the radar set for measuring the actual distance L to the object, when used by a uniformly accelerated observer? (L is defined as the distance in the momentary rest frame of the observer at the instant the pulse is reflected, which is at the observer's proper time halfway between emitting and receiving the pulse.) Give a correct formula relating $L_0 \equiv \tau/2$ to the actual distance L. Show that the reading L_0 becomes infinite as L approaches g^{-1}, where g is the observer's acceleration, as measured by an accelerometer he carries.

Solution:

The world line of the object is $t = 0$ in its resframe, while the world line of the observer (radar set) is

$$g(x - l) = \sqrt{1 + g^2t^2} - 1, \tag{6.5.1}$$

where l is the distance between the object and the observer at time $t = 0$. Suppose the light is reflected at $t = T$. So we can figure out the light is emitted at $t_1 = -(l-T)(2-gl+gT)/(2-2gl+2gT)$ and received at $t_2 = (l+T)(2-gl-gT)/(2-2gl-2gT)$. For uniform accelerated motion, we have

$$2L_0 = \tau = g^{-1} \ln \left(\frac{gt_2 + \sqrt{1 + g^2t_2^2}}{gt_1 + \sqrt{1 + g^2t_1^2}} \right) = g^{-1} \ln \left[\frac{1}{(1 - gl - gT)(1 - gl + gT)} \right]. \tag{6.5.2}$$

The equal time line of the observer which cross the event $x = 0, t = T$ is $t = gTx/(1 - gl) + T$. The equal time line cross the world line of the radar set at t_0 and

$$\frac{gt_0}{\sqrt{1 + g^2t_0^2}} = \frac{gT}{1 - gl}. \tag{6.5.3}$$

And we can get

$$L = \frac{1}{g}(1 - T/t_0) = g^{-1}(1 - e^{-gL_0}), \tag{6.5.4}$$

i.e.

$$L_0 = g \ln \left(\frac{1}{1 - gL} \right). \tag{6.5.5}$$

It is easy to see $L_0 \to \infty$ as $L \to g^{-1}$.

6.6 CLOCK RATES VERSUS COORDINATE TIME IN ACCELERATED COORDINATES

Let a clock be attached to each grid point, $(\xi^{1'}, \xi^{2'}, \xi^{3'}) =$ constant, of the local coordinate system of an accelerated observer. Assume for simplicity that the observer is in hyperbolic motion. Use equation (6.18) to show that proper time as measured by a lattice clock differs from coordinate time at its lattIce point:

$$d\tau/d\xi^{0'} = 1 + g\xi^{1'}. \tag{6.6.1}$$

(Of course, very near the observer, at $\xi^{1'} \ll g^{-1}$, the discrepancy is negligible.)

Solution:

If we set $d\xi^{1'} = d\xi^{2'} = d\xi^{3'} = 0$, we have

$$ds^2 = -(1 + g\xi^{1'})^2(d\xi^{0'})^2. \tag{6.6.2}$$

So we have

$$d\tau/d\xi^{0'} = 1 + g\xi^{1'}. \tag{6.6.3}$$

6.7 ACCELERATION OF LATTICE POINTS IN ACCELERATED COORDINATES

Let an accelerometer be attached to each grid point of the local coordinates of an observer in hyperbolic motion. Calculate the magnitude of the acceleration measured by the accelerometer at $(\xi^{1'}, \xi^{2'}, \xi^{3'})$.

Solution:

We have

$$\begin{aligned} x^0 &= (g^{-1} + \xi^{1'}) \sinh\left(g\xi^{0'}\right) \\ x^1 &= (g^{-1} + \xi^{1'}) \cosh\left(g\xi^{0'}\right) \\ x^2 &= \xi^{2'} \\ x^3 &= \xi^{3'} \end{aligned} \tag{6.7.1}$$

and

$$d\tau = (1 + g\xi^{1'})d\xi^{0'}. \tag{6.7.2}$$

We have

$$u^\mu = (\cosh g\xi^{0'}, \sinh g\xi^{0'}, 0, 0) \tag{6.7.3}$$

and

$$a^\mu = \frac{g}{1 + g\xi^{1'}}(\sinh g\xi^{0'}, \cosh g\xi^{0'}, 0, 0) \tag{6.7.4}$$

in the inertial frame. The magnitude of the acceleration is

$$|a| = \sqrt{a^\mu a_\mu} = \frac{g}{1 + g\xi^{1'}}. \tag{6.7.5}$$

6.8 OBSERVER WITH ROTATING TETRAD

An observer moving along an arbitrarily accelerated world line chooses not to Fermi-Walker transport his orthonormal tetrad. Instead, he allows it to rotate. The antisymmetric rotation tensor $\boldsymbol{\Omega}$ that enters into his transport law

$$d\boldsymbol{e}_{\alpha'}/d\tau = -\boldsymbol{\Omega} \cdot \boldsymbol{e}_{\alpha'} \tag{6.8.1}$$

splits into a Fermi-Walker part plus a spatial rotation part:

$$\Omega^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu + u_\alpha \omega_\beta \epsilon^{\alpha\beta\mu\nu} = \Omega^{\mu\nu}_{\rm FW} + \Omega^{\mu\nu}_{\rm SR}, \quad \boldsymbol{\omega} \text{ is a vector orthogonal to 4-velocity } \boldsymbol{u}. \tag{6.8.2}$$

(a) The observer chooses his time basis vector to be $\boldsymbol{e}_{0'} = \boldsymbol{u}$. Show that this choice is permitted by his transport law 6.8.1, 6.8.2.

(b) Show that $\Omega^{\mu\nu}_{\rm SR}$ produces a rotation in the plane perpendicular to $\boldsymbol{u}$ and $\boldsymbol{\omega}$, i.e., that

$$\boldsymbol{\Omega}_{\rm SR} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{\Omega}_{\rm SR} \cdot \boldsymbol{\omega} = 0. \tag{6.8.3}$$

(c) Suppose the accelerated observer Fermi-Walker transports a second orthonormal tetrad $\boldsymbol{e}_{\alpha''}$. Show that the space vectors of his first tetrad rotate relative to those of his second tetrad with angular velocity vector equal to $\boldsymbol{\omega}$.

(d) The observer uses the same prescription [equation (6.16)] to set up local coordinates based on his rotating tetrad as for his Fermi-Walker tetrad. Pick an event $\mathcal{L}$ on the observer's world line, set $\tau = 0$ there, and choose the original inertial frame of prescription (6.16) so (1) it comoves with the accelerated observer at $\mathcal{L}$, (2) its origin is at $\mathcal{L}$, and (3) its axes coincide with the accelerated axes at $\mathcal{L}$. Show that these conditions translate into

$$z^{\mu}(0) = 0, \quad \boldsymbol{e}_{\alpha'}(0) = \boldsymbol{e}_{\alpha}. \tag{6.8.4}$$

(e) Show that near $\mathcal{L}$, equations (6.16) for the rotating, accelerated coordinates reduce to

$$\begin{aligned} x^0 &= \xi^{0'} + a_{k'}\xi^{k'}\xi^{0'} + O([\xi^{\alpha'}]^3); \\ x^j &= \xi^{j'} + \frac{1}{2}a^j\xi^{0'2} + \epsilon^{jkl}\omega^k\xi^{l'}\xi^{0'} + O([\xi^{\alpha'}]^3). \end{aligned} \tag{6.8.5}$$

(f) A freely moving particle passes through the event $\mathcal{L}$ with ordinary velocity $\boldsymbol{v}$ as measured in the inertial frame. By transforming its straight world line $x^j = v^j x^0$ to the accelerated, rotating coordinates, show that its coordinate velocity and acceleration there are:

$$\begin{aligned} (d\xi^{j'}/d\xi^{0'})_{\mathrm{at}\mathcal{L}} &= v^j; \\ (d^2\xi^{j'}/d\xi^{0'2})_{\mathrm{at}\mathcal{L}} &= -a^j - 2\epsilon^{jkl}\omega^k v^l + 2v^j a^k v^k. \end{aligned} \tag{6.8.6}$$

Solution:

(a)

$$-\Omega^{\mu\nu}u_\nu = -(a^\mu u^\nu u_\nu - a^\nu u_\nu u^\mu + u_\alpha u_\nu \omega_\beta \epsilon^{\alpha\beta\mu\nu}) = a^\mu. \tag{6.8.7}$$

So we have

$$\frac{d\boldsymbol{u}}{d\tau} = -\boldsymbol{\Omega}\cdot\boldsymbol{u}. \tag{6.8.8}$$

(b)

$$\Omega^{\mu\nu}_{\mathrm{SR}}u_\nu = u_\alpha u_\nu \omega_\beta \epsilon^{\alpha\beta\mu\nu} = 0, \quad \Omega^{\mu\nu}_{\mathrm{SR}}\omega_\nu = u_\alpha \omega_\beta \omega_\nu \epsilon^{\alpha\beta\mu\nu} = 0. \tag{6.8.9}$$

(c)

$$\begin{aligned} \frac{de^\mu_{j'}}{d\tau} &= -\Omega^{\mu\nu}e_{\nu j'} = a^\nu e_{\nu j'}u^\mu - u_\alpha e_{\nu j'}\omega_\beta\epsilon^{\alpha\beta\mu\nu}; \\ \frac{de^\mu_{j''}}{d\tau} &= -\Omega^{\mu\nu}_{\mathrm{FW}}e_{\nu j''} = a^\nu e_{\nu j''}u^\mu. \end{aligned} \tag{6.8.10}$$

At a moment when the tetrads coincide, we have

$$\frac{d(e^\mu_{j'} - e^\mu_{j''})}{d\tau} = -u_\alpha e_{\nu j'}\omega_\beta\epsilon^{\alpha\beta\mu\nu}. \tag{6.8.11}$$

If we constrain the equation in the space plane perpendicular to $\boldsymbol{u}$, we have

$$\frac{d(e^l_{j'} - e^l_{j''})}{d\tau} = e_{mj'}\omega_n\epsilon^{0nlm} = e_{mj'}\omega_n\epsilon_{nml}, \tag{6.8.12}$$

i.e.

$$\frac{d(\boldsymbol{e}_{j'} - \boldsymbol{e}_{j''})}{d\tau} = \boldsymbol{\omega}\times\boldsymbol{e}_{j'}. \tag{6.8.13}$$

(d) If the inertial frame comoves with the accelerated observer at $\mathcal{L}$, we have

$$\boldsymbol{e}_0 = \boldsymbol{u} \tag{6.8.14}$$

or $u = (1, 0, 0, 0)$ in this frame. If the origin of the inertial frame is at $\mathcal{L}$, we have

$$x^\mu = z^\mu(0) = 0. \tag{6.8.15}$$

If its axes coincide with the accelerated axes at $\mathcal{L}$. Then we have

$$\boldsymbol{e}_{\alpha'}(0) = \boldsymbol{e}_\alpha. \tag{6.8.16}$$

(e) In the inertial frame we take in (d), we have at the origin that

$$\Omega^{0i} = -a^i, \quad \Omega^{jk} = u_0 \omega_l \epsilon^{0ljk} = \epsilon_{jkl}\omega_l. \tag{6.8.17}$$

The tetrads at the origin satisfy that

$$\boldsymbol{e}_{1'} = (0, 1, 0, 0), \quad \boldsymbol{e}_{2'} = (0, 0, 1, 0), \quad \boldsymbol{e}_{3'} = (0, 0, 0, 1). \tag{6.8.18}$$

The velocity and acceleration at the origin are

$$\boldsymbol{u} = (1, 0, 0, 0), \quad \boldsymbol{a} = (0, a^1, a^2, a^3). \tag{6.8.19}$$

Near the origin, we have

$$u^0 = 1 + O([\xi^{\alpha'}]^2), \quad u^i = a^i\tau + O([\xi^{\alpha'}]^2). \tag{6.8.20}$$

So

$$z^0 = \tau + O([\xi^{\alpha'}]^3), \quad z^i = \frac{1}{2}a^i\tau^2 + O([\xi^{\alpha'}]^3). \tag{6.8.21}$$

Since

$$\frac{de^0_{j'}}{d\tau} = -\Omega^{0i}e_{ij'} = a^{j'}, \quad \frac{de^m_{j'}}{d\tau} = -\Omega^{mn}e_{nj'} = -\epsilon_{mj'l}\omega_l, \tag{6.8.22}$$

we have

$$e^0_{j'} = a^j\tau + O([\xi^{\alpha'}]^2), \quad e^m_{j'} = \delta^m_{j'} - \epsilon_{mj'l}\omega_l\tau + O([\xi^{\alpha'}]^2). \tag{6.8.23}$$

Note that $\xi^{0'} = \tau$, now (6.16) becomes

$$\begin{aligned} x^0 &= \xi^{j'}e^0_{j'} + z^0 = \xi^{0'} + a_{j'}\xi^{j'}\xi^{0'} + O([\xi^{\alpha'}]^3), \\ x^j &= \xi^{k'}e^j_{k'} + z^j = \xi^{j'} - \epsilon_{jk'l}\omega_l\xi^{0'}\xi^{k'} + \frac{1}{2}a^j\xi^{0'2} + O([\xi^{\alpha'}]^3) \\ &= \xi^{j'} + \frac{1}{2}a^j\xi^{0'2} + \epsilon^{jlk}\omega^l\xi^{k'}\xi^{0'} + O([\xi^{\alpha'}]^3). \end{aligned} \tag{6.8.24}$$

(f) The world line of the particle in ξ frame is

$$\xi^{j'} + \frac{1}{2}a^j\xi^{0'2} + \epsilon^{jlk}\omega^l\xi^{k'}\xi^{0'} = v^j(\xi^{0'} + a_{k'}\xi^{k'}\xi^{0'}) + O([\xi^{\alpha'}]^3). \tag{6.8.25}$$

Take the first order dirivative with $\xi^{0'}$ and then set $\xi = 0$, we have

$$d\xi^{j'}/d\xi^{0'} = v^j. \tag{6.8.26}$$

Take the second order dirivative with $\xi^{0'}$ and then set $\xi = 0$, we have

$$d^2\xi^{j'}/d\xi^{0'2} + a^j + 2\epsilon^{jlk}\omega^l d\xi^{k'}/d\xi^{0'} = 2v^j a_{k'}d\xi^{k'}/d\xi^{0'}, \tag{6.8.27}$$

i.e.

$$d^2\xi^{j'}/d\xi^{0'2} = -a^j - 2\epsilon^{jlk}\omega^l v^k + 2v^j a^k v^k. \tag{6.8.28}$$

6.9 THOMAS PRECESSION

Consider a spinning body (gyroscope, electron, ...) that accelerates because forces act at its center of mass. Such forces produce no torque; so they leave the body's intrinsic angular-momentum vector $\boldsymbol{S}$ unchanged, except for the unique rotation in the $\boldsymbol{u} \wedge \boldsymbol{a}$ plane required to keep $\boldsymbol{S}$ orthogonal to the 4-velocity $\boldsymbol{u}$. Mathematically speaking, the body Fermi-Walker transports its angular momentum (no rotation in planes other than $\boldsymbol{u} \wedge \boldsymbol{a}$):

$$d\boldsymbol{S}/d\tau = (\boldsymbol{u} \wedge \boldsymbol{a}) \cdot \boldsymbol{S}. \tag{6.9.1}$$

This transport law applies to a spInnIng electron that moves In a circular orbit of radius r around an atomic nucleus. As seen In the laboratory frame, the electron moves in the x, y-plane with constant angular velocity, ω. At time $t = 0$, the electron is at $x = r, y = 0$; and its spin (as treated classically) has components

$$S^0 = 0, \quad S^x = \frac{1}{\sqrt{2}}\hbar, \quad S_y = 0, \qquad S^z = \frac{1}{2}\hbar. \tag{6.9.2}$$

Calculate the subsequent behavior of the spin as a function of laboratory time, $S^\mu(t)$.

Solution:

The world line of the electron is

$$\begin{cases} x = r\cos\omega t; \\ y = r\sin\omega t; \\ z = 0. \end{cases} \tag{6.9.3}$$

So we have

$$d\tau = \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1-\omega^2 r^2}dt \equiv \frac{dt}{\gamma}. \tag{6.9.4}$$

So the four velocity of the electron is

$$u^\mu = \gamma(1, -\omega r\sin\omega t, \omega r\cos\omega t, 0). \tag{6.9.5}$$

The four acceleration of the electron is

$$a^\mu = \gamma^2(0, -\omega^2 r\cos\omega t, -\omega^2 r\sin\omega t, 0). \tag{6.9.6}$$

Now we have

$$\begin{aligned} \frac{dS^0}{dt} &= \frac{dS^0}{\gamma d\tau} = a^\nu S_\nu = -\gamma^2\omega^2 r(\cos\omega t S^x + \sin\omega t S^y); \\ \frac{dS^x}{dt} &= -\gamma^2\omega^2 r\cos\omega t S^0 + \gamma^2\omega^3 r^2 S_y; \\ \frac{dS^y}{dt} &= -\gamma^2\omega^2 r\sin\omega t S^0 - \gamma^2\omega^3 r^2 S_x; \\ \frac{dS^z}{dt} &= 0, \end{aligned} \tag{6.9.7}$$

i.e.

$$\frac{d}{dt}\begin{pmatrix} S^0 \\ S^x \\ S^y \end{pmatrix} = -\gamma^2\omega^2 r \begin{pmatrix} 0 & \cos\omega t & \sin\omega t \\ \cos\omega t & 0 & -\omega r \\ \sin\omega t & \omega r & 0 \end{pmatrix} \cdot \begin{pmatrix} S^0 \\ S^x \\ S^y \end{pmatrix}. \tag{6.9.8}$$

The solution is

$$\begin{aligned} S^0 &= -\frac{\hbar}{\sqrt{2}}\gamma\omega r\sin\gamma\omega t, \\ S^x &= \frac{\hbar}{\sqrt{2}}(\cos\omega t\cos\gamma\omega t + \gamma\sin\omega t\sin\gamma\omega t), \\ S^y &= \frac{\hbar}{\sqrt{2}}(\sin\omega t\cos\gamma\omega t - \gamma\cos\omega t\sin\gamma\omega t), \\ S^z &= \frac{\hbar}{2}. \end{aligned} \tag{6.9.9}$$

Chapter 7

INCOMPATIBILITY OF GRAVITY AND SPECIAL RELATIVITY

7.1 SCALAR GRAVITATIONAL FIELD, Φ

A. Consider the variational principle $\delta I = 0$, where

$$I = -m \int e^{\Phi} \left(-\eta_{\alpha\beta} \frac{dz^\alpha}{d\lambda} \frac{dz^\beta}{d\lambda} \right)^{1/2} d\lambda, \tag{7.1.1}$$

Here $m =$ (rest mass) and $z^\alpha(\lambda) =$ (parametrized world line) for a test particle in the scalar gravitational field Φ. By varying the particle's world line, derive differential equations governing the particle's motion. Write them using the particle's proper time as the path parameter,

$$d\tau = \left(-\eta_{\alpha\beta} \frac{dz^\alpha}{d\lambda} \frac{dz^\beta}{d\lambda} \right)^{1/2} d\lambda, \tag{7.1.2}$$

so that $u^\alpha = dz^\alpha/d\tau$ satisfies $u^\alpha u^\beta \eta_{\alpha\beta} = -1$.

B. Obtain the field equation for $\Phi(x)$ implied by the variational principle $\delta I = 0$, where $I = \int \mathcal{L} d^4x$ and

$$\mathcal{L} = -\frac{1}{8\pi G} \eta^{\alpha\beta} \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial \Phi}{\partial x^\beta} - \int m e^{\Phi} \delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)] d\tau. \tag{7.1.3}$$

Show that the second term here gives the same integral as that studied in part A (equation 7.3).

Discussion: The field equations obtained describe how a single particle of mass m generates the scalar field. If many particles are present, one includes in $\mathcal{L}$ a term $-\int m e^{\Phi} \delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)] d\tau$ for each particle.

C. Solve the field equation of part B, assuming a single source particle at rest. Also assume that $e^\Phi = 1$ is an adequate approximation in the neighborhood of the particle. Then check this assumption from your solution; i.e., what value does it assign to e^Φ at the surface of the earth? (Units with $c = 1$ are used throughout; one may also set $G = 1$, if one wishes.)

D. Now treat the static, spherically symmetric field Φ from part C as the field of the sun acting as a given external field in the variattonal principle of part A, and study the motion of a planet determined by this variational principle. Constants of motion are available from the spherical symmetry and time-independence of the integrand. Use spherical coordinates and assume motion in a plane. Derive a formula for the perihelion precession of a planet.

E. Pass to the limit of a zero rest-mass particle in the equations of motion of part A. Do this by using a parameter λ different from proper time, so chosen that $k^\mu = dx^\mu/d\lambda$ is the energy-momentum vector, and by taking the limit $m \to 0$ with $k^0 = \gamma m = E$ remaining finite (so $u^0 = \gamma \to \infty$. Use these equations to show that the quantities $q^\mu = k^\mu e^\Phi$ are constants of motion, and from this deduce that there is no bending of light by the sun in this scalar theory.

Solution:

A.

$$\begin{aligned}\delta I &= -m\int e^\Phi \Phi_{,\alpha}\delta z^\alpha d\tau - m\int e^\Phi \frac{1}{2}\frac{d\lambda}{d\tau}\left(-2\eta_{\alpha\beta}\frac{d\delta z^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)d\lambda \\ &= -m\int e^\Phi\left(\Phi_{,\alpha} + a_\alpha + \Phi_{,\beta}u^\beta u_\alpha\right)\delta z^\alpha d\tau = 0.\end{aligned} \tag{7.1.4}$$

The differential equations governing the particle's motion are

$$a^\alpha + \Phi_{,\beta}u^\beta u^\alpha = -\eta^{\alpha\beta}\Phi_{,\beta}. \tag{7.1.5}$$

B.

$$-\int me^{\Phi(\boldsymbol{x})}\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau d^4x = -\int me^{\Phi(\boldsymbol{z}(\tau))}d\tau = -m\int e^\Phi\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda. \tag{7.1.6}$$

C.

$$\begin{aligned}\delta I &= -\frac{1}{4\pi G}\int \eta^{\alpha\beta}\frac{\partial\delta\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\beta}d^4x - \int me^\Phi\delta\Phi\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau d^4x \\ &= \int\left(\frac{1}{4\pi G}\Phi^{,\alpha}{}_{\alpha} - \int me^\Phi\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau\right)\delta\Phi d^4x \\ &= 0.\end{aligned} \tag{7.1.7}$$

The field equations are

$$\Phi^{,\alpha}{}_{\alpha} = 4\pi G\int me^\Phi\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau. \tag{7.1.8}$$

If we assume $\Phi \ll 1$, then we have

$$\Phi^{,\alpha}{}_{\alpha} = 4\pi G\int m\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau. \tag{7.1.9}$$

In the frame where the earth and gravtion field is static, we have

$$\Phi^{,i}{}_{i} = 4\pi Gm\delta^3(\boldsymbol{x} - \boldsymbol{z}) = 4\pi G\rho. \tag{7.1.10}$$

So at the surface of earth, we have

$$\Phi = -\frac{GM}{R} = -7.0\times10^{-10} \tag{7.1.11}$$

and so we can assume $e^\Phi \approx 1 - 7.0\times10^{-10}$ at the surface of the earth.

D. Outside of the sun, we have $\rho = 0$, so we have

$$\nabla^2\Phi = 0, \quad r > R_\mathrm{s}, \tag{7.1.12}$$

i.e.

$$\Phi = -\frac{GM}{r}, \quad r > R_\mathrm{s}. \tag{7.1.13}$$

Because the field is spherically symmetric, the orbit of the planet will be confined to a plane. We will choose the coordinate system to make the plane be $\theta = \pi/2$. The action of the particle can now be written as

$$I = \int L dt, \tag{7.1.14}$$

where

$$L = -me^{\Phi}\sqrt{1 - \left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2}. \tag{7.1.15}$$

It is easy to calculate the canonical momentum for the planet,

$$p_r = \gamma m e^{\Phi}\frac{dr}{dt} \tag{7.1.16}$$

and

$$p_\phi = \gamma m e^{\Phi} r^2 \frac{d\phi}{dt}, \tag{7.1.17}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2}} = \frac{dt}{d\tau}. \tag{7.1.18}$$

The Hamiltonian of the planet is

$$H = p_r\frac{dr}{dt} + p_\phi\frac{d\phi}{dt} - L = \gamma m e^{\Phi}. \tag{7.1.19}$$

Because there is no t and ϕ in L, so H and p_ϕ will be constant. If we let

$$H = mE, \quad p_\phi = mL, \tag{7.1.20}$$

we have

$$\frac{dt}{d\tau} = Ee^{-\Phi}, \quad \frac{d\phi}{d\tau} = \frac{L}{r^2}e^{-\Phi}. \tag{7.1.21}$$

Recall that

$$\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 - 1. \tag{7.1.22}$$

We have

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}e^{-2\Phi} = E^2e^{-2\Phi} - 1. \tag{7.1.23}$$

Define

$$x \equiv \frac{L^2}{GMr}. \tag{7.1.24}$$

Time the equation above by

$$\left(\frac{d\tau}{d\phi}\right)^2 = \frac{r^4}{L^2}e^{2\Phi}. \tag{7.1.25}$$

We have

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = \frac{E^2r^4}{L^2} - \frac{r^4}{L^2}e^{2\Phi}. \tag{7.1.26}$$

Define

$$x \equiv \frac{L^2}{GMr}. \tag{7.1.27}$$

We have

$$\left(\frac{dx}{d\phi}\right)^2 + x^2 = \frac{E^2L^2}{G^2M^2} - \frac{L^2}{G^2M^2}\exp\left(-\frac{2G^2M^2}{L^2}x\right). \tag{7.1.28}$$

Take the derivative with ϕ, we have

$$\frac{d^2x}{d\phi^2} + x = \exp\left(-\frac{2G^2M^2}{L^2}x\right) \approx 1 - \frac{2G^2M^2}{L^2}x \tag{7.1.29}$$

The solution is

$$x = 1 - 2\alpha + e\cos\left[(1+\alpha)\phi\right], \quad \alpha = \frac{G^2M^2}{L^2}. \tag{7.1.30}$$

We have therefore found that, during each orbit of the planet, perihelion advances by an angle

$$\Delta\Phi = 2\pi\alpha = \frac{2\pi G^2M^2}{L^2}. \tag{7.1.31}$$

To convert from the angular momentum L to more conventional quantities, we may use expressions valid for Newtonian orbits, since the quantity we're looking at is already a small perturbation. An ordinary ellipse satisfies

$$r = \frac{(1-e^2)a}{1+e\cos\phi}, \tag{7.1.32}$$

where a is the semi-major axis. Comparing to

$$r = \frac{L^2}{GMx}, \tag{7.1.33}$$

we have

$$L^2 \approx GM(1-e^2)a. \tag{7.1.34}$$

Plugging this into 7.1.31 and restoring explicit factors of the speed of light, we obtain

$$\Delta\phi = \frac{2\pi GM}{c^2(1-e^2)a}. \tag{7.1.35}$$

E. Since we have

$$\delta I = -\int \left[e^{\Phi}m\Phi_{,\alpha}d\tau + d\left(\eta_{\alpha\beta}e^{\Phi}m\frac{dz^\beta}{d\tau}\right)\right]\delta z^\alpha. \tag{7.1.36}$$

By taking the limit $m \to 0$ with $k^0 = \gamma m = E$ remaining finite, we have

$$\delta I = -\int d\left(\eta_{\alpha\beta}e^{\Phi}\frac{dz^\beta}{d\lambda}\right)\delta z^\alpha = 0, \tag{7.1.37}$$

i.e. $q^\mu = k^\mu e^\Phi$ are constants of motion and there is no bending of light by the sun in this scalar theory.

7.2 VECTOR GRAVITATIONAL FIELD, Φ_μ

A. Verify that the variational principle $\delta I = 0$ gives Maxwell's equations by varying A_μ and the Lorentz force law by varying $z^\mu(\tau)$, when

$$I = \frac{-1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}d^4x + \frac{1}{2}m\int\frac{dz^\mu}{d\tau}\frac{dz_\mu}{d\tau}d\tau + e\int\frac{dz^\mu}{d\tau}A_\mu(z)d\tau. \tag{7.2.1}$$

Here, $F_{\mu\nu}$ is an abbrevIation for $A_{\nu,\mu} - A_{\mu,\nu}$.

B. Define, by analogy to the above, a vector gravitational field Φ_μ with $G_{\mu\nu} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}$ using a variational principle with

$$I = \frac{1}{16\pi G}\int G_{\mu\nu}G^{\mu\nu}d^4x + \frac{1}{2}m\int\frac{dz^\mu}{d\tau}\frac{dz_\mu}{d\tau}d\tau + m\int\frac{dz^\mu}{d\tau}\Phi_\mu(z)d\tau. \tag{7.2.2}$$

Find the "Coulomb" law in this theory, and verify that the coefficients of the terms in the variational principle have been chosen reasonably.

C. Compute the perihelion precession in this theory.

D. Compute the bending of light in this theory (i.e., scattering of a highly relativistic particle $u^0 = \gamma \to \infty$, as it passes by the sun, because of the sun's Φ_μ field.

E. Obtain a formula for the total field energy corresponding to the Lagrangian implicit in part B. Use the standard method of Hamiltonian mechanics, with

$$I_{\text{field}} = \frac{1}{16\pi G}\int G_{\mu\nu}G^{\mu\nu}d^4x \equiv \int \mathcal{L}d^4x; \tag{7.2.3}$$

$\mathcal{L}$ is the Lagrangian density and $L \equiv \int \mathcal{L}d^3x$ IS the Lagrangian. The corresponding Hamiltonian density (energy density) is

$$\mathcal{H} = \sum_\mu \Phi_{\mu,0}\frac{\partial \mathcal{L}}{\partial \Phi_{\mu,0}} - \mathcal{L}. \tag{7.2.4}$$

Show that vector gravitational waves carry negative energy.

Solution:

A. First we vary A_ν to get Maxwells equations,

$$\begin{aligned}
\delta I &= -\frac{1}{8\pi}\int F^{\mu\nu}\delta F_{\mu\nu}d^4x + e\int \frac{dz^\nu}{d\tau}\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]d\tau\delta A_\nu d^4x \\
&= -\frac{1}{8\pi}\int (F^{\mu\nu}\partial_\mu\delta A_\nu - F^{\nu\mu}\partial_\mu\delta A_\nu)d^4x + e\int \frac{dz^\nu}{d\tau}\delta^3[\boldsymbol{x} - \boldsymbol{z}(\tau)]\delta[t - z^0(\tau)]dz^0(\tau)\frac{d\tau}{dz^0(\tau)}\delta A_\nu d^4x \\
&= \frac{1}{4\pi}\int \partial_\mu F^{\mu\nu}\delta A_\nu d^4x + e\int \frac{dz^\nu}{dt}\delta^3[\boldsymbol{x} - \boldsymbol{z}(t)]\delta A_\nu d^4x \\
&= \int \left\{\frac{1}{4\pi}\partial_\mu F^{\mu\nu} + e\frac{dz^\nu}{dt}\delta^3[\boldsymbol{x} - \boldsymbol{z}(t)]\right\}\delta A_\nu d^4x = 0.
\end{aligned} \tag{7.2.5}$$

So we have Maxwells equations

$$\frac{1}{4\pi}\partial_\mu F^{\nu\mu} = j^\nu \equiv e\frac{dz^\nu}{dt}\delta^3[\boldsymbol{x} - \boldsymbol{z}(t)]. \tag{7.2.6}$$

Now we vary $z^\mu(\tau)$ to get Lorentz's law,

$$\begin{aligned}
\delta I &= m\int \frac{d\delta z^\mu}{d\tau}\frac{dz_\mu}{d\tau}d\tau + e\int \frac{d\delta z^\mu}{d\tau}A_\mu d\tau + e\int \frac{dz^\nu}{d\tau}A_{\nu,\mu}\delta z^\mu d\tau \\
&= -m\int \delta z^\mu \frac{d^2 z_\mu}{d\tau^2}d\tau - e\int \delta z^\mu A_{\mu,\nu}\frac{dz^\nu}{d\tau}d\tau + e\int \frac{dz^\nu}{d\tau}A_{\nu,\mu}\delta z^\mu d\tau \\
&= -\int (m\frac{d^2 z_\mu}{d\tau^2} - eF_{\mu\nu}u^\nu)\delta z^\mu d\tau = 0.
\end{aligned} \tag{7.2.7}$$

So we have Lorentz's law,

$$ma_\mu = eF_{\mu\nu}u^\nu. \tag{7.2.8}$$

B. By similar procedure, we can get the field equation for the gravitational field

$$-\frac{1}{4\pi G}\partial_\mu G^{\nu\mu} = m\frac{dz^\nu}{dt}\delta^3[\boldsymbol{x} - \boldsymbol{z}(t)]. \tag{7.2.9}$$

and equation of motion for a particle in the gravitational field,

$$a_\mu = G_{\mu\nu}u^\nu. \tag{7.2.10}$$

For a special case where $z^\mu = (t, 0, 0, 0)$, we have

$$\partial_\mu G^{0\mu} = -4\pi G M\delta^3(\boldsymbol{x}). \tag{7.2.11}$$

Suppose there is no time dependent componet in the solution, we have

$$\nabla^2\Phi^0 = 4\pi GM\delta^3(\boldsymbol{x}), \tag{7.2.12}$$

i.e.

$$\Phi^0 = -\frac{GM}{r}. \tag{7.2.13}$$

C. Because the field is spherically symmetric, the orbit of the planet will be confined to a plane. We will choose the coordinate system to make the plane be $\theta = \pi/2$. The action of the particle can now be written as

$$I = \int L dt, \tag{7.2.14}$$

$$L = -m(\sqrt{1 - \dot{r}^2 - r^2\dot{\phi}^2} - \frac{GM}{r}). \tag{7.2.15}$$

It is easy to calculate the canonical momentum for the planet,

$$p_r = \gamma m e^{\Phi} \cdot r \tag{7.2.16}$$

and

$$p_\phi = \gamma m e^{\Phi} r^2 \cdot \phi, \tag{7.2.17}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \dot{r}^2 - r^2\dot{\phi}^2}} = \frac{dt}{d\tau}. \tag{7.2.18}$$

The Hamiltonian of the planet is

$$H = p_r\frac{dr}{dt} + p_\phi\frac{d\phi}{dt} - L = \gamma m - \frac{GMm}{r}. \tag{7.2.19}$$

Because there is no t and ϕ in L, so H and p_ϕ will be constant. If we let

$$H = mE, \quad p_\phi = mL, \tag{7.2.20}$$

we have

$$\frac{dt}{d\tau} = E + \frac{GM}{r}, \quad \frac{d\phi}{d\tau} = \frac{L}{r^2}. \tag{7.2.21}$$

Recall that

$$\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 - 1. \tag{7.2.22}$$

We have

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} = \left(E + \frac{GM}{r}\right)^2 - 1. \tag{7.2.23}$$

Time the equation above by

$$\left(\frac{d\tau}{d\phi}\right)^2 = \frac{r^4}{L^2}. \tag{7.2.24}$$

We have

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = \frac{r^4}{L^2}\left(E + \frac{GM}{r}\right)^2 - \frac{r^4}{L^2}. \tag{7.2.25}$$

Define

$$x \equiv \frac{L^2}{GMr}. \tag{7.2.26}$$

We have

$$\left(\frac{dx}{d\phi}\right)^2 + x^2 = \frac{L^2}{G^2M^2}\left(E + \frac{G^2M^2}{L^2}x\right)^2 - \frac{L^2}{G^2M^2}. \tag{7.2.27}$$

Take the derivative with ϕ, we have

$$\frac{d^2x}{d\phi^2} + x = E + \frac{G^2M^2}{L^2}x. \tag{7.2.28}$$

The solution is

$$x = E(1+2\alpha) + e\cos\left[(1-\alpha)\phi\right], \quad \alpha = \frac{G^2M^2}{2L^2}. \tag{7.2.29}$$

To convert from the angular momentum L to more conventional quantities, we may use expressions valid for Newtonian orbits,

$$L^2 \approx GM(1-e^2)a. \tag{7.2.30}$$

Rrestoring explicit factors of the speed of light, we obtain

$$\Delta\phi = \frac{\pi GM}{c^2(1-e^2)a}. \tag{7.2.31}$$

D. Take the limit $m \to 0$ and keep $\tau/m = \lambda$ finite, we have

$$\delta I = -\int (\frac{d^2z_\mu}{d\lambda^2} - mG_{\mu\nu}\frac{dz^\nu}{d\lambda})\delta z^\mu d\lambda \to -\int \frac{d^2z_\mu}{d\lambda^2}\delta z^\mu d\lambda = 0. \tag{7.2.32}$$

So light will not bent in gravitational field.

E. Since

$$\mathcal{L} = \frac{1}{16\pi G}G_{\mu\nu}G^{\mu\nu} = \frac{1}{8\pi G}(\Phi_{\mu,\nu}\Phi^{\mu,\nu} - \Phi_{\mu,\nu}\Phi^{\nu,\mu}) \tag{7.2.33}$$

So

$$\begin{aligned}\mathcal{H} &= \frac{1}{4\pi G}(\Phi_{\mu,0}\Phi^{\mu,0} - \Phi_{\mu,0}\Phi^{0,\mu}) - \frac{1}{16\pi G}G_{\mu\nu}G^{\mu\nu} \\ &= \frac{1}{4\pi G}\Phi_{\mu,0}G^{0\mu} - \frac{1}{16\pi G}G_{\mu\nu}G^{\mu\nu}\end{aligned} \tag{7.2.34}$$

For free field, we have

$$\partial^\rho\Phi^0G^0{}_\rho = \partial^\rho(\Phi^0G^0{}_\rho) = \partial^i(\Phi^0G^0{}_i) \tag{7.2.35}$$

If we add a divergence to the $\mathcal{H}$, the value of H will not be changed, so we have

$$\begin{aligned}\mathcal{H} &= \frac{1}{4\pi G}\Phi_{\mu,0}G^{0\mu} - \frac{1}{16\pi G}G_{\mu\nu}G^{\mu\nu} + \frac{1}{4\pi G}\Phi^{0,\mu}G^0{}_\mu \\ &= \frac{1}{4\pi G}G_{0\mu}G^{0\mu} - \frac{1}{16\pi G}G_{\mu\nu}G^{\mu\nu} \\ &= -\frac{1}{8\pi G}G^{0i}G^{0i} - \frac{1}{16\pi G}G^{ij}G^{ij} < 0,\end{aligned} \tag{7.2.36}$$

i.e. vector gravitational waves carry negative energy.

7.3 SYMMETRIC TENSOR GRAVITATIONAL FIELD, $h_{\mu\nu} = h_{\nu\mu}$

This problem is used as an sample for discussing the possible special-relativity-compatible gravity theory in the textbook. The solution of the exercise is presented in Box 7.1 of the textbook.

Chapter 8

GRAVITATIONAL COLLAPSE

8.1 PRACTICE WITH TENSOR ALGEBRA

All calculations on tensor algebra in this exercise are straightforward, and the answer to this exercise are provided in the textbook. So I will not repeat it here.

8.2 COMMUTATORS

Take the mathematician's viewpoint that tangent vectors and directional derivatives are the same thing, $\boldsymbol{u} \equiv \partial_{\boldsymbol{u}}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vector fields, and define their commutator in the manner familiar from quantum mechanics

$$[\boldsymbol{u}, \boldsymbol{v}] \equiv [\partial_{\boldsymbol{u}}, \partial_{\boldsymbol{v}}] \equiv \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}} - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}} \tag{8.2.1}$$

(a) Derive the following expression for $[\boldsymbol{u}, \boldsymbol{v}]$, valid in any coordinate basis,

$$[\boldsymbol{u}, \boldsymbol{v}] = (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta})\boldsymbol{e}_\alpha. \tag{8.2.2}$$

Thus, despite the fact that it looks like a second-order differential operator, $[\boldsymbol{u}, \boldsymbol{v}]$ is actually of first order-i.e., it is a tangent vector.

(b) For any basis $\{\boldsymbol{e}_\alpha\}$, one defines the "commutation coefficients" $c_{\beta\gamma}{}^\alpha$ and $c_{\beta\gamma\alpha}$ by

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] \equiv c_{\beta\gamma}{}^\alpha \boldsymbol{e}_\alpha; \quad c_{\beta\gamma\alpha} \equiv g_{\alpha\mu} c_{\beta\gamma}{}^\mu. \tag{8.2.3}$$

Show that $c_{\beta\gamma}{}^\alpha = c_{\beta\gamma\alpha} = 0$ for any coordinate basis.

(c) Calculate $c_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}}$ for the spherical noncoordinate basis of exercise 8.1.

Solution:

(a)

$$\begin{aligned}
[\boldsymbol{u}, \boldsymbol{v}]f &= \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}f - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}f = \partial_{\boldsymbol{u}}(f_{,\alpha}v^\alpha) - \partial_{\boldsymbol{v}}(f_{,\alpha}u^\alpha) \\
&= f_{,\alpha\beta}v^\alpha u^\beta + f_{,\alpha}v^\alpha{}_{,\beta}u^\beta - f_{,\alpha\beta}u^\alpha v^\beta + f_{,\alpha}u^\alpha{}_{,\beta}v^\beta \\
&= f_{,\alpha}(v^\alpha{}_{,\beta}u^\beta - u^\alpha{}_{,\beta}v^\beta).
\end{aligned} \tag{8.2.4}$$

So we have

$$[\boldsymbol{u},\boldsymbol{v}] = (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta})\boldsymbol{e}_\alpha. \tag{8.2.5}$$

(b) For coordinate basis, we have

$$[\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta]f = f_{,\beta\alpha} - f_{,\alpha\beta} = 0. \tag{8.2.6}$$

So $c_{\beta\gamma}{}^\alpha = c_{\beta\gamma\alpha} = 0$ for any coordinate basis.

(c) In spherical coordinate basis, we have

$$\boldsymbol{e}_{\hat{0}} = (1,0,0,0); \quad \boldsymbol{e}_{\hat{r}} = (0,1,0,0); \quad \boldsymbol{e}_{\hat{\theta}} = (0,0,1/r,0); \qquad \boldsymbol{e}_{\hat{\phi}} = (0,0,0,1/r\sin\theta). \tag{8.2.7}$$

It is easy to see that commutators between $\boldsymbol{e}_{\hat{0}}$ and other basis vanish. And we have

$$[\boldsymbol{e}_{\hat{r}},\boldsymbol{e}_{\hat{\theta}}] = \left(\frac{1}{r}\right)_{,r}\boldsymbol{e}_\theta = -\frac{1}{r}\boldsymbol{e}_{\hat{\theta}}; \tag{8.2.8}$$

$$[\boldsymbol{e}_{\hat{r}},\boldsymbol{e}_{\hat{\phi}}] = \left(\frac{1}{r\sin\theta}\right)_{,r}\boldsymbol{e}_\phi = -\frac{1}{r\sin\theta}\boldsymbol{e}_{\hat{\phi}}; \tag{8.2.9}$$

$$[\boldsymbol{e}_{\hat{\theta}},\boldsymbol{e}_{\hat{\phi}}] = \frac{1}{r}\left(\frac{1}{r\sin\theta}\right)_{,\theta}\boldsymbol{e}_\phi = -\frac{\cot\theta}{r}\boldsymbol{e}_{\hat{\phi}}; \tag{8.2.10}$$

So the nonvanishing components of $c_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}}$ are

$$\begin{aligned} c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} &= -c_{\hat{\theta}\hat{r}}{}^{\hat{\theta}} = -1/r; \\ c_{\hat{r}\hat{\phi}}{}^{\hat{\phi}} &= -c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}} = -1/r\sin\theta; \\ c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} &= -c_{\hat{\phi}\hat{\theta}}{}^{\hat{\phi}} = -\cot\theta/r. \end{aligned} \tag{8.2.11}$$

8.3 COMPONENTS OF LEVI-CIVITA TENSOR IN NONORTHONORMAL FRAME

(a) Show that expressions (8.10) are the components of $\boldsymbol{\epsilon}$ in an arbitrary basis, with $\boldsymbol{e}_0$ pointing toward the future and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ right-handed.

(b) Show that the components of the permutation tensors [defined by equations (3.50h)-(3.50j)] have the same values [equations (3.50k)-(3.50m)] in arbitrary frames as in Lorentz frames.

Solution:

(a)

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} &= L^{\hat{\mu}}{}_\alpha L^{\hat{\nu}}{}_\beta L^{\hat{\lambda}}{}_\gamma L^{\hat{\rho}}{}_\delta \epsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}} \\ &= \sum_\pi L^{\pi(0)}{}_\alpha L^{\pi(1)}{}_\beta L^{\pi(2)}{}_\gamma L^{\pi(3)}{}_\delta [\pi(0)\pi(1)\pi(2)\pi(3)] \end{aligned} \tag{8.3.1}$$

Suposse that $\pi_1 : 0123 \to \alpha\beta\gamma\delta$, we than have

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} &= \sum_\pi (-1)^\pi L^{\pi(0)}{}_{\pi_1(0)} L^{\pi(1)}{}_{\pi_1(1)} L^{\pi(2)}{}_{\pi_1(2)} L^{\pi(3)}{}_{\pi_1(3)} \\ &= (-1)^{\pi_1} \sum_\pi (-1)^{\pi_1^{-1}\pi} L^{\pi_1^{-1}\pi(0)}{}_0 L^{\pi_1^{-1}\pi(1)}{}_1 L^{\pi_1^{-1}\pi(2)}{}_2 L^{\pi_1^{-1}\pi(3)}{}_3 \\ &= (-1)^{\pi_1} \det L \\ &= \det L[\alpha\beta\gamma\delta]. \end{aligned} \tag{8.3.2}$$

On the hand, we have

$$g_{\alpha\beta} = L^{\hat{\mu}}{}_{\alpha}\eta_{\hat{\mu}\hat{\nu}}L^{\hat{\nu}}{}_{\beta}. \tag{8.3.3}$$

So

$$g \equiv \det g = \det L^{\mathrm{T}} \det\eta \det L = -(\det L)^2 \tag{8.3.4}$$

We now have

$$\epsilon_{\alpha\beta\gamma\delta} = (-g)^{1/2}[\alpha\beta\gamma\delta]. \tag{8.3.5}$$

As for $\epsilon^{\alpha\beta\gamma\delta}$, we have

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta} &= g^{\alpha\mu}g^{\beta\nu}g^{\gamma\lambda}g^{\delta\rho}\epsilon_{\mu\nu\lambda\rho}\\ &= (-g)^{1/2}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\lambda}g^{\delta\rho}[\mu\nu\lambda\rho]\\ &= (-g)^{1/2}\det g^{-1}[\alpha\beta\gamma\delta]\\ &= -(-g)^{-1/2}[\alpha\beta\gamma\delta].\end{aligned} \tag{8.3.6}$$

(b) Since

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\lambda\rho} = -[\alpha\beta\gamma\delta][\mu\nu\lambda\rho] = \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\epsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}, \tag{8.3.7}$$

the components of the permutation tensors have the same values in arbitrary frames as in Lorentz frames.

8.4 PRACTICE IN WRITING COMPONENTS OF GRADIENT

All calculations on tensor calculus in this exercise are straightforward, and the answer to this exercise are provided in the textbook. So I will not repeat it here.

8.5 A SHEET OF PAPER IN POLAR COORDINATES

The two-dimensional metnc for a flat sheet of paper in polar coordinates (r, θ) is $ds^2 = dr^2 + r^2 d\phi^2$, or in modern notation, $\boldsymbol{g} = \boldsymbol{d}r \otimes \boldsymbol{d}r + r^2 \boldsymbol{d}\phi \otimes \boldsymbol{d}\phi$.

(a) Calculate the connection coefficients using equations (8.24).

(b) Write down the geodesic equation in (r, θ) coordinates.

(c) Solve this geodesic equation for $r(\lambda)$ and $\phi(\lambda)$, and show that the solution is a uniformly parametrized straight line.

(d) Verify: that the noncoordinate basis $\boldsymbol{e}_{\hat{r}} \equiv \boldsymbol{e}_r = \partial\mathcal{P}/\partial r$, $\boldsymbol{e}_{\hat{\phi}} \equiv r^{-1}\boldsymbol{e}_\phi = r^{-1}\partial\mathcal{P}/\partial\phi$, $\boldsymbol{\omega}^{\hat{r}} = \boldsymbol{d}r$, $\boldsymbol{\omega}^{\hat{\phi}} = r\boldsymbol{d}\phi$ is orthonormal, and that $\langle \boldsymbol{\omega}^{\hat{\alpha}}, \boldsymbol{e}_{\hat{\beta}}\rangle = \delta^{\hat{\alpha}}{}_{\hat{\beta}}$. Then calculate the connection coefficients of this basis from a knowledge [part (a)] of the connection of the coordinate basis.

(e) Consider the Keplerian orbit of Figure 8.1 and Section 8.3 as a nongeodesic curve in the sun's two-dimensional, Euclidean, equatorial plane. In place of the old notation $d\mathbf{v}/dt$, $d\mathbf{e}_{\hat{r}}/dt$, etc., use the new notation $\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v}$, $\boldsymbol{\nabla}_{\boldsymbol{e}_{\hat{r}}}\boldsymbol{v}$, etc. Then $\boldsymbol{v} = d\mathcal{P}/dt$ is the tangent to the orbit, and $\boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v}$ is the acceleration. Derive equations (8.4) for $a^{\hat{r}}$ and $a^{\hat{\phi}}$ using component manipulations and connection coefficients in the orthonormal basis.

Solution:

(a)

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}). \tag{8.5.1}$$

The only non-vanishing component of $g_{\alpha\beta,\gamma}$ is $g_{\phi\phi,r} = 2r$. Then the non-vanishing components of $\Gamma_{\alpha\beta\gamma}$ is $\Gamma_{\phi\phi r} = \Gamma_{\phi r\phi} = -\Gamma_{r\phi\phi} = r$. So

$$\Gamma^{\phi}{}_{\phi r} = \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}, \quad \Gamma^{r}{}_{\phi\phi} = -r; \tag{8.5.2}$$

all others vanish.

(b)

$$\begin{aligned} &\frac{d^2r}{d\lambda^2} + \Gamma^{r}{}_{\phi\phi}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = \frac{d^2r}{d\lambda^2} - r\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = 0; \\ &\frac{d^2\phi}{d\lambda^2} + 2\Gamma^{\phi}{}_{r\phi}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0. \end{aligned} \tag{8.5.3}$$

(c) Define $x \equiv r\cos\phi$ and $y \equiv r\sin\phi$. Then we have

$$\begin{aligned} x'' &= (r'\cos\phi - r\sin\phi\phi')' \\ &= r''\cos\phi - 2r'\sin\phi\phi' - r\cos\phi(\phi')^2 - r\sin\phi\phi'' \\ &= r''\cos\phi - 2r'\sin\phi\phi' - r\cos\phi\cdot\frac{1}{r}r'' - r\sin\phi\cdot -\frac{2}{r}r'\phi' \\ &= 0. \end{aligned} \tag{8.5.4}$$

Similarly, we can also get $y'' = 0$. So the general solution of the geodesic equation is $x = a\lambda + b$ for some a and b, $y = j\lambda + k$ for some j and k, which is a straight line in R^2 plane.

(d)

$$\begin{aligned} &\boldsymbol{\nabla}\boldsymbol{e}_{\hat{r}} = \boldsymbol{\nabla}\boldsymbol{e}_r = \Gamma^{\phi}{}_{r\phi}\boldsymbol{d}\phi\otimes\boldsymbol{e}_\phi = r^{-1}\boldsymbol{\omega}^{\hat{\phi}}\otimes\boldsymbol{e}_{\hat{\phi}}, \\ &\boldsymbol{\nabla}\boldsymbol{e}_{\hat{\phi}} = \boldsymbol{\nabla}r^{-1}\boldsymbol{e}_\phi = -r^{-2}\boldsymbol{d}r\otimes\boldsymbol{e}_\phi + r^{-1}(\Gamma^{r}{}_{\phi\phi}\boldsymbol{d}\phi\otimes\boldsymbol{e}_r + \Gamma^{\phi}{}_{\phi r}\boldsymbol{d}r\otimes\boldsymbol{e}_\phi) = -r^{-1}\boldsymbol{\omega}^{\hat{\phi}}\otimes\boldsymbol{e}_{\hat{r}}. \end{aligned} \tag{8.5.5}$$

So we have

$$\Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = r^{-1}; \quad \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -r^{-1}; \tag{8.5.6}$$

all others vanish.

(e)

$$\boldsymbol{v} = \frac{dr\boldsymbol{e}_{\hat{r}}}{dt} = \frac{dr}{dt}\boldsymbol{e}_{\hat{r}} + r\frac{d\phi}{dt}\boldsymbol{e}_{\hat{\phi}}. \tag{8.5.7}$$

$$\boldsymbol{a}^{\hat{r}} = \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}}v^{\hat{\phi}}v^{\hat{\phi}} + \frac{dv^{\hat{r}}}{dt} = -r\frac{d\phi}{dt}\frac{d\phi}{dt} + \frac{d^2r}{dt^2}, \tag{8.5.8}$$

$$\boldsymbol{a}^{\hat{\phi}} = \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}}v^{\hat{r}}v^{\hat{\phi}} + \frac{dv^{\hat{\phi}}}{dt} = \frac{dr}{dt}\frac{d\phi}{dt} + \frac{d}{dt}\left(r\frac{d\phi}{dt}\right). \tag{8.5.9}$$

8.6 SPHERICAL COORDINATES IN FLAT SPACETIME

All calculations in this exercise are straightforward, and the answer to this exercise are provided in the textbook. So I will not repeat it here.

8.7 SYMMETRIES OF CONNECTION COEFFICIENTS

From equation (8.24b), the symmetry of the metric, and the antisymmetry ($c_{\beta\gamma\mu} = -c_{\gamma\beta\mu}$) of the commutation coefficients, show that: $\Gamma_{\alpha[\beta\gamma]} = 0$ (last two indices are symmetric) in a coordinate basis; $\Gamma_{(\hat{\alpha}\hat{\beta})\hat{\gamma}} = 0$ (first two indices are antlsymmetrIc) in a globally orthonormal basis, $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$.

Solution:

In general, we have

$$\Gamma_{\mu\beta\gamma} = \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu}). \tag{8.7.1}$$

In a coordinate basis, $c_{\alpha\beta\gamma} = 0$. So

$$2\Gamma_{\alpha[\beta\gamma]} = \frac{1}{2}[(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) - (g_{\alpha\gamma,\beta} + g_{\alpha\beta,\gamma} - g_{\gamma\beta,\alpha})] = 0. \tag{8.7.2}$$

In a globally orthonormal basis, we have $g_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = 0$. So

$$2\Gamma_{(\hat{\alpha}\hat{\beta})\hat{\gamma}} = \frac{1}{2}[(c_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + c_{\hat{\alpha}\hat{\gamma}\hat{\beta}} - c_{\hat{\beta}\hat{\gamma}\hat{\alpha}}) + (c_{\hat{\beta}\hat{\alpha}\hat{\gamma}} + c_{\hat{\beta}\hat{\gamma}\hat{\alpha}} - c_{\hat{\alpha}\hat{\gamma}\hat{\beta}})] = 0. \tag{8.7.3}$$

8.8 NEW DEFINITION OF ∇T COMPARED WITH OLD DEFINITION

The new definition of $\boldsymbol{\nabla T}$ is given by equations (8.16) and (8.17). Use the fact that parallel transport keeps local-Lorentz components fixed to derive, from (8.16), the Lorentz-frame equation $\boldsymbol{\nabla_u T} = T^{\beta}{}_{\alpha,\gamma}\, u^{\gamma} \boldsymbol{e}_{\beta} \otimes \boldsymbol{\omega}^{\alpha}$. From this and equation (8.17), infer that the Lorentz-frame components of $\boldsymbol{\nabla T}$ are $T^{\beta}{}_{\alpha,\gamma}$ which accords with the old definition of $\boldsymbol{\nabla T}$.

Solution:

Suppose the components of $\boldsymbol{T}$ at $\mathcal{P}(0)$ is $T^{\beta}{}_{\alpha}$. The components of $\boldsymbol{T}$ at $\mathcal{P}(\xi)$ is $T^{\beta}{}_{\alpha} + T^{\beta}{}_{\alpha,\gamma}\, u^{\gamma}\xi$. Because parallel transport keeps local-Lorentz components fixed, in a local-Lorentz frame, we have

$$T^{\beta}{}_{\alpha}\,[P(\xi)]_{\text{parallel-transported to } \mathcal{P}_0} = T^{\beta}{}_{\alpha} + T^{\beta}{}_{\alpha,\gamma}\, u^{\gamma}\xi \tag{8.8.1}$$

According to (8.16), the components of $\boldsymbol{\nabla_u T}$ is $T^{\beta}{}_{\alpha,\gamma}\, u^{\gamma}$, i.e. $\boldsymbol{\nabla_u T} = T^{\beta}{}_{\alpha,\gamma}\, u^{\gamma} \boldsymbol{e}_{\beta} \otimes \boldsymbol{\omega}^{\alpha}$. As $\boldsymbol{\nabla_u T} \equiv \boldsymbol{\nabla T}(\cdots, \cdots, \boldsymbol{u})$, we have

$$\boldsymbol{\nabla T} = T^{\beta}{}_{\alpha,\gamma}\, \boldsymbol{e}_{\beta} \otimes \boldsymbol{\omega}^{\alpha} \otimes \boldsymbol{\omega}^{\gamma}. \tag{8.8.2}$$

8.9 CHAIN RULE FOR $\nabla_u T$

(a) Use calculations in a local Lorentz frame to show that "$\boldsymbol{\nabla_u}$" obeys the standard chain rule for derivatives:

$$\boldsymbol{\nabla_u}(f\boldsymbol{A} \otimes \boldsymbol{B}) = \boldsymbol{\nabla_u} f(\boldsymbol{A} \otimes \boldsymbol{B}) + f(\boldsymbol{\nabla_u A}) \otimes \boldsymbol{B} + f\boldsymbol{A} \otimes (\boldsymbol{\nabla_u B}). \tag{8.9.1}$$

Here $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary vectors, 1-forms, or tensors; and f is an arbitrary function.

(b) Rewrite equation (8.27) in component notation in an arbitrary basis.

Solution:

Assume for concreteness that $\boldsymbol{A}$ is a $(1,1)$ tensor and $\boldsymbol{B}$ is a vector. In a local Lorentz frame, the components of $\boldsymbol{\nabla_u}(f\boldsymbol{A}\otimes\boldsymbol{B})$ are

$$(fA^{\alpha}{}_{\beta}B^{\gamma})_{,\delta}u^{\delta}. \tag{8.9.2}$$

In light of Leibniz rule, the components of $\boldsymbol{\nabla_u}(f\boldsymbol{A}\otimes\boldsymbol{B})$ can be expanded as

$$(f_{,\delta}u^{\delta})A^{\alpha}{}_{\beta}B^{\gamma}+f(A^{\alpha}{}_{\beta,\delta}u^{\delta})B^{\gamma}+fA^{\alpha}{}_{\beta}(B^{\gamma}{}_{,\delta}u^{\delta}). \tag{8.9.3}$$

i.e.

$$\boldsymbol{\nabla_u}(f\boldsymbol{A}\otimes\boldsymbol{B})=\boldsymbol{\nabla_u}f(\boldsymbol{A}\otimes\boldsymbol{B})+f(\boldsymbol{\nabla_u}\boldsymbol{A})\otimes\boldsymbol{B}+f\boldsymbol{A}\otimes(\boldsymbol{\nabla_u}\boldsymbol{B}) \tag{8.9.4}$$

in terms of geometric objects.

(b) As for an arbitrary coordinate system, the ordinary derivatives must be replaced by covariant derivatives, i.e.

$$(fA^{\alpha}{}_{\beta}B^{\gamma})_{;\delta}u^{\delta}=(f_{,\delta}u^{\delta})A^{\alpha}{}_{\beta}B^{\gamma}+f(A^{\alpha}{}_{\beta;\delta}u^{\delta})B^{\gamma}+fA^{\alpha}{}_{\beta}(B^{\gamma}{}_{;\delta}u^{\delta}). \tag{8.9.5}$$

8.10 COVARIANT DERIVATIVE COMMUTES WITH CONTRACTION

Let $\boldsymbol{S}$ be a $(1,2)$ tensor. Using components in a local Lorentz frame show that

$$\boldsymbol{\nabla_u}=(\text{contraction on slots 1 and 2 of }\boldsymbol{S})=(\text{contraction on slots 1 and 2 of }\boldsymbol{\nabla_u}\boldsymbol{S}). \tag{8.10.1}$$

Solution:

In a local Lorentz frame, equation 8.10.1 can be written explicitly as

$$(\sum_{\alpha}S^{\alpha}{}_{\alpha\beta})_{,\gamma}u^{\gamma}=sum_{\alpha}S^{\alpha}{}_{\alpha\beta,\gamma}u^{\gamma}. \tag{8.10.2}$$

8.11 ALGEBRAIC PROPERTIES OF ∇

Use calculations in a local Lorentz frame to show that

$$\boldsymbol{\nabla}_{a\boldsymbol{u}+b\boldsymbol{v}}\boldsymbol{S}=a\boldsymbol{\nabla_u}\boldsymbol{S}+b\boldsymbol{\nabla_v}\boldsymbol{S} \tag{8.11.1}$$

for all tangent vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and numbers a, b; also that

$$\boldsymbol{\nabla_u}(\boldsymbol{S}+\boldsymbol{M})=\boldsymbol{\nabla_u}\boldsymbol{S}+\boldsymbol{\nabla_u}\boldsymbol{M} \tag{8.11.2}$$

for any two tensor fields $\boldsymbol{S}$ and $\boldsymbol{M}$ of the same rank; also that

$$\boldsymbol{\nabla_u}\boldsymbol{w}-\boldsymbol{\nabla_w}\boldsymbol{u}=[\boldsymbol{u},\boldsymbol{w}] \tag{8.11.3}$$

for any two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$.

Solution:

Assume for concreteness that $\boldsymbol{S}$ and $\boldsymbol{M}$ is a $(1,1)$ tensor. In local Lorentz frame, we have

$$\begin{aligned}S^{\alpha}{}_{\beta,\gamma}(au^{\gamma}+bv^{\gamma})&=a(S^{\alpha}{}_{\beta,\gamma}u^{\gamma})+b(S^{\alpha}{}_{\beta,\gamma}v^{\gamma});\\(S^{\alpha}{}_{\beta}+M^{\alpha}{}_{\beta})_{,\gamma}&=(S^{\alpha}{}_{\beta,\gamma}u^{\gamma})+M^{\alpha}{}_{\beta,\gamma}u^{\gamma},\end{aligned} \tag{8.11.4}$$

i.e. 8.11.1 and 8.11.2 hold. Also, the components of $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w} - \boldsymbol{\nabla}_{\boldsymbol{w}}\boldsymbol{u}$ are

$$w^{\alpha}{}_{,\beta}u^{\beta} - u^{\alpha}{}_{,\beta}v^{\beta}. \tag{8.11.5}$$

Comparing with 8.2.2, 8.11.3 holds in local Lorentz frame as well. 8.11.1, 8.11.2 and 8.11.3 are all composed of geometry objects which are invariant under coordinate transformation. Thus they all holds in an arbitrary frame.

8.12 CONNECTION COEFFICIENTS FOR 1-FORM BASIS

8.13 "ΓT" CORRECTION TERMS FOR $T^{\beta}{}_{\alpha;\gamma}$

8.14 METRIC IS COVARIANTLY CONSTANT

8.15 CONNECTION COEFFICIENTS IN TERMS OF METRIC

Detailed answers to exercises 8.12, 8.13, 8.14 and 8.15 are provided in the textbook. So I will not repeat them here.

8.16 SOME USEFUL FORMULAS IN COORDINATE FRAMES

In any coordinate frame, define g to be the determinant of the matrix $g_{\alpha\beta}$ [equation 8.11]. Derive the following relatIons, valid in any coordinate frame.

(a) Contraction of connection coefficients:

$$\Gamma^{\alpha}{}_{\beta\alpha} = (\ln\sqrt{-g})_{,\beta}. \tag{8.16.1}$$

(b) Components of Ricci tensor:

$$R_{\alpha\beta} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\Gamma^{\mu}{}_{\alpha\beta})_{,\mu} - (\ln\sqrt{-g})_{,\alpha\beta} - \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\nu}{}_{\beta\mu}. \tag{8.16.2}$$

(c) Divergence of a vector A^{α} or antisymmetric tensor $F^{\alpha\beta}$:

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{\alpha})_{,\alpha}, \quad F^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\alpha\beta})_{,\beta}. \tag{8.16.3}$$

(d) Integral of a scalar field Ψ over the proper volume of a 4-dimensional region $\mathcal{V}$:

$$\int_{\mathcal{V}} \Psi d(\text{proper volume}) = \int_{\mathcal{V}} \Psi\sqrt{-g}dtdxdydz. \tag{8.16.4}$$

Solution:

(a) From Exercise 5.5, we know

$$\boldsymbol{d}\ln|\det A| = (A^{-1})^{\mu}{}_{\alpha}\boldsymbol{d}A^{\mu}{}_{\alpha}. \tag{8.16.5}$$

So

$$(\ln\sqrt{-g})_{,\beta} = \frac{1}{2}(\ln(-g))_{,\beta} = \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\beta}. \tag{8.16.6}$$

On the other hand, we have

$$\Gamma^{\alpha}{}_{\beta\alpha} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}) = \frac{1}{2}g^{\alpha\mu}g_{\mu\alpha,\beta}. \tag{8.16.7}$$

Now we have

$$\Gamma^{\alpha}{}_{\beta\alpha} = (\ln\sqrt{-g})_{,\beta}. \tag{8.16.8}$$

(b)

$$\begin{aligned} R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta} \\ &= \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha\mu} \\ &= \Gamma^{\mu}{}_{\alpha\beta,\mu} - (\ln\sqrt{-g})_{,\alpha\beta} + (\ln\sqrt{-g})_{,\nu}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\mu\beta}\Gamma^{\mu}{}_{\alpha\nu} \\ &= \frac{1}{\sqrt{-g}}(\sqrt{-g}\Gamma^{\mu}{}_{\alpha\beta})_{,\mu} - (\ln\sqrt{-g})_{,\alpha\beta} - \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\nu}{}_{\beta\mu}. \end{aligned} \tag{8.16.9}$$

(c)

$$A^{\alpha}{}_{;\alpha} = A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}{}_{\mu\alpha}A^{\mu} = A^{\alpha}{}_{,\alpha} + (\ln\sqrt{-g})_{,\alpha}A^{\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{\alpha})_{,\alpha}. \tag{8.16.10}$$

$$F^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}F^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}F^{\alpha\mu} = F^{\alpha\beta}{}_{,\beta} + (\ln\sqrt{-g})_{,\mu}F^{\alpha\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\alpha\beta})_{,\beta}. \tag{8.16.11}$$

(d) In a local Lorentz frame, d(proper volume) $= d\hat{t}d\hat{x}d\hat{y}d\hat{z}$. When we transform this volume element to the given coordinate frame, we have

$$d(\text{proper volume}) = \det\left|\frac{\partial x^{\hat{\alpha}}}{\partial x^{\beta}}\right| dtdxdydz. \tag{8.16.12}$$

Since we have

$$g_{\alpha\beta} = \frac{\partial x^{\hat{\mu}}}{\partial x^{\alpha}}\frac{\partial x^{\hat{\nu}}}{\partial x^{\beta}}\eta_{\hat{\mu}\hat{\nu}}, \tag{8.16.13}$$

we can get

$$g = -\det\left|\frac{\partial x^{\hat{\alpha}}}{\partial x^{\beta}}\right|^2. \tag{8.16.14}$$

Thus

$$d(\text{proper volume}) = \sqrt{-g}dtdxdydz. \tag{8.16.15}$$

The integration will be

$$\int_{\mathcal{V}} \Psi d(\text{proper volume}) = \int_{\mathcal{V}} \Psi\sqrt{-g}dtdxdydz. \tag{8.16.16}$$

Chapter 9

DIFFERENTIAL TOPOLOGY

9.1 COMPONENT MANIPULATIONS

Derive equations (9.11c) through (9.11g) from (9.10), (9.1 la, b), (9.6), (9.7), and (9.8).

Solution:

$$
\begin{aligned}
\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{u} \rangle &= \langle \boldsymbol{\omega}^{\alpha}, u^{\beta} \boldsymbol{e}_{\beta} \rangle = u^{\beta} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = u^{\beta} \delta^{\alpha}{}_{\beta} = u^{\alpha}; \\
\langle \boldsymbol{\omega}, \boldsymbol{e}_{\beta} \rangle &= \langle \sigma_{\alpha} \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = \sigma_{\alpha} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = \sigma_{\alpha} \delta^{\alpha}{}_{\beta} = \sigma_{\beta}; \\
\langle \boldsymbol{\omega}, \boldsymbol{u} \rangle &= \langle \sigma_{\alpha} \boldsymbol{\omega}^{\alpha}, u^{\beta} \boldsymbol{e}_{\beta} \rangle = \sigma_{\alpha} u^{\beta} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = \sigma_{\alpha} u^{\beta} \delta^{\alpha}{}_{\beta} = \sigma_{\alpha} u^{\alpha}; \\
\langle L^{\alpha'}{}_{\beta} \boldsymbol{\omega}^{\beta}, \boldsymbol{e}_{\beta'} \rangle &= \langle L^{\alpha'}{}_{\beta} \boldsymbol{\omega}^{\beta}, \boldsymbol{e}_{\gamma} L^{\gamma}{}_{\beta'} \rangle = L^{\alpha'}{}_{\beta} L^{\gamma}{}_{\beta'} \langle \boldsymbol{\omega}^{\beta}, \boldsymbol{e}_{\gamma} \rangle = L^{\alpha'}{}_{\beta} L^{\gamma}{}_{\beta'} \delta^{\beta}{}_{\gamma} = L^{\alpha'}{}_{\beta} L^{\beta}{}_{\beta'} = \delta^{\alpha'}{}_{\beta'}; \\
\sigma_{\alpha'} &= \langle \boldsymbol{\sigma}, \boldsymbol{e}_{\alpha'} \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{e}_{\beta} L^{\beta}{}_{\alpha'} \rangle = \langle \boldsymbol{\sigma}, \boldsymbol{e}_{\beta} \rangle L^{\beta}{}_{\alpha'} = \sigma_{\beta} L^{\beta}{}_{\alpha'}.
\end{aligned} \tag{9.1.1}
$$

9.2 COMPONENTS OF GRADIENT, AND DUALITY OF COORDINATE BASES

In an arbitrary basis, define $f_{,\alpha}$ by the expansion (9.13a). Then combine equations (9.11d) and (9.12) to obtain the method (9.13b) of computing $f_{,\alpha}$. Finally, combine equations (9.12) and (9.l3b) to show that the bases $\{\boldsymbol{d}x^{\alpha}\}$ and $\{\partial/\partial x^{\beta}\}$ are the duals of each other.

Solution:

$$
f_{,\alpha} = \langle \boldsymbol{d}f, \boldsymbol{e}_{\alpha} \rangle = \partial_{\boldsymbol{e}_{\alpha}} f = \boldsymbol{e}_{\alpha}[f]. \tag{9.2.1}
$$

If $\{\boldsymbol{e}_{\alpha}\}$ is a coordinate basis, we have

$$
f_{,\alpha} = \frac{\partial f}{\partial x^{\alpha}}. \tag{9.2.2}
$$

Particularly,

$$
\langle \boldsymbol{d}x^{\alpha}, \frac{\partial}{\partial x^{\beta}} \rangle = \frac{\partial x^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}{}_{\beta}. \tag{9.2.3}
$$

So the bases $\{\boldsymbol{d}x^\alpha\}$ and $\{\partial/\partial x^\beta\}$ are the duals of each other.

9.3 PRACTICE MANIPULATING TANGENT VECTORS

Let $\mathcal{P}_0$ be the point with coordinates $(x=0, y=1, z=0)$ in a three-dimensional space; and define three curves through $\mathcal{P}_0$ 0 by

$$\begin{aligned}\mathcal{P}(\lambda) &= (\lambda, 1, \lambda),\\ \mathcal{P}(\xi) &= (\sin\xi, \cos\xi, \xi),\\ \mathcal{P}(\rho) &= (\sinh\rho, \cosh\rho, \rho+\rho^3).\end{aligned} \tag{9.3.1}$$

(a) Compute $(d/d\lambda)f$, $(d/d\xi)f$ and $(d/d\rho)f$ for the definition $f = x^2 - y^2 + z^2$ at the point $\mathcal{P}_0$.

(b) Calculate the components of the tangent vectors $(d/d\lambda)$, $(d/d\xi)$ and $(d/d\rho)$ at $\mathcal{P}_0$, using the basis $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$.

Solution:

(a)

$$\begin{aligned}\frac{df}{d\lambda} &= \frac{d}{d\lambda}(2\lambda^2 - 1) = 4\lambda;\\ \frac{df}{d\xi} &= \frac{d}{d\xi}(\cos^2\xi - \sin^2\xi + \xi^2) = -4\sin\xi\cos\xi + 2\xi;\\ \frac{df}{d\rho} &= \frac{d}{d\rho}[-1 + (\rho+\rho^3)^2] = 2(\rho+\rho^3)(1+3\rho^2).\end{aligned} \tag{9.3.2}$$

So we have $(d/d\lambda)f = 0$, $(d/d\xi)f = 0$ and $(d/d\rho)f = 0$ at the point $\mathcal{P}_0$.

(b)

$$\begin{aligned}\frac{d}{d\lambda} &= \frac{\partial}{\partial x} + \frac{\partial}{\partial z};\\ \frac{d}{d\xi} &= \cos\xi\frac{\partial}{\partial x} - \sin\xi\frac{\partial}{\partial y} + \frac{\partial}{\partial z};\\ \frac{d}{d\rho} &= \cosh\rho\frac{\partial}{\partial x} + \sinh\rho\frac{\partial}{\partial y} + (1+3\rho^2)\frac{\partial}{\partial z}.\end{aligned} \tag{9.3.3}$$

The components of the tangent vectors are $(d/d\lambda) = (1,0,1)$, $(d/d\xi) = (1,0,1)$ and $(d/d\rho) = (1,0,1)$ at $\mathcal{P}_0$.

9.4 MORE PRACTICE WITH TANGENT VECTORS

In a three-dimensional space with coordinates (x, y, z), introduce the vector field $\boldsymbol{v} = y^2\partial/\partial x - x\partial/\partial z$, and the functions $f = xy$, $g = z^3$. Compute

$$(a)\boldsymbol{v}[f];\quad (b)\boldsymbol{v}[g];\quad (c)\boldsymbol{v}[fg];\quad (d)f\boldsymbol{v}[g] - g\boldsymbol{v}f;\quad (e)\boldsymbol{v}[f^2+g^2];\quad (f)\boldsymbol{v}\{\boldsymbol{v}[f]\}. \tag{9.4.1}$$

Solution:

$$
\begin{aligned}
\boldsymbol{v}[f] &= y^3; \\
\boldsymbol{v}[g] &= -3xz^2; \\
\boldsymbol{v}[fg] &= y^3z^3 - 3x^2yz^2; \\
f\boldsymbol{v}[g] - g\boldsymbol{v}f &= -3x^2yz^2 - y^3z^3; \\
\boldsymbol{v}[f^2+g^2] &= 2xy^4 - 6xz^5; \\
\boldsymbol{v}\{\boldsymbol{v}[f]\} &= 0.
\end{aligned} \tag{9.4.2}
$$

9.5 PICTURE OF BASIS 1-FORMS INDUCED BY COORDINATES

In the tangent space of Figure 9.1, draw the basis 1-forms $\boldsymbol{d}\psi$ and $\boldsymbol{d}\chi$ induced by the ψ, χ-coordinate system.

Solution:

$\boldsymbol{d}\psi$ and $\boldsymbol{d}\chi$ can be drawn as vector perpendicular to the ψ = constant and χ = constant plane. The drawing is eliminated here.

9.6 PRACTICE WITH DUAL BASES

In a three-dimensional space with spherical coordinates r, θ, ϕ, one often likes to use, Instead of the basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$, the basis

$$
\boldsymbol{e}_{\hat{r}} = \frac{\partial}{\partial r}, \quad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{9.6.1}
$$

(a) What is the 1-form basis $\{\boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\phi}}\}$ dual to this tangent-vector basis?

(b) On the sphere $r = 1$, draw pictures of the bases $\{\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi\}$, $\{\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$, $\{\boldsymbol{d}r, \boldsymbol{d}\theta, \boldsymbol{d}\phi\}$, and $\{\boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\phi}}\}$.

Solution:

(a)

$$
\boldsymbol{\omega}^{\hat{r}} = \boldsymbol{d}r, \quad \boldsymbol{\omega}^{\hat{\theta}} = r\boldsymbol{d}\theta, \quad \boldsymbol{\omega}^{\hat{\phi}} = r\sin\theta\boldsymbol{d}\phi. \tag{9.6.2}
$$

(b) The drawing is eliminated here.

9.7 PRACTICE WITH COMMUTATORS

Compute the commutator $[\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}]$ of the vector fields

$$
\boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{9.7.1}
$$

Express your result as a linear combination of $\boldsymbol{e}_{\hat{\theta}}$ and $\boldsymbol{e}_{\hat{\phi}}$.

Solution:

$$[\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}] = -\frac{\cot\theta}{r}\boldsymbol{e}_{\hat{\phi}}. \tag{9.7.2}$$

9.8 ANGULAR MOMENTUM OPERATORS

In Cartesian coordinates of three-dimensional Euclidean space, one defines three "angular momentum operators" (vector fields) $\boldsymbol{L}_j$ by

$$\boldsymbol{L}_j \equiv \epsilon_{jkl}x^k(\partial/\partial x^l). \tag{9.8.1}$$

Draw a picture of these three vector fields. Calculate their commutators both pictorially and analytically.

Solution:

The drawing is eliminated here.

$$[\boldsymbol{L}_i, \boldsymbol{L}_j] = \epsilon_{imn}\epsilon_{jkl}[x^m\frac{\partial}{\partial x^n}, x^k\frac{\partial}{\partial x^l}] = \epsilon_{imn}\epsilon_{jkl}\left(x^m\delta^k_n\frac{\partial}{\partial x^l} - x^k\delta^m_l\frac{\partial}{\partial x^n}\right) = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} = -\epsilon_{ijk}\boldsymbol{L}_k. \tag{9.8.2}$$

9.9 COMMUTATORS AND COORDINATE-INDUCED BASES

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vector fields in spacetime. Show that in some neighborhood of any given point there exists a coordinate system for which

$$\boldsymbol{u} = \frac{\partial}{\partial x^1}, \quad \boldsymbol{v} = \frac{\partial}{\partial x^2}, \tag{9.9.1}$$

if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent and commute:

$$[\boldsymbol{u}, \boldsymbol{v}] = 0. \tag{9.9.2}$$

Solution:

Then necessity of the proposition is obvious. For sufficiency, we prove the following lemma first.

Lemma

Suppose $\boldsymbol{u}$ is a smooth tangent vector field on a manifold M. If $\boldsymbol{u}_p \neq 0$ at a point $p \in M$, then there exists a local coordinate system $(Z; z^i)$ such that

$$\boldsymbol{u}|_Z = \frac{\partial}{\partial z^1}. \tag{9.9.3}$$

The proof of the lemma is given below.

There exists a local coordinate system $(X; x^i)$, $x^i(p) = 0$, such that the restriction of $\boldsymbol{u}$ on X can be expressed as

$$\boldsymbol{u}|_X = \xi^i \frac{\partial}{\partial x^i}, \tag{9.9.4}$$

where the ξ^i are smooth functions on X. Since $\boldsymbol{u}_p \neq 0$, we may assume that $\xi^1(p) \neq 0$. By the continuity of ξ^1, we may assume that X is a sufficiently small neighborhood of p such that ξ^1 is nonzero in X. Consider the system of ordinary differential equations

$$\frac{dx^j}{dx^1} = \frac{\xi^j(x^1, \cdots, x^m)}{\xi^1(x^1, \cdots, x^m)}, \quad 2 \leq j \leq m, \tag{9.9.5}$$

where x^1 is the independent variable and x^j, $2 \leq j \leq m$, are unknown functions. By the theory of ordinary differential equations, there exists a positive number δ such that $\{(x^1, \cdots, x^m) | |x^i| < \delta\} \in X$, and for any given initial value $(y^2, \cdots, y^m)$, $|y^j| < \delta$, $2 \leq j \leq m$, the system has a unique solution

$$x^j = \phi^j(x^1; y^2, \cdots, y^m), -\delta < x^1 < \delta, \tag{9.9.6}$$

which satisfies the initial conditions

$$\phi^j(0; y^2, \cdots, y^m) = y^j, \tag{9.9.7}$$

where the functions ϕ^j depend on x^1 and the initial values v^j smoothly. Consider the change of coordinates

$$\begin{cases} x^1 = y^1, \\ x^j = \phi^j(x^1; y^2, \cdots, y^m), \quad 2 \leq j \leq m. \end{cases} \tag{9.9.8}$$

Then its Jacobian is

$$\left.\frac{\partial(x^1, \cdots, x^m)}{\partial(y^1, \cdots, y^m)}\right|_{y^1=0} = 1. \tag{9.9.9}$$

Hence there exists a neighborhood $Z \subset X$ of p that has y^i, $1 \leq j \leq m$, as its local coordinate system. Under this system,

$$\boldsymbol{u}|_Z = \xi^i \frac{\partial}{\partial x^i} = \xi^1 \frac{\partial x^i}{\partial y^1} \frac{\partial}{\partial x^i} = \xi^1 \frac{\partial}{\partial y^1}. \tag{9.9.10}$$

Let

$$\begin{cases} z^1 = \int_0^{y^1} \frac{dy^1}{\xi^1}, \\ z^j = y^j, \quad 2 \leq j \leq m. \end{cases} \tag{9.9.11}$$

Then z^i, $1 \leq j \leq m$, is a local coordinate system of Z and

$$\boldsymbol{u}|_Z = \frac{\partial}{\partial z^1}. \tag{9.9.12}$$

This completes the proof.

Now we prove the sufficiency of the proposition.

From the lemma, there exists a local coordinate system $(Z; z^i)$ such that

$$\boldsymbol{u}|_Z = \frac{\partial}{\partial z^1} \tag{9.9.13}$$

and

$$\boldsymbol{v}|_Z = \xi^i \frac{\partial}{\partial z^i}, \tag{9.9.14}$$

where the ξ^i are smooth functions on Z. Then we have

$$[\boldsymbol{u}, \boldsymbol{v}]|_Z = \frac{\partial \xi^i}{\partial z^1} \frac{\partial}{\partial z^i} = 0, \tag{9.9.15}$$

i.e. $(\partial\xi^i/\partial z^1) = 0$, $1 \le i \le m$. This implies that ξ^i, $1 \le i \le m$ is a function of $z^2, \cdots, z^m$ only. By Lemma 1, there is a change of local coordinates from $(z^2, \cdots, z^m)$ to $(y^2, \cdots, y^m)$ such that

$$\sum_{j=2}^{m} \xi^j \frac{\partial}{\partial z^j} = \frac{\partial}{\partial y^2}. \tag{9.9.16}$$

Now let $y^1 = z^1$, $(y^1, \cdots, y^m)$ then becomes a local coordinate system at p and under this system,

$$\boldsymbol{u}|_Y = \frac{\partial}{\partial y^1}, \quad \boldsymbol{v}|_Y = f(y^2, \cdots, y^m)\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}. \tag{9.9.17}$$

Now let $x^1 = y^1 - \int_0^{y^2} f(t, \cdots, y^m)dt$, $x^j = y^j$, $2 \le j \le m$. Then x^i, $1 \le i \le m$, is a local coordinate system of X and

$$\boldsymbol{u}|_X = \frac{\partial}{\partial x^1}, \quad \boldsymbol{v}|_X = \frac{\partial}{\partial x^2}. \tag{9.9.18}$$

9.10 COMPONENTS OF COMMUTATOR IN NON-COORDINATE BASIS

Derive equation (9.23).

Solution:

$$\begin{aligned}
[\boldsymbol{u}, \boldsymbol{v}]f &= [u^\alpha \boldsymbol{e}_\alpha, v^\beta \boldsymbol{e}_\beta]f \\
&= u^\alpha \boldsymbol{e}_\alpha(v^\beta \boldsymbol{e}_\beta(f)) - v^\beta \boldsymbol{e}_\beta(u^\alpha \boldsymbol{e}_\alpha(f)) \\
&= u^\alpha \boldsymbol{e}_\alpha(v^\beta)\boldsymbol{e}_\beta(f) - v^\beta \boldsymbol{e}_\beta(u^\alpha)\boldsymbol{e}_\alpha(f) + u^\alpha v^\beta [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta](f) \\
&= \boldsymbol{u}(v^\beta)\boldsymbol{e}_\beta(f) - \boldsymbol{v}(u^\beta)\boldsymbol{e}_\beta(f) + u^\mu v^\nu c_{\mu\nu}{}^\beta \boldsymbol{e}_\beta(f).
\end{aligned} \tag{9.10.1}$$

So we have

$$[\boldsymbol{u}, \boldsymbol{v}] = [\boldsymbol{u}(v^\beta) - \boldsymbol{v}(u^\beta) + u^\mu v^\nu c_{\mu\nu}{}^\beta]\boldsymbol{e}_\beta. \tag{9.10.2}$$

9.11 LIE DERIVATIVE

The "Lie derivative" of a vector field $\boldsymbol{v}(\mathcal{P})$ along a vector field $\boldsymbol{u}(\mathcal{P})$ is defined by

$$\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v} = [\boldsymbol{u}, \boldsymbol{v}]. \tag{9.11.1}$$

Draw a space-filling family of curves (a "congruence") on a sheet of paper. Draw an arbitrary vector $\boldsymbol{v}$ at an arbitrary point $\mathcal{P}_0$ on the sheet. Transport that vector along the curve through $\mathcal{P}_0$ by means of the "Lie transport" law $\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v} = 0$, where $\boldsymbol{u} = d/dt$ is the tangent to the curve. Draw the resulting vector $\boldsymbol{v}$ at various points $\mathcal{P}(t)$ along the curve.

The drawing is eliminated here.

9.12 A CHIP OFF THE OLD BLOCK

(a) Prove the Jacobi identity

$$[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]]+[\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]]+[\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]]=0 \tag{9.12.1}$$

by picking out all terms of the form $\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}\partial_{\boldsymbol{w}}$, showing that they add to zero, and arguing from symmetry that all other terms, e.g., $\partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}$ terms, must similarly cancel.

(b) State this identity in index form.

(c) Draw a picture corresponding to this identity (see Box 9.2).

Solution:

(a)

$$[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]]=\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}\partial_{\boldsymbol{w}}-\partial_{\boldsymbol{u}}\partial_{\boldsymbol{w}}\partial_{\boldsymbol{v}}-\partial_{\boldsymbol{v}}\partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}+\partial_{\boldsymbol{w}}\partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}; \tag{9.12.2}$$

$$[\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]]=\partial_{\boldsymbol{v}}\partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}-\partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}\partial_{\boldsymbol{w}}-\partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}+\partial_{\boldsymbol{u}}\partial_{\boldsymbol{w}}\partial_{\boldsymbol{v}}; \tag{9.12.3}$$

$$[\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]]=\partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}-\partial_{\boldsymbol{w}}\partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}-\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}\partial_{\boldsymbol{w}}+\partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}\partial_{\boldsymbol{w}}. \tag{9.12.4}$$

Combine all together and we can find out

$$[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]]+[\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]]+[\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]]=0. \tag{9.12.5}$$

(b) The component of $[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]]$ is

$$\begin{aligned}&u^{\gamma}(v^{\beta}w^{\alpha}{}_{,\beta}-w^{\beta}v^{\alpha}{}_{,\beta})_{,\gamma}-(v^{\gamma}w^{\beta}{}_{,\gamma}-w^{\gamma}v^{\beta}{}_{,\gamma})u^{\alpha}{}_{,\beta}\\&=w^{\alpha}{}_{,\beta}v^{\beta}{}_{,\gamma}u^{\gamma}-v^{\alpha}{}_{,\beta}w^{\beta}{}_{,\gamma}u^{\gamma}-u^{\alpha}{}_{,\beta}w^{\beta}{}_{,\gamma}v^{\gamma}+u^{\alpha}{}_{,\beta}v^{\beta}{}_{,\gamma}w^{\gamma}+w^{\alpha}{}_{,\beta\gamma}v^{\beta}u^{\gamma}-v^{\alpha}{}_{,\beta\gamma}w^{\beta}u^{\gamma}.\end{aligned} \tag{9.12.6}$$

By similar method, we write down the components for other two terms and combine them together, we would find out

$$[\boldsymbol{u},[\boldsymbol{v},\boldsymbol{w}]]+[\boldsymbol{v},[\boldsymbol{w},\boldsymbol{u}]]+[\boldsymbol{w},[\boldsymbol{u},\boldsymbol{v}]]=0. \tag{9.12.7}$$

(c) The drawing is eliminated here.

9.13 ROTATION GROUP: GENERATORS

Let $\mathcal{H}_l$ be three 3×3 matrices whose components are $(\mathcal{H}_l)_{mn}=\epsilon_{lmn}$.

(a) Display the matrices $\mathcal{H}_1$, $(\mathcal{H}_1)^2$, $(\mathcal{H}_1)^3$, and $(\mathcal{H}_1)^4$.

(b) Sum the series

$$\mathcal{R}_x(\theta)\equiv\exp(\mathcal{H}_1\theta)=\sum_{n=0}^{\infty}\frac{\theta^n}{n!}(\mathcal{H}_1)^n. \tag{9.13.1}$$

Show that $\mathcal{R}_x(\theta)$ is a rotation matrix and that it produces a rotation through an angle θ about the x-axis.

(c) Show similarly that $\mathcal{R}_z(\phi)=\equiv\exp(\mathcal{H}_3\phi)$ and $\mathcal{R}_y(\chi)=\equiv\exp(\mathcal{H}_2\chi)$ are rotation matrices, and that they produce rotations through angles ϕ and χ about the z- and y-axes, respecti vely.

(d) Explain why $\mathcal{P}=\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)$ defines the Euler-angle coordinates, ψ, θ, ϕ for the generic element $\mathcal{P}\in SO(3)$ of the rotation group.

(e) Let $\mathcal{L}$ be the curve $\mathcal{P} = \mathcal{R}_z(t)$ through the identity matrix, $\mathcal{L}(0) = I \in SO(3)$. Show that its tangent, $(d\mathcal{L}/dt)(0) \equiv \dot{\mathcal{L}}(0)$ does not vanish by computing $\dot{\mathcal{L}}(0)f_{12}$ where f_{12} is the function $f_{12}(\mathcal{P}) = P_{12}$ whose value is the 12 matrix element of $\mathcal{P}$.

(f) Define a vector field $\boldsymbol{e}_3$ on $SO(3)$ by letting $\boldsymbol{e}_3(\mathcal{P})$ be the tangent (at $t = 0$) to the curve $\mathcal{L}(t) = \mathcal{R}_z(t)\mathcal{P}$ through $\mathcal{P}$. Show that $\boldsymbol{e}_3(\mathcal{P})$ is nowhere zero. Note: $\boldsymbol{e}_3(\mathcal{P})$ is called the "generator of rotations about the z-axis", because it points from $\mathcal{P}$ toward neighboring rotations,which differ from $\mathcal{R}_z(t)\mathcal{P}$ by a rotation about the z-axis.

(g) Show that $\boldsymbol{e}_3 = (\partial/\partial\psi)_{\theta\phi}$.

(h) Derive the following formulas, valid for $t \ll 1$:

$$\begin{aligned}
&\mathcal{R}_x(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi - t\sin\psi\cot\theta)\mathcal{R}_x(\theta + t\cos\psi)\mathcal{R}_z(\phi + t\sin\psi/\sin\theta);\\
&\mathcal{R}_y(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi - t\cos\psi\cot\theta)\mathcal{R}_x(\theta - t\sin\psi)\mathcal{R}_z(\phi + t\cos\psi/\sin\theta).
\end{aligned} \tag{9.13.2}$$

(i) Define $\boldsymbol{e}_1(\mathcal{P})$ and $\boldsymbol{e}_2(\mathcal{P})$ to be the tangent vectors (at $t = 0$) to the curves $\mathcal{L}(t) = \mathcal{R}_x(t)\mathcal{P}$ and $\mathcal{L}(t) = \mathcal{R}_y(t)\mathcal{P}$, respectively. Show that

$$\begin{aligned}
\boldsymbol{e}_1 &= \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right);\\
\boldsymbol{e}_2 &= -\sin\psi\frac{\partial}{\partial\theta} - \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right).
\end{aligned} \tag{9.13.3}$$

$\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the "generators of rotations about the x- and y-axes."

Solution:

The equation 9.13.2 and 9.13.3 might not be correct in the textbook. Please check it.

(a)

$$\mathcal{H}_1 = \begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}; (\mathcal{H}_1)^2 = \begin{pmatrix}0&0&0\\0&-1&0\\0&0&-1\end{pmatrix}; (\mathcal{H}_1)^3 = \begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}; (\mathcal{H}_1)^4 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{9.13.4}$$

(b)

$$\begin{aligned}
\mathcal{R}_x(\theta) &= I + \sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k+1}}{(2k+1)!}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} + \sum_{k=1}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&\sin\theta\\0&-\sin\theta&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&\cos\theta-1&0\\0&0&\cos\theta-1\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix}.
\end{aligned} \tag{9.13.5}$$

Now it is easy to see that $\mathcal{R}_x(\theta)$ is a rotation matrix and that it produces a rotation through an angle θ about the x-axis.

(c) By similar procedure in (b), we can figure out that

$$\mathcal{R}_y(\chi) = \begin{pmatrix}\cos\chi&0&-\sin\chi\\0&1&0\\\sin\chi&0&\cos\chi\end{pmatrix}; \mathcal{R}_z(\phi) = \begin{pmatrix}\cos\phi&\sin\phi&0\\-\sin\phi&\cos\phi&0\\0&0&1\end{pmatrix}. \tag{9.13.6}$$

(d) The first step of Euler rotation is to rotate the x and y axis around z axis by ϕ to x' and y' axis. Then we rotate the y' and z axis around x' axis by θ to y'' and z' axis. Finally, we rotate the x' and y'' axis around z' axis by ψ to x'' and y''' axis. So mathematically, we have

$$\mathcal{P} = \mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi). \tag{9.13.7}$$

(e)

$$\dot{\mathcal{L}}(0)f_{12} = \frac{d}{dt}\{f_{12}[\mathcal{R}_z(t)]\}_{t=0} = \frac{d}{dt}(\sin t)_{t=0} = 1. \tag{9.13.8}$$

(f) (g)

$$\mathcal{L}(t) = \mathcal{R}_z(t)\mathcal{P} = \mathcal{R}_z(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(t+\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) \tag{9.13.9}$$

So we have

$$\boldsymbol{e}_3 = (\partial/\partial\psi)_{\theta\phi} \neq 0. \tag{9.13.10}$$

(h) From the definition of $\mathcal{H}_l$, we have the commutation relation

$$[\mathcal{H}_i, \mathcal{H}_j] = -\epsilon_{ijk}\mathcal{H}_k. \tag{9.13.11}$$

The Baker-Hausdorff lemma states that

$$\exp(G\lambda)A\exp(-G\lambda) = A + \lambda[G,A] + \cdots + \frac{\lambda^n}{n!}[G,[G,[G,\cdots[G,A]]]\cdots] + \cdots \tag{9.13.12}$$

We can figure out that

$$\mathcal{R}_z(\phi)\mathcal{H}_1\mathcal{R}_z(-\phi) = \cos\phi\mathcal{H}_1 - \sin\phi\mathcal{H}_2; \quad \mathcal{R}_z(\phi)\mathcal{H}_2\mathcal{R}_z(-\phi) = \cos\phi\mathcal{H}_2 + \sin\phi\mathcal{H}_1; \cdots \tag{9.13.13}$$

Keep the first order of t, we have

$$\begin{aligned}
&\mathcal{R}_z(\psi - t\sin\psi\cot\theta)\mathcal{R}_x(\theta + t\cos\psi)\mathcal{R}_z(\phi + t\sin\psi/\sin\theta)\\
&= (I - t\sin\psi\cot\theta\mathcal{H}_3)\mathcal{R}_z(\psi)(I + t\cos\psi\mathcal{H}_1)\mathcal{R}_x(\theta)(I + t\sin\psi/\sin\theta\mathcal{H}_3)\mathcal{R}_z(\phi)\\
&= \mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) - t\sin\psi\cot\theta\mathcal{H}_3\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&+ t\cos\psi\mathcal{R}_z(\psi)\mathcal{H}_1\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) + t\sin\psi/\sin\theta\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{H}_3\mathcal{R}_z(\phi)
\end{aligned} \tag{9.13.14}$$

Note that

$$\mathcal{R}_z(\psi)\mathcal{H}_1\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = (\cos\psi\mathcal{H}_1 - \sin\psi\mathcal{H}_2)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi). \tag{9.13.15}$$

$$\begin{aligned}
\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{H}_3\mathcal{R}_z(\phi) &= \mathcal{R}_z(\psi)(\cos\theta\mathcal{H}_3 + \sin\theta\mathcal{H}_2)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&= [\cos\theta\mathcal{H}_3 + \sin\theta(\cos\psi\mathcal{H}_2 + \sin\psi\mathcal{H}_1)]\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi).
\end{aligned} \tag{9.13.16}$$

Now we have

$$\begin{aligned}
&\mathcal{R}_z(\psi - t\sin\psi\cot\theta)\mathcal{R}_x(\theta + t\cos\psi)\mathcal{R}_z(\phi + t\sin\psi/\sin\theta)\\
&= (I - t\sin\psi\cot\theta\mathcal{H}_3 + t\cos^2\psi\mathcal{H}_1 - t\cos\psi\sin\psi\mathcal{H}_2 + t\frac{\sin\psi}{\sin\theta}\cos\theta\mathcal{H}_3\\
&+ t\sin\psi\cos\psi\mathcal{H}_2 + t\sin^2\psi\mathcal{H}_1)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&= (I + t\mathcal{H}_1)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&= \mathcal{R}_x(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi).
\end{aligned} \tag{9.13.17}$$

By similar procedure, we would arrive that

$$\begin{aligned}
&\mathcal{R}_z(\psi - t\cos\psi\cot\theta)\mathcal{R}_x(\theta - t\sin\psi)\mathcal{R}_z(\phi + t\cos\psi/\sin\theta)\\
&= (I - t\cos\psi\cot\theta\mathcal{H}_3 - t\sin\psi\cos\psi\mathcal{H}_1 + t\sin^2\psi\mathcal{H}_2 + t\frac{\cos\psi}{\sin\theta}\cos\theta\mathcal{H}_3\\
&+ t\cos^2\psi\mathcal{H}_2 + t\sin\psi\cos\psi\mathcal{H}_1)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&= (I + t\mathcal{H}_2)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)\\
&= \mathcal{R}_y(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi).
\end{aligned} \tag{9.13.18}$$

(i)

$$\mathcal{R}_x(t)\mathcal{P} = \mathcal{R}_x(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi - t\sin\psi\cot\theta)\mathcal{R}_x(\theta + t\cos\psi)\mathcal{R}_z(\phi + t\sin\psi/\sin\theta). \quad (9.13.19)$$

So we have

$$\boldsymbol{e}_1 = \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \quad (9.13.20)$$

$$\mathcal{R}_y(t)\mathcal{P} = \mathcal{R}_y(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi - t\cos\psi\cot\theta)\mathcal{R}_x(\theta - t\sin\psi)\mathcal{R}_z(\phi + t\cos\psi/\sin\theta). \quad (9.13.21)$$

So

$$\boldsymbol{e}_2 = -\sin\psi\frac{\partial}{\partial\theta} - \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \quad (9.13.22)$$

9.14 ROTATION GROUP: STRUCTURE CONSTANTS

Use the three vector fields constructed in the last exercise,

$$\begin{aligned}
\boldsymbol{e}_1 &= \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right);\\
\boldsymbol{e}_2 &= -\sin\psi\frac{\partial}{\partial\theta} - \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right);\\
\boldsymbol{e}_3 &= \frac{\partial}{\partial\psi},
\end{aligned} \quad (9.14.1)$$

as basis vectors for the manifold of the rotation group. The above equations express this "basis of generators" in terms of the Euler-angle basis. Show that the commutation coefficients for this basis are

$$c^{\alpha\beta}{}_{\gamma} = \epsilon_{\alpha\beta\gamma}, \quad (9.14.2)$$

independently of location $\mathcal{P}$ in the rotation group. These coefficients are also called the structure constants of the rotation group.

Solution:

$$\begin{aligned}
[\boldsymbol{e}_1, \boldsymbol{e}_2] = {} & [-\sin\psi\cot\theta(-\sin\psi)_{,\psi} - (-\cos\psi\cot\theta)(\cos\psi)_{,\psi}]\frac{\partial}{\partial\theta}\\
& + [-\sin\psi\cot\theta(-\cos\psi\cot\theta)_{,\psi} + \cos\psi(-\cos\psi\,c\\
& \quad -(-\cos\psi\cot\theta)(-\sin\psi\cot\theta)_{,\psi} - (-\sin\psi)(-\sin\psi\cot\theta)\\
& + [-\sin\psi\cot\theta(\cos\psi\csc\theta)_{,\psi} + \cos\psi(\cos\psi\,c\\
& \quad -(-\cos\psi\cot\theta)(\sin\psi\csc\theta)_{,\psi} - (-\sin\psi)(\sin\psi\csc\theta)\\
= {} & \frac{\partial}{\partial\psi}.
\end{aligned} \quad (9.14.3)$$

$$\begin{aligned}
[\boldsymbol{e}_2, \boldsymbol{e}_3] &= [-(-\sin\psi)_{,\psi}]\frac{\partial}{\partial\theta} + [-(-\cos\psi\cot\theta)_{,\psi}]\frac{\partial}{\partial\psi} + [-(\cos\psi\csc\theta)_{,\psi}]\frac{\partial}{\partial\phi}\\
&= \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right).
\end{aligned} \quad (9.14.4)$$

$$[\boldsymbol{e}_3, \boldsymbol{e}_1] = [(\cos\psi)_{,\psi}]\frac{\partial}{\partial\theta} + [(-\sin\psi\cot\theta)_{,\psi}]\frac{\partial}{\partial\psi} + [(\sin\psi\csc\theta)_{,\psi}]\frac{\partial}{\partial\phi}$$
$$= -\sin\psi\frac{\partial}{\partial\theta} - \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{9.14.5}$$

So the commutation coefficients for this basis are

$$c^{\alpha\beta}{}_{\gamma} = \epsilon_{\alpha\beta\gamma}, \tag{9.14.6}$$

independently of location $\mathcal{P}$ in the rotation group.

Chapter 10

AFFINE GEOMETRY: GEODESICS, PARALLEL TRANSPORT, AND COVARIANT DERIVATIVE

10.1 ADDITIVITY OF COVARIANT DIFFERENTIATION

Show that the commutator ("closer of quadrilaterals") is additive:

$$[\boldsymbol{u}, \boldsymbol{v}+\boldsymbol{w}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, \boldsymbol{w}]; \quad [\boldsymbol{u}+\boldsymbol{n}, \boldsymbol{v}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{n}, \boldsymbol{v}]. \tag{10.1.1}$$

Use this result, the additivity condition $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}$, and symmetry of the covariant derivative, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$, to prove that

$$\boldsymbol{\nabla}_{\boldsymbol{u}+\boldsymbol{n}}\boldsymbol{v} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{v}. \tag{10.1.2}$$

Solution:

$$[\boldsymbol{u}, \boldsymbol{v}+\boldsymbol{w}]f = \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}+\boldsymbol{w}}f - \partial_{\boldsymbol{v}+\boldsymbol{w}}\partial_{\boldsymbol{u}}f = \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}f - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}f + \partial_{\boldsymbol{u}}\partial_{\boldsymbol{w}}f - \partial_{\boldsymbol{w}}\partial_{\boldsymbol{u}}f = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, \boldsymbol{w}]f. \tag{10.1.3}$$

$$[\boldsymbol{u}+\boldsymbol{n}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{n}] = -[\boldsymbol{v}, \boldsymbol{u}] - [\boldsymbol{v}, \boldsymbol{n}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{n}, \boldsymbol{v}]. \tag{10.1.4}$$

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}+\boldsymbol{n}}\boldsymbol{v} &= \boldsymbol{\nabla}_{\boldsymbol{v}}(\boldsymbol{u}+\boldsymbol{n}) + [\boldsymbol{u}+\boldsymbol{n}, \boldsymbol{v}] = \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{n} + [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{n}, \boldsymbol{v}] \\ &= \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{n} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{n} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{v}. \end{aligned} \tag{10.1.5}$$

10.2 CHAIN RULE FOR COVARIANT DIFFERENTIATION

Use pictures, and the "take-the-difference-and-take-the-limit" definition of $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ to show that

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(f\boldsymbol{v}) = f\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{v}\partial_{\boldsymbol{u}}[f]. \tag{10.2.1}$$

Solution:

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(f\boldsymbol{v}) = \lim_{\epsilon\to 0}\frac{(f+\Delta f)(\boldsymbol{v}+\Delta\boldsymbol{v})-f\boldsymbol{v}}{\epsilon} = f\lim_{\epsilon\to 0}\frac{(\boldsymbol{v}+\Delta\boldsymbol{v})-\boldsymbol{v}}{\epsilon} + \lim_{\epsilon\to 0}\frac{(f+\Delta f)-f}{\epsilon}\boldsymbol{v} = f\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{v}\partial_{\boldsymbol{u}}[f]. \tag{10.2.2}$$

10.3 ANOTHER CHAIN RULE

Derive equation (10.7), using the "take-the-difference-and-take-the-limit" definitions of derivatives. Hint: Before taking the differences, parallel transport $\boldsymbol{\sigma}[\mathcal{P}(\lambda)]$ and $\boldsymbol{v}[\mathcal{P}(\lambda)]$ back from $\mathcal{P}(\lambda)$ to $\mathcal{P}(0)$.

Solution:

$$\begin{aligned}\frac{d\langle\boldsymbol{\sigma},\boldsymbol{v}\rangle}{d\lambda} &= \frac{\langle\boldsymbol{\sigma}[\mathcal{P}(\lambda)],\boldsymbol{v}[\mathcal{P}(\lambda)]\rangle - \langle\boldsymbol{\sigma}[\mathcal{P}(0)],\boldsymbol{v}[\mathcal{P}(0)]\rangle}{d\lambda}\\ &= \lim_{\lambda\to 0}\frac{\langle\boldsymbol{\sigma}+\lambda\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{v}+\lambda\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma}\rangle - \langle\boldsymbol{\sigma},\boldsymbol{v}\rangle}{\lambda}\\ &= \langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{v}\rangle + \langle\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle.\end{aligned} \tag{10.3.1}$$

10.4 STILL ANOTHER CHAIN RULE

Show that, as in flat spacetime, so also in curved spacetime,

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v}\otimes\boldsymbol{w}) = (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v})\otimes\boldsymbol{w} + \boldsymbol{v}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}). \tag{10.4.1}$$

Write down the more familiar component version of this equation in flat spacetime.

Solution:

Choose 1-forms $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ at the event $\mathcal{P}_0$ in question, and extend them along the vector $\boldsymbol{u} = d/d\lambda$ by parallel transport, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\rho} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma} = 0$. Then

$$\begin{aligned}[\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v}\otimes\boldsymbol{w})](\boldsymbol{\rho},\boldsymbol{\sigma}) &= \frac{d}{d\lambda}[(\boldsymbol{v}\otimes\boldsymbol{w})(\boldsymbol{\rho},\boldsymbol{\sigma})]\\ &= \frac{d}{d\lambda}[\langle\boldsymbol{\rho},\boldsymbol{v}\rangle\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle]\\ &= \frac{d\langle\boldsymbol{\rho},\boldsymbol{v}\rangle}{d\lambda}\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle + \langle\boldsymbol{\rho},\boldsymbol{v}\rangle\frac{d\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle}{d\lambda}\\ &= \langle\boldsymbol{\rho},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle + \langle\boldsymbol{\rho},\boldsymbol{v}\rangle\langle\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle\\ &= [(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v})\otimes\boldsymbol{w}](\boldsymbol{\rho},\boldsymbol{\sigma}) + [\boldsymbol{v}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w})](\boldsymbol{\rho},\boldsymbol{\sigma}).\end{aligned} \tag{10.4.2}$$

The more familiar component version of this equation in flat spacetime is

$$(v^\alpha w^\beta)_{,\gamma}u^\gamma = v^\alpha{}_{,\gamma}u^\gamma w^\beta + v^\alpha w^\beta{}_{,\gamma}u^\gamma. \tag{10.4.3}$$

10.5 ONE MORE CHAIN RULE

Show, using techniques similar to those in exercise 10.4, that

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{\sigma}\otimes\boldsymbol{\rho}\otimes\boldsymbol{v}) = (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma})\otimes\boldsymbol{\rho}\otimes\boldsymbol{v} + \boldsymbol{\sigma}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\rho})\otimes\boldsymbol{v} + \boldsymbol{\sigma}\otimes\boldsymbol{\rho}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}). \tag{10.5.1}$$

Solution:

Choose 1-forms $\boldsymbol{\xi}$ and vector $\boldsymbol{w}$ and $\boldsymbol{x}$ at the event $\mathcal{P}_0$ in question, and extend them along the vector $\boldsymbol{u} = d/d\lambda$ by parallel transport, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\xi} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{x} = 0$. Then

$$\begin{aligned}
&[\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{\sigma}\otimes\boldsymbol{\rho}\otimes\boldsymbol{v})](\boldsymbol{w},\boldsymbol{x},\boldsymbol{\xi})\\
&= \frac{d}{d\lambda}[(\boldsymbol{\sigma}\otimes\boldsymbol{\rho}\otimes\boldsymbol{v})(\boldsymbol{w},\boldsymbol{x},\boldsymbol{\xi})]\\
&= \frac{d}{d\lambda}[\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle\langle\boldsymbol{\rho},\boldsymbol{x}\rangle\langle\boldsymbol{\xi},\boldsymbol{v}\rangle]\\
&= \frac{d\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle}{d\lambda}\langle\boldsymbol{\rho},\boldsymbol{x}\rangle\langle\boldsymbol{\xi},\boldsymbol{v}\rangle + \langle\boldsymbol{\sigma},\boldsymbol{w}\rangle\frac{d\langle\boldsymbol{\rho},\boldsymbol{x}\rangle}{d\lambda}\langle\boldsymbol{\xi},\boldsymbol{v}\rangle + \langle\boldsymbol{\sigma},\boldsymbol{w}\rangle\langle\boldsymbol{\rho},\boldsymbol{x}\rangle\frac{d\langle\boldsymbol{\xi},\boldsymbol{v}\rangle}{d\lambda}\\
&= \langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{w}\rangle\langle\boldsymbol{\rho},\boldsymbol{x}\rangle\langle\boldsymbol{\xi},\boldsymbol{v}\rangle + \langle\boldsymbol{\sigma},\boldsymbol{w}\rangle\langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\rho},\boldsymbol{x}\rangle\langle\boldsymbol{\xi},\boldsymbol{v}\rangle + \langle\boldsymbol{\sigma},\boldsymbol{w}\rangle\langle\boldsymbol{\rho},\boldsymbol{x}\rangle\langle\boldsymbol{\xi},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle\\
&= [(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma})\otimes\boldsymbol{\rho}\otimes\boldsymbol{v}](\boldsymbol{w},\boldsymbol{x},\boldsymbol{\xi}) + [\boldsymbol{\sigma}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\rho})\otimes\boldsymbol{v}](\boldsymbol{w},\boldsymbol{x},\boldsymbol{\xi}) + [\boldsymbol{\sigma}\otimes\boldsymbol{\rho}\otimes(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v})](\boldsymbol{w},\boldsymbol{x},\boldsymbol{\xi}).
\end{aligned} \tag{10.5.2}$$

10.6 GEODESIC EQUATION

Use the "Schild's ladder" construction process for parallel transport (beginning of Box 10.2) to show that a geodesic parallel transports its own tangent vector along itself (end of Box 10.2).

Solution:

In the "Schild's ladder", if $\mathscr{A}\mathscr{X}$ is the tangent vector of the curve $\mathscr{A}\mathscr{M}$, then $\mathscr{M}$ and $\mathscr{X}$ will be the same point. So the middle point $\mathscr{N}$ of $\mathscr{X}\mathscr{M}$ is the same as $\mathscr{M}$. Now Take geodesic that starts at $\mathscr{A}$ and passes through $\mathscr{N}$, and extend it by an equal parameter increment to point $\mathscr{P}$. Because the curve $\mathscr{A}\mathscr{M}$ is a geodesic, $\mathscr{A}\mathscr{P}$ will overlap the original curve. So $\mathscr{N}\mathscr{P}$ is the tangent vector of the curve at $\mathscr{N}$, i.e. a geodesic parallel transports its own tangent vector along itself.

10.7 COMPUTATION OF CONNECTION COEFFICIENTS

Equation (10.14) for ${\Gamma^{\mu}}_{\alpha\beta}$ from equation (10.13).

Solution:

$$\langle\boldsymbol{\omega}^{\mu}, \boldsymbol{\nabla}_{\beta}\boldsymbol{e}_{\alpha}\rangle = \langle\boldsymbol{\omega}^{\mu}, \boldsymbol{e}_{\nu}{\Gamma^{\nu}}_{\alpha\beta}\rangle = {\Gamma^{\nu}}_{\alpha\beta}{\delta^{\mu}}_{\nu} = {\Gamma^{\mu}}_{\alpha\beta}. \tag{10.7.1}$$

10.8 CONNECTION FOR 1-FORM BASIS

Derive equations (10.15) and (10.16), which relate $\boldsymbol{\nabla}_\beta \boldsymbol{\omega}^\nu$ to $\Gamma^\nu{}_{\alpha\beta}$ from equation (10.14).

Solution:

$$0 = \frac{d}{dx^\beta}\langle \boldsymbol{\omega}^\nu, \boldsymbol{e}_\alpha\rangle = \langle \boldsymbol{\nabla}_\beta \boldsymbol{\omega}^\nu, \boldsymbol{e}_\alpha\rangle + \langle \boldsymbol{\omega}^\nu, \boldsymbol{\nabla}_\beta \boldsymbol{e}_\alpha\rangle = \langle \boldsymbol{\nabla}_\beta \boldsymbol{\omega}^\nu, \boldsymbol{e}_\alpha\rangle + \Gamma^\nu{}_{\alpha\beta}. \tag{10.8.1}$$

So

$$\langle \boldsymbol{\nabla}_\beta \boldsymbol{\omega}^\nu, \boldsymbol{e}_\alpha\rangle = -\Gamma^\nu{}_{\alpha\beta} \tag{10.8.2}$$

and

$$\boldsymbol{\nabla}_\beta \boldsymbol{\omega}^\nu = -\Gamma^\nu{}_{\alpha\beta}\boldsymbol{\omega}^\alpha. \tag{10.8.3}$$

10.9 SYMMETRY OF CONNECTION COEFFICIENTS

Show that the symmetry of spacetime's covariant derivative (equation 10.2a) is equivalent to the following symmetry condition on the connection coefficients:

$$\Gamma^\mu{}_{[\alpha\beta]} = -\frac{1}{2}\langle \boldsymbol{\omega}^\mu, [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta]\rangle \equiv -\frac{1}{2}c_{\alpha\beta}{}^\mu \tag{10.9.1}$$

As a special case, $\Gamma^\mu{}_{[\alpha\beta]}$ is symmetric in α and β when a coordinate basis is used. Show that in a coordinate basis this symmetry reduces the number of independent connection coefficients at each event from 64 to 40.

Solution:

$$\Gamma^\mu{}_{[\alpha\beta]} = \frac{1}{2}(\langle \boldsymbol{\omega}^\mu, \boldsymbol{\nabla}_\beta \boldsymbol{e}_\alpha\rangle - \langle \boldsymbol{\omega}^\mu, \boldsymbol{\nabla}_\alpha \boldsymbol{e}_\beta\rangle) = \frac{1}{2}\langle \boldsymbol{\omega}^\mu, \boldsymbol{\nabla}_\beta \boldsymbol{e}_\alpha - \boldsymbol{\nabla}_\alpha \boldsymbol{e}_\beta\rangle = -\frac{1}{2}\langle \boldsymbol{\omega}^\mu, [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta]\rangle. \tag{10.9.2}$$

If $\boldsymbol{e}_\alpha$s are coordinate basises, we have $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$. So the number of independent connection coefficients at each event is $C_4^1(C_4^2 + C_4^1) = 40$.

10.10 COMPONENTS OF GRADIENT

Derive equation (10.18) for the components of the gradient, $S^\alpha{}_{\beta\gamma;\delta}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}_\delta \boldsymbol{S} &= \boldsymbol{\nabla}_\delta(S^\alpha{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma) \\
&= S^\alpha{}_{\beta\gamma,\delta}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma + S^\alpha{}_{\beta\gamma}\boldsymbol{\nabla}_\delta \boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma + S^\alpha{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \boldsymbol{\nabla}_\delta \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma + S^\alpha{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\nabla}_\delta \boldsymbol{\omega}^\gamma \\
&= S^\alpha{}_{\beta\gamma,\delta}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma + S^\alpha{}_{\beta\gamma}\Gamma^\mu{}_{\alpha\delta}\boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma - S^\alpha{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \Gamma^\beta{}_{\mu\delta}\boldsymbol{\omega}^\mu \otimes \boldsymbol{\omega}^\gamma - S^\alpha{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \Gamma^\gamma{}_{\mu\delta}\boldsymbol{\omega}^\mu \\
&= (S^\alpha{}_{\beta\gamma,\delta} + S^\mu{}_{\beta\gamma}\Gamma^\alpha{}_{\mu\delta} - S^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - S^\alpha{}_{\beta\mu}\Gamma^\mu{}_{\gamma\delta})\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma.
\end{aligned} \tag{10.10.1}$$

So, we have

$$S^{\alpha}{}_{\beta\gamma;\delta} = S^{\alpha}{}_{\beta\gamma,\delta} + S^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta} - S^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\beta\mu}\Gamma^{\mu}{}_{\gamma\delta}. \tag{10.10.2}$$

10.11 DIVERGENCE

Let $\boldsymbol{T}$ be a $(0,2)$ tensor field, and define the divergence on its second slot by the same process as in flat spacetime: $\boldsymbol{\nabla}\cdot\boldsymbol{T}$ contraction of $\boldsymbol{\nabla T}$; i.e.,

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})^{\alpha} = T^{\alpha\beta}{}_{;\beta}\,. \tag{10.11.1}$$

Write the components $T^{\alpha\beta}{}_{;\beta}$ In terms of $T^{\alpha\beta}{}_{,\beta}$ plus correction terms for each of the two indices of $\boldsymbol{T}$.

Solution:

$$T^{\alpha\beta}{}_{;\gamma} = T^{\alpha\beta}{}_{,\gamma} + T^{\mu\beta}\Gamma^{\alpha}{}_{\mu\gamma} + T^{\alpha\mu}\Gamma^{\beta}{}_{\mu\gamma}. \tag{10.11.2}$$

After contraction, we have

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + T^{\mu\beta}\Gamma^{\alpha}{}_{\mu\beta} + T^{\alpha\mu}\Gamma^{\beta}{}_{\mu\beta}. \tag{10.11.3}$$

10.12 VERIFICATION OF CHAIN RULE

Let $S^{\alpha\beta}{}_{\gamma}$ be components of a $(2,1)$ tensor field, and $M_{\beta}{}^{\gamma}$ be components of a $(1,1)$ tensor field. By contracting these tensor fields, one obtains a vector field $S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}$. The chain rule for the divergence of this vector field reads

$$(S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{;\alpha} = S^{\alpha\beta}{}_{\gamma;\alpha}M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}{}_{;\alpha}\,. \tag{10.12.1}$$

Verify the validity of this chain rule by expressing both sides of the equation in terms of directional derivatives plus connection-coefficient corrections.

Solution:

$$(S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{;\alpha} = (S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{,\alpha} + S^{\mu\beta}{}_{\gamma}M_{\beta}{}^{\gamma}\Gamma^{\alpha}{}_{\mu\alpha}. \tag{10.12.2}$$

$$\begin{aligned}
&S^{\alpha\beta}{}_{\gamma;\alpha}M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}{}_{;\alpha}\\
&= (S^{\alpha\beta}{}_{\gamma,\alpha} + S^{\mu\beta}{}_{\gamma}\Gamma^{\alpha}{}_{\mu\alpha} + S^{\alpha\mu}{}_{\gamma}\Gamma^{\beta}{}_{\mu\alpha} - S^{\alpha\beta}{}_{\mu}\Gamma^{\mu}{}_{\gamma\alpha})M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}(M_{\beta}{}^{\gamma}{}_{,\alpha} + M_{\beta}{}^{\mu}\Gamma^{\gamma}{}_{\mu\alpha} - M_{\mu}{}^{\gamma}\Gamma^{\mu}{}_{\beta\alpha})\\
&= (S^{\alpha\beta}{}_{\gamma,\alpha} + S^{\mu\beta}{}_{\gamma}\Gamma^{\alpha}{}_{\mu\alpha})M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}{}_{,\alpha}\\
&= (S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{,\alpha} + S^{\mu\beta}{}_{\gamma}M_{\beta}{}^{\gamma}\Gamma^{\alpha}{}_{\mu\alpha}\\
&= (S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{;\alpha}.
\end{aligned} \tag{10.12.3}$$

10.13 TRANSFORMATION LAW FOR CONNECTION COEFFICIENTS

Let $\{\boldsymbol{e}_\alpha\}$ and $\{\boldsymbol{e}_{\mu'}\}$ be two different fields of basis vectors related by the transformation law

$$\boldsymbol{e}_{\mu'}(\mathcal{P}) = L^{\alpha}{}_{\mu'}(\mathcal{P})\boldsymbol{e}_\alpha(\mathcal{P}). \tag{10.13.1}$$

Show that the corresponding connection coefficients are related by

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta'}L^{\nu}{}_{\gamma'}\Gamma^{\rho}{}_{\mu\nu} + L^{\alpha'}{}_{\mu}L^{\mu}{}_{\beta',\gamma'}. \tag{10.13.2}$$

Solution:

$$\begin{aligned}
\Gamma^{\alpha'}{}_{\beta'\gamma'} &= \langle \boldsymbol{\omega}^{\alpha'}, \boldsymbol{\nabla}_{\gamma'}\boldsymbol{e}_{\beta'}\rangle \\
&= \langle L^{\alpha'}{}_{\rho}\boldsymbol{\omega}^{\rho}, \boldsymbol{\nabla}_{\gamma'}L^{\mu}{}_{\beta'}\boldsymbol{e}_\mu\rangle \\
&= L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta'}\langle \boldsymbol{\omega}^{\rho}, \boldsymbol{\nabla}_{\gamma'}\boldsymbol{e}_\mu\rangle + L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta',\gamma'}\langle \boldsymbol{\omega}^{\rho}, \boldsymbol{e}_\mu\rangle \\
&= L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta'}L^{\nu}{}_{\gamma'}\langle \boldsymbol{\omega}^{\rho}, \boldsymbol{\nabla}_{\nu}\boldsymbol{e}_\mu\rangle + L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta',\gamma'}\delta^{\rho}{}_{\mu} \\
&= L^{\alpha'}{}_{\rho}L^{\mu}{}_{\beta'}L^{\nu}{}_{\gamma'}\Gamma^{\rho}{}_{\mu\nu} + L^{\alpha'}{}_{\mu}L^{\mu}{}_{\beta',\gamma'}.
\end{aligned} \tag{10.13.3}$$

10.14 POLAR COORDINATES IN FLAT 2-DIMENSIONAL SPACE

On a sheet of paper draw an (r, ϕ) polar coordinate system. At neighboring points, draw the basis vectors $\boldsymbol{e}_{\hat{r}} = \partial/\partial r$ and $\boldsymbol{e}_{\hat{\phi}} = r^{-1}\partial/\partial\phi$. (a) Use this picture, and Euclid's version of parallel transport, to justify the relations

$$\boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{r}} = 0, \quad \boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{\phi}} = 0, \quad \boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{r}} = r^{-1}\boldsymbol{e}_{\hat{\phi}}, \quad \boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}} = -r^{-1}\boldsymbol{e}_{\hat{r}}. \tag{10.14.1}$$

(b) From these relations write down the connection coefficients. (c) Let $\boldsymbol{A} = A^{\hat{r}}\boldsymbol{e}_{\hat{r}} + A^{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}}$ be a vector field. Show that its divergence, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = A^{\hat{\alpha}}{}_{;\hat{\alpha}} = A^{\hat{\alpha}}{}_{,\hat{\alpha}} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}}A^{\hat{\mu}}$, can be calculated using the formula

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial A^{\hat{\phi}}}{\partial\phi} + \frac{1}{r}\frac{\partial(rA^{\hat{r}})}{\partial r} \tag{10.14.2}$$

(which should be familiar to most readers).

Solution:

(a) From figure 1, it is easy to see that

$$\boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{r}} = 0, \quad \boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{\phi}} = 0, \quad \boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{r}} = \lim_{\Delta\phi\to 0}\frac{\Delta\phi\boldsymbol{e}_{\hat{\phi}}}{r\Delta\phi} = r^{-1}\boldsymbol{e}_{\hat{\phi}}, \quad \boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}} = \lim_{\Delta\phi\to 0}\frac{-\Delta\phi\boldsymbol{e}_{\hat{r}}}{r\Delta\phi} = -r^{-1}\boldsymbol{e}_{\hat{r}}. \tag{10.14.3}$$

(b)

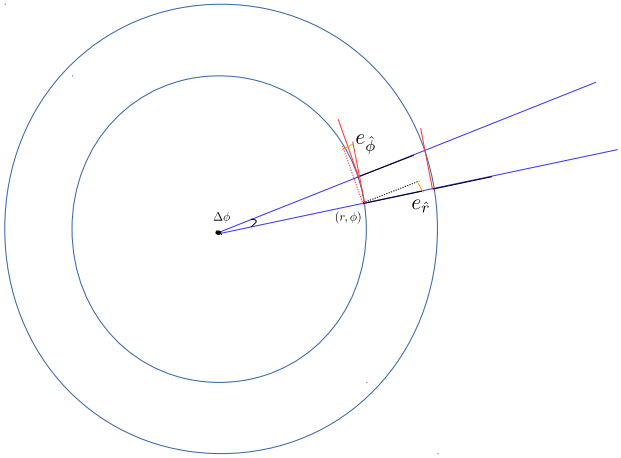
$$\Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = r^{-1}, \quad \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -r^{-1}. \tag{10.14.4}$$

All other connection coefficients vanish.

(c)

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = A^{\hat{r}}{}_{,\hat{r}} + A^{\hat{\phi}}{}_{,\hat{\phi}} + \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}}A^{\hat{r}} = A^{\hat{r}}{}_{,\hat{r}} + A^{\hat{\phi}}{}_{,\hat{\phi}} + \frac{A^{\hat{r}}}{r} = \frac{1}{r}\frac{\partial A^{\hat{\phi}}}{\partial\phi} + \frac{1}{r}\frac{\partial(rA^{\hat{r}})}{\partial r}. \tag{10.14.5}$$

Figure 10.1: Polar coordinate system



10.15 COMPONENTS OF PARALLEL-TRANSPORT LAW

Show that equation (10.28) is the component version of the law for parallel transporting a vector $\boldsymbol{v}$ along the curve $\mathcal{P}(\lambda)$ with tangent vector $\boldsymbol{u} = d\mathcal{P}/d\lambda$.

Solution:

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = v^{\alpha}{}_{;\beta}u^{\beta} = (v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}v^{\mu})u^{\beta} = \frac{dv^{\alpha}}{d\lambda} + \Gamma^{\alpha}{}_{\mu\beta}v^{\mu}\frac{dx^{\beta}}{d\lambda} = 0. \tag{10.15.1}$$

10.16 GEODESICS IN POLAR COORDINATES

In rectangular coordinates on a flat sheet of paper, Euclid's straight lines (geodesics) satisfy $d^2x/d\lambda^2 = d^2y/d\lambda^2 = 0$. Transform this geodesic equation into polar coordinates ($x = r\cos\phi$, $y = r\sin\phi$); and read off the resulting connection coefficients by comparison with equation (10.27). These are the connection coefficients for the coordinate basis $(\partial/\partial r, \partial/\partial\phi)$. From them calculate the connection coefficients for the basis

$$\boldsymbol{e}_{\hat{r}} = \frac{\partial}{\partial r}, \quad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r}\frac{\partial}{\partial\phi}. \tag{10.16.1}$$

The answer should agree with the answer to part (b) of Exercise 10.14.

Solution:

$$\frac{d^2(r\cos\phi)}{d\lambda^2} = \frac{d^2r}{d\lambda^2}\cos\phi - 2\sin\phi\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} - r\cos\phi\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} - r\sin\phi\frac{d^2\phi}{d\lambda^2} = 0. \tag{10.16.2}$$

$$\frac{d^2(r\sin\phi)}{d\lambda^2} = \frac{d^2r}{d\lambda^2}\sin\phi + 2\cos\phi\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} - r\sin\phi\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} + r\cos\phi\frac{d^2\phi}{d\lambda^2} = 0. \tag{10.16.3}$$

From 10.16.2 $\times\cos\phi +$ 10.16.3 $\times\sin\phi$, we can get

$$\frac{d^2r}{d\lambda^2} - r\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda} = 0. \tag{10.16.4}$$

From $-10.16.2 \times \sin\phi + 10.16.3 \times \cos\phi$, we can get

$$r\frac{d^2\phi}{d\lambda^2} + 2\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = 0. \tag{10.16.5}$$

So we have

$$\Gamma^r{}_{\phi\phi} = -r; \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}. \tag{10.16.6}$$

Now we can get

$$\boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{r}} = \boldsymbol{\nabla}_r\partial_r = 0, \quad \boldsymbol{\nabla}_{\hat{r}}\boldsymbol{e}_{\hat{\phi}} = \boldsymbol{\nabla}_r r^{-1}\partial_\phi = -\frac{1}{r^2}\partial_\phi + r^{-1}\Gamma^\phi{}_{\phi r}\partial_\phi = 0, \tag{10.16.7}$$

$$\boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{r}} = r^{-1}\boldsymbol{\nabla}_\phi\partial_r = r^{-1}\Gamma^\phi{}_{\phi r}\partial_\phi = r^{-1}\boldsymbol{e}_{\hat{\phi}}, \quad \boldsymbol{\nabla}_{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}} = r^{-2}\boldsymbol{\nabla}_\phi\partial_\phi = r^{-2}\Gamma^r{}_{\phi\phi}\partial_r = -r^{-1}\boldsymbol{e}_{\hat{r}}. \tag{10.16.8}$$

So

$$\Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = r^{-1}, \quad \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -r^{-1}. \tag{10.16.9}$$

All other connection coefficients vanish.

10.17 ROTATION GROUP: GEODESICS AND CONNECTION COEFFICIENTS

[Continuation of exercises 9.13 and 9.14.] In discussing the rotation group, one must make a clear distinction between the Euclidean space (coordinates x, y, z; basis vectors $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) in which the rotation matrices act, and the group manifold $SO(3)$ (coordinates ψ, θ, ϕ, coordinate basis $\partial/\partial\psi$, $\partial/\partial\theta$, $\partial/\partial\phi$; basis of "generators" $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$), whose points $\mathcal{P}$ are rotation matrices.

(a) Pick a vector

$$\boldsymbol{n} = n^x\partial/\partial x + n^x\partial/\partial y + n^x\partial/\partial z \tag{10.17.1}$$

in Euclidean space. Show that

$$\mathcal{R}_{\boldsymbol{n}}(t) \equiv \exp[(n^x\mathcal{H}_1 + n^y\mathcal{H}_2 + n^z\mathcal{H}_3)t] \tag{10.17.2}$$

is a rotation matrix that rotates the axes of Euclidean space by an angle

$$t|\boldsymbol{n}| = t[(n^x)^2 + (n^y)^2 + (n^z)^2]^{1/2}. \tag{10.17.3}$$

about the direction $\boldsymbol{n}$. ($\mathcal{H}_j$ are matrices defined in exercise 9.13.)

(b) In the group manifold $SO(3)$, pick a point (rotation matrix) $\mathcal{P}$, and pick a tangent vector $\boldsymbol{u} = u^\alpha\boldsymbol{e}_\alpha$ at $\mathcal{P}$. Let $\boldsymbol{u}$ be a vector in Euclidean space with the same components as $\boldsymbol{u}$ has in $SO(3)$:

$$\boldsymbol{u} = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3; \quad \boldsymbol{u} = u^1\partial/\partial x + u^2\partial/\partial y + u^3\partial/\partial z. \tag{10.17.4}$$

Show that u is the tangent vector (at $t = 0$) to the curve

$$\mathcal{L}(t) = \mathcal{R}_{\boldsymbol{u}}\mathcal{P}. \tag{10.17.5}$$

The curve $\mathcal{L}(t)$ through the arbitrary point $\mathcal{P}$ with arbitrary tangent vector $\boldsymbol{u} = (d\mathcal{L}/dt)_{t=0}$ is a very special curve: every point on it differs from $\mathcal{P}$ by a rotation $\mathcal{R}_{\boldsymbol{u}}(t)$ about one and the same direction $\boldsymbol{u}$. No other curve in $SO(3)$ with "starting conditions" $\{\mathcal{P}, \boldsymbol{u}\}$ has such beautiful simplicity. Hence it is natural to decree that each such $\mathcal{L}(t)$ is a geodesic of the group manifold $SO(3)$. This decree adds new geometric structure to $SO(3)$; it converts $SO(3)$ from a differentiable manifold into something more special: an affine manifold.

One has no guarantee that an arbitrarily chosen family of curves in an arbitrary manifold can be decreed to be geodesics. Most families of curves simply do not possess the right geometric properties to function as geodesics. Most will lead to covariant derivatives that violate one or more of the fundamental conditions

(10.2). To learn whether a given choice of geodesics is possible, one can try to derive connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ (for some given basis) corresponding to the chosen geodesics. If the derivation is successful, the choice of geodesics was a possible one. If the derivation produces inconsistencies, the chosen family of curves have the wrong geometrIc properties to function as geodesics.

(c) For the basis of generators $\{\boldsymbol{e}_\alpha\}$ derive connection coefficients corresponding to the chosen geodesics,$\mathcal{L}(t) = \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}$ of $SO(3)$.

Solution:

(a) $\mathcal{H}_l$ are antisymmetric matrix, i.e. $\mathcal{H}_l + \mathcal{H}_l^{\mathrm{T}} = 0$, so we have

$$\mathcal{R}_{\boldsymbol{n}}(t)\mathcal{R}_{\boldsymbol{n}}^{\mathrm{T}}(t) = \exp[(n^x\mathcal{H}_1 + n^y\mathcal{H}_2 + n^z\mathcal{H}_3)t]\exp[-(n^x\mathcal{H}_1 + n^y\mathcal{H}_2 + n^z\mathcal{H}_3)t] = I. \tag{10.17.6}$$

Because the determinant of $\mathcal{R}_{\boldsymbol{n}}(t)$ can not change discontinually as t varies, we must have $\det \mathcal{R}_{\boldsymbol{n}}(t) = 1$. So $\mathcal{R}_{\boldsymbol{n}}(t)$ is a rotation matrix. Since

$$(n^l\mathcal{H}_l)\boldsymbol{n} = (n^l\mathcal{H}_l)_{ij}\boldsymbol{n}^j = \epsilon_{lij}n^l n^j = 0, \tag{10.17.7}$$

$\mathcal{R}_{\boldsymbol{n}}(t)\boldsymbol{n} = \boldsymbol{n}$, i.e. the rotation axes of $\mathcal{R}_{\boldsymbol{n}}(t)\boldsymbol{n}$ is $\boldsymbol{n}$. In a well chosen orthogonal bases ($\boldsymbol{e}_1 = \boldsymbol{n}$),

$$\mathcal{R}'_{\boldsymbol{n}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{pmatrix}, \tag{10.17.8}$$

where $\cos\Theta$ is rotation angle. Thus $1 + 2\cos\Theta = \mathrm{Tr}\mathcal{R}'_{\boldsymbol{n}}(t) = \mathrm{Tr}\mathcal{R}_{\boldsymbol{n}}(t)$. We can calculate by brute force that

$$(n^l\mathcal{H}_l)_{ab} = n^l\epsilon_{lab}; \quad (n^l\mathcal{H}_l)^2_{ab} = (n_a n_b - n^2\delta_{ab}); \quad (n^l\mathcal{H}_l)^3_{ab} = -n^2 n^l\epsilon_{lab}\cdots \tag{10.17.9}$$

So

$$\mathrm{Tr}(n^l\mathcal{H}_l)^{2k+1} = 0; \quad \mathrm{Tr}(n^l\mathcal{H}_l)^{2k} = 2(-n^2)^k; \tag{10.17.10}$$

and the trace of rotation matrix is

$$3 + 2\sum_{k=1}^{\infty}(-1)^k(t|\boldsymbol{n}|)^{2k} = 1 + 2\cos(|\boldsymbol{n}|t). \tag{10.17.11}$$

So the rotation angle is

$$\Theta = t|\boldsymbol{n}| = t[(n^x)^2 + (n^y)^2 + (n^z)^2]^{1/2}. \tag{10.17.12}$$

(b) At the vicinity of $t = 0$, we have

$$\mathcal{R}_{\boldsymbol{u}}(t) \approx I + u^l\mathcal{H}_l t. \tag{10.17.13}$$

The definition of $\{\boldsymbol{e}_l\}$ implies that

$$\lim_{t\to 0}\frac{\mathcal{R}_l(t)\mathcal{P} - \mathcal{P}}{t} = \mathcal{H}_l\mathcal{P} = \boldsymbol{e}_l. \tag{10.17.14}$$

So the tagent vector of the curve $\mathcal{L}(t) = \mathcal{R}_{\boldsymbol{u}}\mathcal{P}$ is

$$\lim_{t\to 0}\frac{\mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P} - \mathcal{P}}{t} = u^l\mathcal{H}_l\mathcal{P} = u^l\boldsymbol{e}_l. \tag{10.17.15}$$

(c) The geodesic equation of the manifold is

$$\frac{du^\alpha}{dt} + \Gamma^{\alpha}{}_{\beta\gamma}u^\beta u^\gamma = 0. \tag{10.17.16}$$

The tagent vector of $\mathcal{L}(t)$ at an arbitrary point of the curve is

$$\lim_{\tau\to 0}\frac{\mathcal{R}_{\boldsymbol{u}}(t+\tau)\mathcal{P} - \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}}{\tau} = \lim_{\tau\to 0}\frac{\mathcal{R}_{\boldsymbol{u}}(\tau)\mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P} - \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}}{\tau} = u^l\boldsymbol{e}_l(t), \tag{10.17.17}$$

i.e. $du^\alpha/dt = 0$. So we have $\Gamma^{\alpha}{}_{(\beta\gamma)} = 0$. On the other hand, $\Gamma^{\alpha}{}_{[\beta\gamma]} = -c^{\alpha}{}_{\beta\gamma}/2 = \epsilon_{\alpha\beta\gamma}/2$. Finally, we can get $\Gamma^{\alpha}{}_{\beta\gamma} = -\epsilon_{\alpha\beta\gamma}/2$.

Chapter 11

GEODESIC DEVIATION AND SPACETIME CURVATURE

11.1 $[\nabla_A, \nabla_B]C$ DEPENDS ON DERIVATIVES OF C

Let $\boldsymbol{C}_{\rm NEW}$ and $\boldsymbol{C}_{\rm OLD}$ be vector fields related by

$$\boldsymbol{C}_{\rm NEW}(\mathcal{P}) = f(\mathcal{P})\boldsymbol{C}_{\rm OLD}(\mathcal{P}), \tag{11.1.1}$$

$f(\mathcal{P})$ is an arbitrary function, except $f(\mathcal{P}_0) = 1$. Show that

$$\{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm NEW}\}_{\mathcal{P}_0} - \{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm OLD}\}_{\mathcal{P}_0} = \boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f. \tag{11.1.2}$$

Solution:

$$\begin{aligned}
\{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm NEW}\}_{\mathcal{P}_0} &= \{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]f(\mathcal{P})\boldsymbol{C}_{\rm OLD}\}_{\mathcal{P}_0} \\
&= \{\boldsymbol{\nabla_A}\boldsymbol{\nabla_B}[f(\mathcal{P})\boldsymbol{C}_{\rm OLD}] - \boldsymbol{\nabla_B}\boldsymbol{\nabla_A}[f(\mathcal{P})\boldsymbol{C}_{\rm OLD}]\}_{\mathcal{P}_0} \\
&= \{\boldsymbol{\nabla_A}[\boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla_B}f(\mathcal{P}) + f(\mathcal{P})\boldsymbol{\nabla_B}\boldsymbol{C}_{\rm OLD}] - \boldsymbol{\nabla_B}[\boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla_A}f(\mathcal{P}) + f(\mathcal{P})\boldsymbol{\nabla_A}\boldsymbol{C}_{\rm OLD}]\}_{\mathcal{P}_0} \\
&= \{\boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla_A}\boldsymbol{\nabla_B}f(\mathcal{P}) + f(\mathcal{P})\boldsymbol{\nabla_A}\boldsymbol{\nabla_B}\boldsymbol{C}_{\rm OLD} - \boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla_B}\boldsymbol{\nabla_A}f(\mathcal{P}) - f(\mathcal{P})\boldsymbol{\nabla_B}\boldsymbol{\nabla_A}\boldsymbol{C}_{\rm OLD}\}_{\mathcal{P}_0} \\
&= f(\mathcal{P}_0)\{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm OLD}\}_{\mathcal{P}_0} + \boldsymbol{C}_{\rm OLD}(\mathcal{P}_0)\{\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f\}_{\mathcal{P}_0}.
\end{aligned} \tag{11.1.3}$$

So we have

$$\{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm NEW}\}_{\mathcal{P}_0} - \{[\boldsymbol{\nabla_A}, \boldsymbol{\nabla_B}]\boldsymbol{C}_{\rm OLD}\}_{\mathcal{P}_0} = \boldsymbol{C}_{\rm OLD}\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f. \tag{11.1.4}$$

11.2 PROOF THAT RIEMANN IS A TENSOR

Show from its definition (11.8, 11.9) that $\boldsymbol{Riemann}$ is a tensor.

Hint: Use the following procedure.

(a) If $f(\mathcal{P})$ is an arbitrary function, show that

$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})f\boldsymbol{C} = f\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}. \tag{11.2.1}$$

(b) Similarly show that

$$\mathcal{R}(f\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}=f\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}\text{ and }\mathcal{R}(\boldsymbol{A},f\boldsymbol{B})\boldsymbol{C}=f\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}. \tag{11.2.2}$$

(c) Show that $\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}$ is linear; i.e.,

$$\begin{aligned}\mathcal{R}(\boldsymbol{A}+\boldsymbol{a},\boldsymbol{B})\boldsymbol{C}&=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{a},\boldsymbol{B})\boldsymbol{C};\\ \mathcal{R}(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{b})\boldsymbol{C}&=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{C};\\ \mathcal{R}(\boldsymbol{A},\boldsymbol{B})(\boldsymbol{C}+\boldsymbol{c})&=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{c}.\end{aligned} \tag{11.2.3}$$

(d) Now use the above properties to prove the most crucial feature of $\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}$: Modify the variations (gradients) of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ in an arbitrary manner, but leave $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ unchanged at $\mathcal{P}_0$:

$$\left.\begin{aligned}\boldsymbol{A}&\to\boldsymbol{A}+a^\alpha\boldsymbol{e}_\alpha\\ \boldsymbol{B}&\to\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha\\ \boldsymbol{C}&\to\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha\end{aligned}\right\}\quad a^\alpha(\mathcal{P}),\,b^\alpha(\mathcal{P}),\,c^\alpha(\mathcal{P})\text{ arbitrary except they all vanish at }\mathcal{P}=\mathcal{P}_0. \tag{11.2.4}$$

Show that this modification leaves $\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}$ unchanged at $\mathcal{P}_0$.

(e) From these facts, conclude that $\boldsymbol{Riemann}$ is a tensor.

Solution:

From exercise 11.1 we have

$$[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]f\boldsymbol{C}=f[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}+\boldsymbol{C}\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f. \tag{11.2.5}$$

From the definition $\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\equiv[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]-\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}$, we have

$$\mathcal{R}(\boldsymbol{A},\boldsymbol{B})f\boldsymbol{C}=f[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}+\boldsymbol{C}\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f-f\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}\boldsymbol{C}-\boldsymbol{C}\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f=f\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}. \tag{11.2.6}$$

Because $\boldsymbol{\nabla}_{f\boldsymbol{A}+g\boldsymbol{B}}=f\boldsymbol{\nabla}_{\boldsymbol{A}}+g\boldsymbol{\nabla}_{\boldsymbol{B}}$ and $[f\boldsymbol{A},\boldsymbol{B}]=f[\boldsymbol{A},\boldsymbol{B}]-(\boldsymbol{\nabla}_{\boldsymbol{B}}f)\boldsymbol{A}$, we have

$$\begin{aligned}\mathcal{R}(f\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}&=[f\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}-\boldsymbol{\nabla}_{f[\boldsymbol{A},\boldsymbol{B}]-(\boldsymbol{\nabla}_{\boldsymbol{B}}f)\boldsymbol{A}}\boldsymbol{C}\\ &=f[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}-(\boldsymbol{\nabla}_{\boldsymbol{B}}f)\boldsymbol{\nabla}_{\boldsymbol{A}}\boldsymbol{C}-f\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}\boldsymbol{C}+(\boldsymbol{\nabla}_{\boldsymbol{B}}f)\boldsymbol{\nabla}_{\boldsymbol{A}}\boldsymbol{C}\\ &=f\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}.\end{aligned} \tag{11.2.7}$$

By similar procedure, we can also prove that $\mathcal{R}(\boldsymbol{A},f\boldsymbol{B})\boldsymbol{C}=f\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}$.

$$\begin{aligned}\mathcal{R}(\boldsymbol{A}+\boldsymbol{a},\boldsymbol{B})\boldsymbol{C}&=[\boldsymbol{\nabla}_{\boldsymbol{A}+\boldsymbol{a}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}-\boldsymbol{\nabla}_{[\boldsymbol{A}+\boldsymbol{a},\boldsymbol{B}]}\boldsymbol{C}\\ &=[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}+[\boldsymbol{\nabla}_{\boldsymbol{a}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}-\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}\boldsymbol{C}-\boldsymbol{\nabla}_{[\boldsymbol{a},\boldsymbol{B}]}\boldsymbol{C}\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{a},\boldsymbol{B})\boldsymbol{C};\end{aligned} \tag{11.2.8}$$

By similar procedure, we can also prove that $\mathcal{R}(\boldsymbol{A},\boldsymbol{B}+\boldsymbol{b})\boldsymbol{C}=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{A},\boldsymbol{b})\boldsymbol{C}$. Then we also have

$$\begin{aligned}\mathcal{R}(\boldsymbol{A},\boldsymbol{B})(\boldsymbol{C}+\boldsymbol{c})&=[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}](\boldsymbol{C}+\boldsymbol{c})-\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}(\boldsymbol{C}+\boldsymbol{c})\\ &=[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}+[\boldsymbol{\nabla}_{\boldsymbol{A}},\boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{c}-\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}\boldsymbol{C}-\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}\boldsymbol{c}\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{c}.\end{aligned} \tag{11.2.9}$$

Finally, we have

$$\begin{aligned}&\mathcal{R}(\boldsymbol{A}+a^\alpha\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)+\mathcal{R}(a^\alpha\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)+\mathcal{R}(\boldsymbol{A},b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)+a^\alpha\mathcal{R}(\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+\mathcal{R}(\boldsymbol{A},\boldsymbol{B})c^\alpha\boldsymbol{e}_\alpha+b^\alpha\mathcal{R}(\boldsymbol{A},\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)+a^\alpha\mathcal{R}(\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)\\ &=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}+c^\alpha\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{e}_\alpha+b^\alpha\mathcal{R}(\boldsymbol{A},\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)+a^\alpha\mathcal{R}(\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha).\end{aligned} \tag{11.2.10}$$

Because $a^\alpha(\mathcal{P})$, $b^\alpha(\mathcal{P})$, $c^\alpha(\mathcal{P})$ arbitrary except they all vanish at $\mathcal{P}=\mathcal{P}_0$, we have

$$\mathcal{R}(\boldsymbol{A}+a^\alpha\boldsymbol{e}_\alpha,\boldsymbol{B}+b^\alpha\boldsymbol{e}_\alpha)(\boldsymbol{C}+c^\alpha\boldsymbol{e}_\alpha)(\mathcal{P}_0)=\mathcal{R}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C}(\mathcal{P}_0). \tag{11.2.11}$$

So in conclusion, we have shown that $\boldsymbol{Riemann}$ is a tensor.

11.3 COMPONENTS OF RIEMANN IN COORDINATE BASIS

Derive equation (11.12) for the components of the Riemann tensor in a coordinate basis.

The detailed solution to this exercise has already been provided in the textbook.

11.4 COMPONENTS OF RIEMANN IN NONCOORDINATE BASIS

In a noncoordinate basis with commutation coefficients ${c_{\alpha\beta}}^{\gamma}$ defined by equation (9.22), derive the following equation for the components of Riemann:

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu}{c_{\gamma\delta}}^{\mu}. \tag{11.4.1}$$

Solution:

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \langle \boldsymbol{\omega}^{\alpha}, \mathcal{R}(\boldsymbol{e}_{\gamma}, \boldsymbol{e}_{\delta})\boldsymbol{e}_{\beta}\rangle \\
&= \langle \boldsymbol{\omega}^{\alpha}, (\nabla_{\boldsymbol{e}_{\gamma}}\nabla_{\boldsymbol{e}_{\delta}} - \nabla_{\boldsymbol{e}_{\delta}}\nabla_{\boldsymbol{e}_{\gamma}})\boldsymbol{e}_{\beta} - \nabla_{[\boldsymbol{e}_{\gamma},\boldsymbol{e}_{\delta}]}\boldsymbol{e}_{\beta}\rangle \\
&= \langle \boldsymbol{\omega}^{\alpha}, \nabla_{\boldsymbol{e}_{\gamma}}(\Gamma^{\mu}{}_{\beta\delta}\boldsymbol{e}_{\mu}) - \nabla_{\boldsymbol{e}_{\delta}}(\Gamma^{\mu}{}_{\beta\gamma}\boldsymbol{e}_{\mu}) - \nabla_{{c_{\gamma\delta}}^{\mu}\boldsymbol{e}_{\mu}}\boldsymbol{e}_{\beta}\rangle \\
&= \langle \boldsymbol{\omega}^{\alpha}, (\Gamma^{\mu}{}_{\beta\delta,\gamma}\boldsymbol{e}_{\mu} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\nu}{}_{\mu\gamma}\boldsymbol{e}_{\nu}) - (\Gamma^{\mu}{}_{\beta\gamma,\delta}\boldsymbol{e}_{\mu} + \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\nu}{}_{\mu\delta}\boldsymbol{e}_{\nu}) - {c_{\gamma\delta}}^{\mu}\Gamma^{\nu}{}_{\beta\mu}\boldsymbol{e}_{\nu}\rangle \\
&= \Gamma^{\alpha}{}_{\beta\delta,\gamma} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\alpha}{}_{\mu\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} - \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta} - {c_{\gamma\delta}}^{\mu}\Gamma^{\alpha}{}_{\beta\mu}.
\end{aligned} \tag{11.4.2}$$

11.5 COPLANARITY OF CLOSED CURVES

Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be the bivectors (see Box 11.7) for two small closed curves at the same event. Show that the curves are coplanar if and only if $\boldsymbol{f}_1 = a\boldsymbol{f}_2$ for some number a.

Solution:

Suppose $\boldsymbol{f}_1 = \boldsymbol{u}_1 \wedge \boldsymbol{v}_1$ and $\boldsymbol{f}_2 = \boldsymbol{u}_2 \wedge \boldsymbol{v}_2$. If the curves are coplanar, we have $\boldsymbol{u}_1 = a_{uu}\boldsymbol{u}_2 + a_{uv}\boldsymbol{v}_2$ and $\boldsymbol{v}_1 = a_{vu}\boldsymbol{u}_2 + a_{vv}\boldsymbol{v}_2$. So

$$\boldsymbol{f}_1 = (a_{uu}\boldsymbol{u}_2 + a_{uv}\boldsymbol{v}_2) \wedge (a_{vu}\boldsymbol{u}_2 + a_{vv}\boldsymbol{v}_2) = (a_{uu}a_{vv} - a_{uv}a_{vu})\boldsymbol{u}_2 \wedge \boldsymbol{v}_2 = (a_{uu}a_{vv} - a_{uv}a_{vu})\boldsymbol{f}_2, \tag{11.5.1}$$

i.e. $\boldsymbol{f}_1 = a\boldsymbol{f}_2$, where $a \equiv a_{uu}a_{vv} - a_{uv}a_{vu}$.

If $\boldsymbol{f}_1 = a\boldsymbol{f}_2$, $\boldsymbol{f}_2 = \boldsymbol{u}_2 \wedge \boldsymbol{v}_2$, then we have $\boldsymbol{f}_1 = (a\boldsymbol{u}_1) \wedge (\boldsymbol{v})_2$. Thus the curves are coplanar.

11.6 SYMMETRIES OF Riemann

Show that Riemann has the following symmetries:

$$R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}; \quad R^{\alpha}{}_{[\beta\gamma\delta]} = 0. \tag{11.6.1}$$

Solution:

In a local frame with $\Gamma^{\alpha}{}_{\beta\gamma}(\mathcal{P}_0) = 0$, we have

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}. \tag{11.6.2}$$

It is obvious that $R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\beta\delta\gamma} = 0$, i.e. $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}$. Futhermore,

$$\begin{aligned} 3R^{\alpha}{}_{[\beta\gamma\delta]} &= R^{\alpha}{}_{\beta[\gamma\delta]} + R^{\alpha}{}_{\gamma[\delta\beta]} + R^{\alpha}{}_{\delta[\beta\gamma]} \\ &= R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\delta\beta} + R^{\alpha}{}_{\delta\beta\gamma} \\ &= \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\gamma\beta,\delta} - \Gamma^{\alpha}{}_{\delta\beta,\gamma} + \Gamma^{\alpha}{}_{\delta\gamma,\beta} - \Gamma^{\alpha}{}_{\gamma\delta,\beta}. \end{aligned} \tag{11.6.3}$$

As $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$, we can see that $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$. Because $R^{\alpha}{}_{\beta\gamma\delta}$ is a tensor, the equation will hold in an arbitrary frame.

11.7 GEODESIC DEVIATION MEASURES ALL CURVATURE COMPONENTS

The equation of geodesic deviation, written up to now in the form

$$\boldsymbol{\nabla_u \nabla_u n} + \boldsymbol{Riemann}(\cdots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0 \tag{11.7.1}$$

or

$$\boldsymbol{\nabla_u \nabla_u n} + \mathcal{R}(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} = 0, \tag{11.7.2}$$

also lets itself be written in the Jacobi form $\boldsymbol{\nabla_u \nabla_u n} + \mathcal{J}(\boldsymbol{u}, \boldsymbol{u})\boldsymbol{n} = 0$. Here $\mathcal{J}(\boldsymbol{u}, \boldsymbol{v})$, the "Jacobi curvature operator," is defined by

$$\mathcal{J}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{n} \equiv \frac{1}{2}[\mathcal{R}(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{v} + \mathcal{R}(\boldsymbol{n}, \boldsymbol{v})\boldsymbol{u}], \tag{11.7.3}$$

and is related to the "Jacobi curvature tensor" by

$$\boldsymbol{Jacobi}(\cdots, \boldsymbol{n}, \boldsymbol{u}, \boldsymbol{v}) \equiv \mathcal{J}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{n}, \tag{11.7.4}$$

which implies

$$J^{\mu}{}_{\nu\alpha\beta} = J^{\mu}{}_{\nu\beta\alpha} = \frac{1}{2}(R^{\mu}{}_{\alpha\nu\beta} + R^{\mu}{}_{\beta\nu\alpha}). \tag{11.7.5}$$

(a) Show that $J^{\mu}{}_{(\alpha\beta\gamma)} = 0$ follows from $R^{\mu}{}_{\alpha\beta\gamma} = R^{\mu}{}_{\alpha[\beta\gamma]}$.

(b) Show that by studying geodesic deviation (allowing arbitrary $\boldsymbol{u}$ and $\boldsymbol{n}$ in $\boldsymbol{\nabla_u \nabla_u n} + \mathcal{J}(\boldsymbol{u}, \boldsymbol{u})\boldsymbol{n} = 0$ one can measure all components of $\boldsymbol{Jacobi}$.

(c) Show that $\boldsymbol{Jacobi}$ contains precisely the same information as $\boldsymbol{Riemann}$. Hence, by studying geodesic deviation one can also measure all the components of $\boldsymbol{Riemann}$.

(d) Show that the symmetry of $R^{\mu}{}_{[\nu\alpha\beta]} = 0$ is essential in the equivalence between $\boldsymbol{Jacobi}$ and $\boldsymbol{Riemann}$ by exhibiting proposed values for $R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha}$ for which $R^{\mu}{}_{[\nu\alpha\beta]} \neq 0$, and from which one would find $J^{\mu}{}_{\nu\alpha\beta} = 0$.

Solution:

(a)

$$\begin{aligned} 6J^{\mu}{}_{(\alpha\beta\gamma)} &= (R^{\mu}{}_{\gamma\alpha\beta} + R^{\mu}{}_{\beta\alpha\gamma}) + (R^{\mu}{}_{\alpha\beta\gamma} + R^{\mu}{}_{\gamma\beta\alpha}) + (R^{\mu}{}_{\beta\gamma\alpha} + R^{\mu}{}_{\alpha\gamma\beta}) \\ &= (R^{\mu}{}_{\gamma\alpha\beta} + R^{\mu}{}_{\gamma\beta\alpha}) + (R^{\mu}{}_{\alpha\beta\gamma} + R^{\mu}{}_{\alpha\gamma\beta}) + (R^{\mu}{}_{\beta\gamma\alpha} + R^{\mu}{}_{\beta\alpha\gamma}) \\ &= 0. \end{aligned} \tag{11.7.6}$$

(b) For arbitrary $\boldsymbol{u}$ and $\boldsymbol{n}$ we can get $\mathcal{J}(\boldsymbol{u},\boldsymbol{u})\boldsymbol{n}$ by studying geodesic deviation. So we have

$$\mathcal{J}(\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta)\boldsymbol{e}_\nu=\mathcal{J}(\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha)\boldsymbol{e}_\nu+\mathcal{J}(\boldsymbol{e}_\beta,\boldsymbol{e}_\beta)\boldsymbol{e}_\nu+\mathcal{J}(\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta)\boldsymbol{e}_\nu+\mathcal{J}(\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha)\boldsymbol{e}_\nu. \tag{11.7.7}$$

By the definition of $\boldsymbol{Jacobi}$, we have $\mathcal{J}(\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta)\boldsymbol{e}_\nu=\mathcal{J}(\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha)\boldsymbol{e}_\nu$. Thus

$$\boldsymbol{Jacobi}(\cdots,\boldsymbol{e}_\nu,\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta)=\mathcal{J}(\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta)\boldsymbol{e}_\nu=\frac{1}{2}\left[\mathcal{J}(\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta,\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta)\boldsymbol{e}_\nu-\mathcal{J}(\boldsymbol{e}_\alpha,\boldsymbol{e}_\alpha)\boldsymbol{e}_\nu-\mathcal{J}(\boldsymbol{e}_\beta,\boldsymbol{e}_\beta)\boldsymbol{e}_\nu\right]. \tag{11.7.8}$$

So by studying geodesic deviation one can measure all components of $\boldsymbol{Jacobi}$.

(c)

$$\begin{aligned}J^\mu{}_{\nu\alpha\beta}-J^\mu{}_{\beta\alpha\nu}&=\frac{1}{2}(R^\mu{}_{\alpha\nu\beta}+R^\mu{}_{\beta\nu\alpha}-R^\mu{}_{\alpha\beta\nu}-R^\mu{}_{\nu\beta\alpha})\\&=\frac{1}{2}(R^\mu{}_{\alpha\nu\beta}-R^\mu{}_{\beta\alpha\nu}+R^\mu{}_{\alpha\nu\beta}-R^\mu{}_{\nu\beta\alpha})\\&=\frac{1}{2}[2R^\mu{}_{\alpha\nu\beta}-(R^\mu{}_{\beta\alpha\nu}+R^\mu{}_{\nu\beta\alpha})]\\&=\frac{3}{2}R^\mu{}_{\alpha\nu\beta},\end{aligned} \tag{11.7.9}$$

Thus

$$R^\mu{}_{\alpha\nu\beta}=\frac{2}{3}(J^\mu{}_{\nu\alpha\beta}-J^\mu{}_{\beta\alpha\nu}) \tag{11.7.10}$$

and

$$J^\mu{}_{\nu\alpha\beta}=\frac{1}{2}(R^\mu{}_{\alpha\nu\beta}+R^\mu{}_{\beta\nu\alpha}). \tag{11.7.11}$$

So $\boldsymbol{Jacobi}$ contains precisely the same information as $\boldsymbol{Riemann}$.

(d) If $R^\mu{}_{[\nu\alpha\beta]}\neq 0$ and $R^\mu{}_{\nu\alpha\beta}=R^\mu{}_{\nu[\alpha\beta]}$, we have $R^\mu{}_{\nu\alpha\beta}+R^\mu{}_{\alpha\beta\nu}+R^\mu{}_{\beta\nu\alpha}=3R^\mu{}_{[\nu\alpha\beta]}$. If we let $R^\mu{}_{\nu\alpha\beta}=R^\mu{}_{\alpha\beta\nu}=R^\mu{}_{\beta\nu\alpha}=R^\mu{}_{[\nu\alpha\beta]}$, some

$$J^\mu{}_{\nu\alpha\beta}=\frac{1}{2}(R^\mu{}_{\alpha\nu\beta}+R^\mu{}_{\beta\nu\alpha})=\frac{1}{2}(-R^\mu{}_{\alpha\beta\nu}+R^\mu{}_{\beta\nu\alpha})=0. \tag{11.7.12}$$

Thus $\boldsymbol{Jacobi}$ and $\boldsymbol{Riemann}$ are not equivalent in this case.

11.8 GEODESIC DEVIATION IN GORY DETAIL

Write out the equation of geodesic deviation in component form in a coordinate system. Expand all covariant derivatives (semicolon notation) in terms of ordinary (comma) derivatives and in terms of Γ's to show all Γ and ∂ terms explicitly.

Solution:

The geodesic deviation equations are

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n}+\boldsymbol{Riemann}(\cdots,\boldsymbol{u},\boldsymbol{n},\boldsymbol{u})=0. \tag{11.8.1}$$

In the form of components, we have

$$(n^\alpha{}_{;\beta}u^\beta)_{;\gamma}u^\gamma+R^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta=0. \tag{11.8.2}$$

Expand all covariant derivatives in terms of ordinary derivatives and in terms of Γ's, and we can get

$$\frac{d}{d\lambda}\left(\frac{dn^\alpha}{d\lambda}+\Gamma^\alpha{}_{\nu\beta}n^\nu u^\beta\right)+\Gamma^\alpha{}_{\mu\gamma}u^\gamma\left(\frac{dn^\mu}{d\lambda}+\Gamma^\mu{}_{\nu\beta}n^\nu u^\beta\right)+R^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta=0. \tag{11.8.3}$$

11.9 RIEMANN NORMAL COORDINATES IN GENERAL

Derive properties (11.27), (11.28), (11.29), (11.31), (11.32), and (11.32') of Riemann normal coordinates.

Solution:

From definition (11.26) $\mathcal{P} = \mathcal{G}(1; x^\alpha \boldsymbol{e}_\alpha)$, we have

$$\left.\frac{\partial \mathcal{P}}{\partial x^\alpha}\right|_{\mathcal{P}_0} = \lim_{x^\alpha \to 0} \frac{\mathcal{G}(1; x^\alpha \boldsymbol{e}_\alpha) - G(1;0)}{x^\alpha} = \lim_{x^\alpha \to 0} \frac{\mathcal{G}(x^\alpha; \boldsymbol{e}_\alpha) - G(0; \boldsymbol{e}_\alpha)}{x^\alpha} = \boldsymbol{e}_\alpha. \tag{11.9.1}$$

For the curve $x^\alpha = \lambda v^\alpha$ (where the v^α are constants), it can be wirtten as $\mathcal{G}(1; \lambda v^\alpha \boldsymbol{e}_\alpha)$, i.e. $\mathcal{G}(\lambda; v^\alpha \boldsymbol{e}_\alpha)$. So it is a geodesic through $\mathcal{P}_0$ with affine parameter λ. The geodesic equation is

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0. \tag{11.9.2}$$

By substituting $x^\alpha = \lambda v^\alpha$ into the geodesic equation, we have

$$\Gamma^\alpha{}_{\beta\gamma}(\mathcal{P}_0) v^\beta v^\gamma = 0. \tag{11.9.3}$$

As v^α is arbitrary, we have $\Gamma^\alpha{}_{\beta\gamma}(\mathcal{P}_0) = 0$.

Since the curves $x^\alpha = \lambda v^\alpha$ are geodesics for every choice of the parameters v^α, they provide not only a geodesic tangent $\boldsymbol{u} \equiv (\partial/\partial\lambda)_{v^\alpha}$ but also several deviation vectors $\boldsymbol{N}_{(\alpha)} \equiv (\partial/\partial v^\alpha)_\lambda$. In Riemann normal coordinates, it is easy to get

$$\boldsymbol{u} = v^\alpha \frac{\partial}{\partial x^\alpha}; \quad \boldsymbol{N}_{(\alpha)} = \lambda \frac{\partial}{\partial x^\alpha}. \tag{11.9.4}$$

Substitute 11.9.4 into the geodesic deviation equation 11.8.3, we have

$$\frac{d}{d\lambda}\left(\frac{d\lambda \delta^\alpha_{(\rho)}}{d\lambda} + \Gamma^\alpha{}_{\nu\beta} \lambda \delta^\nu_{(\rho)} v^\beta\right) + \Gamma^\alpha{}_{\mu\gamma} v^\gamma \left(\frac{d\lambda \delta^\mu_{(\rho)}}{d\lambda} + \Gamma^\mu{}_{\nu\beta} \lambda \delta^\nu_{(\rho)} v^\beta\right) + R^\alpha{}_{\beta\gamma\delta} v^\beta \lambda \delta^\gamma_{(\rho)} v^\delta = 0, \tag{11.9.5}$$

i.e.

$$\lambda v^\beta \frac{d\Gamma^\alpha{}_{\rho\beta}}{d\lambda} + 2\Gamma^\alpha{}_{\rho\beta} v^\beta + \lambda \Gamma^\alpha{}_{\mu\gamma} \Gamma^\mu{}_{\rho\beta} v^\beta v^\gamma + \lambda R^\alpha{}_{\beta\rho\delta} v^\beta v^\delta = 0. \tag{11.9.6}$$

Using

$$\Gamma^\alpha{}_{\beta\gamma}\big|_{x^\mu = \lambda z^\mu} = \lambda v^\mu \Gamma^\alpha{}_{\beta\gamma,\mu}(\mathcal{P}_0) + O(\lambda^2), \tag{11.9.7}$$

we have

$$3\lambda \Gamma^\alpha{}_{\rho\beta,\mu}(\mathcal{P}_0) v^\beta v^\mu + \lambda R^\alpha{}_{\beta\rho\mu}(\mathcal{P}_0) v^\beta v^\mu = O(\lambda^2). \tag{11.9.8}$$

So we can read out

$$\Gamma^\alpha{}_{\rho\beta,\mu} + \Gamma^\alpha{}_{\rho\mu,\beta} = -\frac{1}{3}(R^\alpha{}_{\beta\rho\mu} + R^\alpha{}_{\mu\rho\beta}). \tag{11.9.9}$$

By renaming the index, we have

$$\Gamma^\alpha{}_{\beta\rho,\mu} + \Gamma^\alpha{}_{\beta\mu,\rho} = -\frac{1}{3}(R^\alpha{}_{\rho\beta\mu} + R^\alpha{}_{\mu\beta\rho}) \tag{11.9.10}$$

and

$$\Gamma^\alpha{}_{\mu\beta,\rho} + \Gamma^\alpha{}_{\mu\rho,\beta} = -\frac{1}{3}(R^\alpha{}_{\beta\mu\rho} + R^\alpha{}_{\rho\mu\beta}). \tag{11.9.11}$$

11.9.9 + 11.9.10 - 11.9.11 gives

$$2\Gamma^\alpha{}_{\rho\beta,\mu} = -\frac{2}{3}(R^\alpha{}_{\rho\beta\mu} + R^\alpha{}_{\beta\rho\mu}), \tag{11.9.12}$$

i.e.

$$\Gamma^{\alpha}{}_{\beta\gamma,\mu} = -\frac{1}{3}(R^{\alpha}{}_{\beta\gamma\mu} + R^{\alpha}{}_{\gamma\beta\mu}). \tag{11.9.13}$$

If spacetime has a metric (as it does in actuality), and if the observer's frame at $\mathcal{P}_0$ has been chosen orthonormal ($\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \eta_{\alpha\beta}$) then $g_{\alpha\beta}(\mathcal{P}_0) = \eta_{\alpha\beta}$. We get get

$$g_{\alpha\beta,\mu}(\mathcal{P}_0) = \Gamma^{\rho}{}_{\alpha\mu}(\mathcal{P}_0)g_{\rho\beta}(\mathcal{P}_0) + \Gamma^{\rho}{}_{\beta\mu}(\mathcal{P}_0)g_{\alpha\rho}(\mathcal{P}_0) = 0. \tag{11.9.14}$$

Futhermore,

$$\begin{aligned} g_{\alpha\beta,\mu\nu}(\mathcal{P}_0) &= \Gamma^{\rho}{}_{\alpha\mu,\nu}(\mathcal{P}_0)g_{\rho\beta}(\mathcal{P}_0) + \Gamma^{\rho}{}_{\beta\mu,\nu}(\mathcal{P}_0)g_{\alpha\rho}(\mathcal{P}_0) \\ &= -\frac{1}{3}(R_{\beta\alpha\mu\nu} + R_{\beta\mu\alpha\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\mu\beta\nu})(\mathcal{P}_0). \end{aligned} \tag{11.9.15}$$

When the Riemann tensor is deduced from metric, we have $R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\mu\nu}$ and $R_{\beta\mu\alpha\nu} = R_{\alpha\nu\beta\mu}$ (it will be derived in chapter 13). Thus, we have

$$g_{\alpha\beta,\mu\nu}(\mathcal{P}_0) = -\frac{1}{3}(R_{\alpha\nu\beta\mu} + R_{\alpha\mu\beta\nu}) \equiv -\frac{2}{3}J_{\alpha\beta\mu\nu}. \tag{11.9.16}$$

Recall that

$$R^{\mu}{}_{\alpha\nu\beta} = \frac{2}{3}(J^{\mu}{}_{\nu\alpha\beta} - J^{\mu}{}_{\beta\alpha\nu}), \tag{11.9.17}$$

we can get

$$R_{\mu\alpha\nu\beta}(\mathcal{P}_0) = g_{\mu\beta,\alpha\nu}(\mathcal{P}_0) - g_{\mu\nu,\alpha\beta}(\mathcal{P}_0). \tag{11.9.18}$$

11.10 BIANCHI IDENTITIES

Show that the Riemann curvature tensor satisfies the following "Bianchi identities"

$$R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} = 0. \tag{11.10.1}$$

The geometric meaning of these identities will be discussed in Chapter 15.

Solution:

In Riemann normal coordinates, we have $\Gamma^{\alpha}{}_{\beta\gamma}(\mathcal{P}_0) = 0$. Thus

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}. \tag{11.10.2}$$

Futhermore,

$$R^{\alpha}{}_{\beta\gamma\delta;\epsilon} = R^{\alpha}{}_{\beta\gamma\delta,\epsilon} = \Gamma^{\alpha}{}_{\beta\delta,\gamma\epsilon} - \Gamma^{\alpha}{}_{\beta\gamma,\delta\epsilon}. \tag{11.10.3}$$

Then, we can get

$$R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma\epsilon} - \Gamma^{\alpha}{}_{\beta\gamma,\delta\epsilon} + \Gamma^{\alpha}{}_{\beta\epsilon,\delta\gamma} - \Gamma^{\alpha}{}_{\beta\delta,\epsilon\gamma} + \Gamma^{\alpha}{}_{\beta\gamma,\epsilon\delta} - \Gamma^{\alpha}{}_{\beta\epsilon,\gamma\delta} = 0. \tag{11.10.4}$$

Recall that $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}$. Now we arrive at the Bianchi identities

$$R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} = 0. \tag{11.10.5}$$

11.11 CURVATURE OPERATOR ACTS ON 1-FORMS

Let $\mathcal{R}(\boldsymbol{u},\boldsymbol{v})$ be the operator $\mathcal{R}(\boldsymbol{u},\boldsymbol{v}) = [\boldsymbol{\nabla}_{\boldsymbol{u}}, \boldsymbol{\nabla}_{\boldsymbol{v}}] - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}$ when acting on 1-forms $\boldsymbol{\sigma}$ (or other tensors) as well as on tangent vectors. Show that

$$\langle \mathcal{R}(\boldsymbol{u},\boldsymbol{v}), \boldsymbol{w}\rangle = -\langle \boldsymbol{\sigma}, \mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}\rangle. \tag{11.11.1}$$

Solution:

$$\begin{aligned}
\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle &= \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{v}}\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\nabla}_{\boldsymbol{u}}\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle \\
&= \boldsymbol{\nabla}_{\boldsymbol{u}}\langle\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\sigma},\boldsymbol{w}\rangle + \boldsymbol{\nabla}_{\boldsymbol{u}}\langle\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}\rangle - \boldsymbol{\nabla}_{\boldsymbol{v}}\langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{w}\rangle - \boldsymbol{\nabla}_{\boldsymbol{v}}\langle\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}\langle\boldsymbol{\sigma},\boldsymbol{w}\rangle \\
&= \langle[\boldsymbol{\nabla}_{\boldsymbol{u}},\boldsymbol{\nabla}_{\boldsymbol{v}}]\boldsymbol{\sigma},\boldsymbol{w}\rangle + \langle\boldsymbol{\sigma},[\boldsymbol{\nabla}_{\boldsymbol{u}},\boldsymbol{\nabla}_{\boldsymbol{v}}]\boldsymbol{w}\rangle + \langle\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle + \langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}\rangle \\
&\quad - \langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}\rangle - \langle\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\sigma},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle - \langle\boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{\sigma},\boldsymbol{w}\rangle - \langle\boldsymbol{\sigma},\boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}\rangle \\
&= \langle\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{\sigma},\boldsymbol{w}\rangle + \langle\boldsymbol{\sigma},\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}\rangle
\end{aligned} \tag{11.11.2}$$

On the other hand, for any scalar function,

$$[\boldsymbol{\nabla}_{\boldsymbol{u}},\boldsymbol{\nabla}_{\boldsymbol{v}}]f - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}f = [\partial_{\boldsymbol{u}},\partial_{\boldsymbol{v}}]f - \partial_{[\boldsymbol{u},\boldsymbol{v}]}f = [\boldsymbol{u},\boldsymbol{v}]f - [\boldsymbol{u},\boldsymbol{v}]f = 0. \tag{11.11.3}$$

So we have

$$\langle\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{\sigma},\boldsymbol{w}\rangle + \langle\boldsymbol{\sigma},\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}\rangle = 0, \tag{11.11.4}$$

i.e.

$$\langle\mathcal{R}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle = -\langle\boldsymbol{\sigma},\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}\rangle. \tag{11.11.5}$$

11.12 ROTATION GROUP: RIEMANN CURVATURE

[Continuation of exercises 9.13,9.14, and 10.17.] Calculate the components of the Riemann curvature tensor for the rotation group's manifold $SO(3)$; use the basis of generators $\{\boldsymbol{e}_\alpha\}$.

Solution:

We know from 9.14 and 10.17 that ${c_{\alpha\beta}}^{\gamma} = \epsilon_{\alpha\beta\gamma}$ and ${\Gamma^{\alpha}}_{\beta\gamma} = -\epsilon_{\alpha\beta\gamma}/2$. So

$$\begin{aligned}
{R^{\alpha}}_{\beta\gamma\delta} &= {\Gamma^{\alpha}}_{\beta\delta,\gamma} - {\Gamma^{\alpha}}_{\beta\gamma,\delta} + {\Gamma^{\alpha}}_{\mu\gamma}{\Gamma^{\mu}}_{\beta\delta} - {\Gamma^{\alpha}}_{\mu\delta}{\Gamma^{\mu}}_{\beta\gamma} - {\Gamma^{\alpha}}_{\beta\mu}{c_{\gamma\delta}}^{\mu} \\
&= \frac{1}{4}\epsilon_{\alpha\mu\gamma}\epsilon_{\mu\beta\delta} - \frac{1}{4}\epsilon_{\alpha\mu\delta}\epsilon_{\mu\beta\gamma} + \frac{1}{2}\epsilon_{\alpha\beta\mu}\epsilon_{\gamma\delta\mu} \\
&= -\frac{1}{4}\delta^{\alpha\gamma}_{\beta\delta} + \frac{1}{4}\delta^{\alpha\delta}_{\beta\gamma} + \frac{1}{2}\delta^{\alpha\beta}_{\gamma\delta}.
\end{aligned} \tag{11.12.1}$$

The answer given in the textbook might be wrong.

Chapter 12

NEWTONIAN GRAVITY IN THE LANGUAGE OF CURVED SPACETIME

12.1 RIEMANN CURVATURE OF NEWTONIAN SPACETIME

Derive equation (12.5) for ${R^\alpha}_{\beta\gamma\delta}$ from equation (12.4) for ${\Gamma^\alpha}_{\beta\gamma}$.

Solution:

From (12.4), we have

$$ {\Gamma^j}_{00} = \partial\Phi/\partial x^j; \quad \text{all other } {\Gamma^\alpha}_{\beta\gamma} \text{ vanish.} \tag{12.1.1} $$

Firstly, as ${\Gamma^0}_{\beta\gamma} = 0$, we have ${R^0}_{\beta\gamma\delta} = {\Gamma^0}_{\beta\delta,\gamma} - {\Gamma^0}_{\beta\gamma,\delta} + {\Gamma^0}_{\mu\gamma}{\Gamma^\mu}_{\beta\delta} - {\Gamma^0}_{\mu\delta}{\Gamma^\mu}_{\beta\gamma} = 0$.

Secondly, as ${\Gamma^\alpha}_{j\gamma} = 0$, we have ${R^i}_{j\gamma\delta} = {\Gamma^i}_{j\delta,\gamma} - {\Gamma^i}_{j\gamma,\delta} + {\Gamma^i}_{\mu\gamma}{\Gamma^\mu}_{j\delta} - {\Gamma^i}_{\mu\delta}{\Gamma^\mu}_{j\gamma} = 0$.

Thirdly, ${R^i}_{0\gamma\delta} = {\Gamma^i}_{0\delta,\gamma} - {\Gamma^i}_{0\gamma,\delta} + {\Gamma^i}_{\mu\gamma}{\Gamma^\mu}_{0\delta} - {\Gamma^i}_{\mu\delta}{\Gamma^\mu}_{0\gamma} = {\Gamma^i}_{0\delta,\gamma} - {\Gamma^i}_{0\gamma,\delta} + {\Gamma^i}_{0\gamma}{\Gamma^0}_{0\delta} - {\Gamma^i}_{0\delta}{\Gamma^0}_{0\gamma} = {\Gamma^i}_{0\delta,\gamma} - {\Gamma^i}_{0\gamma,\delta}$.
So ${R^i}_{0j0} = -{R^i}_{00j} = {\Gamma^i}_{00,j} = \Phi_{,ij}$.

In conclusion, we have

$$ {R^i}_{0j0} = -{R^i}_{00j} = {\Gamma^i}_{00,j} = \Phi_{,ij}; \quad \text{all other } {R^\alpha}_{\beta\gamma\delta} \text{ vanish.} \tag{12.1.2} $$

12.2 NEWTONIAN FIELD EQUATION

Derive the geometric form (12.8) of the Newtonian field equation from (12.5) through (12.7).

Solution:

$$ 4\pi\rho = \nabla^2\Phi = \Phi_{,ii} = {R^i}_{0i0} = {R^\alpha}_{0\alpha0} = R_{00}. \tag{12.2.1} $$

12.3 GEODESIC DEVIATION DERIVED

Produce a third column for Box 11.4, one that carries out the "geometric analysis" in component notation using the Galilean connection coefficients (12.4) of Newtonian spacetime. Thereby achieve a deeper understanding of how the geometric analysis parallels the old Newtonian analysis.

The detailed solution to this exercise has been provided in Box 12.1 of the textbook.

12.4 CONNECTION COEFFICIENTS FOR ROTATING, ACCELERATING COORDINATES

Beginning with equation (12.4) for the connection coefficients of a Galilean coordinate system $\{x^\alpha(\mathcal{P})\}$, derive the connection coefficients (12.14) of the coordinate system $\{x^{\alpha'}(\mathcal{P})\}$ of equations (12.13). From this, verify that (12.15) are necessary and sufficient for $\{x^{\alpha'}(\mathcal{P})\}$ to be Galilean.

Solution:

The transformation law for Christoffel symbols is

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial^2 x^\mu}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\mu} + \Gamma^\alpha{}_{\beta\gamma}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}. \tag{12.4.1}$$

The coordinates transform as

$$x^{0'} = x^0, \quad x^{j'} = A_{j'k}x^k + a^{j'}, \tag{12.4.2}$$

where $A_{j'k}A_{i'k} = \delta_{i'j'}$. The inverse transformation is

$$x^0 = x^{0'}, \quad x^k = A_{j'k}x^{j'} - a^k, \tag{12.4.3}$$

where $a^k \equiv A_{j'k}a^{j'}$. In Galilean coordinates, we have $\Gamma^i{}_{00} = \Phi_{,i}$ and all other terms vanish. Now, we have

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial^2 x^\mu}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\mu} + \Gamma^i{}_{00}\frac{\partial x^{\alpha'}}{\partial x^i}\frac{\partial t}{\partial x^{\beta'}}\frac{\partial t}{\partial x^{\gamma'}}. \tag{12.4.4}$$

In details, we have

$$\begin{aligned}
\Gamma^{0'}{}_{\beta'\gamma'} &= \frac{\partial^2 x^\mu}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial t'}{\partial x^\mu} + \Gamma^i{}_{00}\frac{\partial t'}{\partial x^i}\frac{\partial t}{\partial x^{\beta'}}\frac{\partial t}{\partial x^{\gamma'}} = 0.\\
\Gamma^{i'}{}_{0'0'} &= \frac{\partial^2 x^\mu}{\partial t'^2}\frac{\partial x^{i'}}{\partial x^\mu} + \Gamma^i{}_{00}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'} = \frac{\partial^2 x^i}{\partial t'^2}A_{i'i} + \Phi_{,i}A_{i'i} = A_{i'i}(\ddot{A}_{j'i}x^{j'} - \ddot{a}^i) + \Phi_{,i'}.\\
\Gamma^{i'}{}_{0'j'} &= \frac{\partial^2 x^\mu}{\partial t'\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^\mu} + \Gamma^i{}_{00}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial t}{\partial t'}\frac{\partial t}{\partial x^{j'}} = \frac{\partial^2 x^i}{\partial t'\partial x^{j'}}A_{i'i} = \dot{A}_{j'i}A_{i'i}.\\
\Gamma^{i'}{}_{j'k'} &= \frac{\partial^2 x^\mu}{\partial x^{j'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^\mu} + \Gamma^i{}_{00}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial t}{\partial x^{j'}}\frac{\partial t}{\partial x^{k'}} = 0.
\end{aligned} \tag{12.4.5}$$

From this, the necessary and sufficient condition for $\{x^{\alpha'}(\mathcal{P})\}$ to be Galilean is

$$\dot{A}_{j'i} = 0, \quad \Phi' = \Phi - A_{i'i}x^{i'}\ddot{a}^i = \Phi - x^i\ddot{a}^i. \tag{12.4.6}$$

12.5 EINSTEIN'S ELEVATOR

Use the formalism of this chapter to discuss "Einstein's elevator" -i.e., the equivalence of "gravity" to an acceleration of one's reference frame. Which aspects of "gravity" are equivalent to an acceleration, and which are not?

Solution:

Gravity and acceleration can both contribute to the connection, but only gravity can contribute to the curvature tensor. i.e. only gravity can cause geodesic deviation.

12.6 GEODESIC DEVIATION ABOVE THE EARTH

A manned orbital laboratory is put into a circular orbit about the Earth [radius of orbit = r_0, angular velocity $= \omega = (M/r_0^3)^{1/2}$] An astronaut jettisons a bag of garbage and watches it move along its geodesic path. He observes its motion relative to (non-Galilean) space coordinates $\{x^{j'}(\mathcal{P})\}$ which (1) are Euclidean at each moment of universal time [$(\partial/\partial x^{j'}) \cdot (\partial/\partial x^{k'}) = \delta_{jk}$], (2) have origin at the laboratory's center, (3) have $\partial/\partial x'$ pointing away from the Earth, (4) have $\partial/\partial x'$ and $\partial/\partial y'$ in the plane of orbit. Use the equation of geodesic deviation to calculate the motion of the garbage bag in this coordinate system. Verify the answer by examining the Keplerian orbits of laboratory and garbage. Hints: (1) Calculate $R^{\alpha'}{}_{\beta'\gamma'\delta'}$ in this coordinate system by a trivial transformation of tensorial components. (2) Use equation (12.14) to calculate $\Gamma^{\alpha'}{}_{\beta'\gamma'}$ at the center of the laboratory (i.e., on the fiducial geodesic).

Solution:

(1) The transformation law of coordinates is

$$t' = t; \quad x' = x\cos\omega t + y\sin\omega t - r_0; \quad y' = -x\sin\omega t + y\cos\omega t; \quad z' = z; \tag{12.6.1}$$

and

$$t = t'; \quad x = x'\cos\omega t' - y'\sin\omega t' + r_0\cos\omega t'; \quad y = x'\sin\omega t' + y'\cos\omega t' + r_0\sin\omega t'; \quad z = z'. \tag{12.6.2}$$

The Jacobian matrix for transformation is

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega y' & \cos\omega t' & \sin\omega t' & 0 \\ -\omega(x'+r_0) & -\sin\omega t' & \cos\omega t' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12.6.3}$$

and

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\omega y & \cos\omega t' & -\sin\omega t' & 0 \\ \omega x & \sin\omega t' & \cos\omega t' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12.6.4}$$

In the original coordinates, we have

$$R^{i}{}_{0j0} = -R^{i}{}_{00j} = \Phi_{,ij} = \frac{M(r^2\delta_{ij} - 3x_i x_j)}{r^5}. \tag{12.6.5}$$

In the new coordinates, we have

$$R^{\alpha'}{}_{\beta'\gamma'\delta'} = R^{i}{}_{0j0}\frac{\partial x^{\alpha'}}{\partial x^{i}}\frac{\partial t}{\partial x^{\beta'}}\left(\frac{\partial x^{j}}{\partial x^{\gamma'}}\frac{\partial t}{\partial x^{\delta'}} - \frac{\partial t}{\partial x^{\gamma'}}\frac{\partial x^{j}}{\partial x^{\delta'}}\right). \tag{12.6.6}$$

The only nonvanishing terms of curvature tensor in the new coordinates are

$$R^{i'}{}_{0'j'0'} = -R^{i'}{}_{0'0'j'} = R^{i}{}_{0j0}\frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{j}}{\partial x^{j'}} = \frac{M[r^2\delta_{i'j'} - 3(x^{i'} - a^{i'})(x^{j'} - a^{j'})]}{r^5}, \tag{12.6.7}$$

where $a^{i'} = (-r_0, 0, 0)$ and $r^2 = (x' + r_0)^2 + y'^2 + z'^2$.

(2) Note that

$$A_{i'i} = \begin{pmatrix} \cos\omega t' & \sin\omega t' & 0 \\ -\sin\omega t' & \cos\omega t' & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.6.8}$$

and $a^{i'} = (-r_0, 0, 0)$. Using the results from exercise 12.4, we have

$$\Gamma^{i'}{}_{00} = \left(-\omega^2(x' + r_0) + \frac{M(x' + r_0)}{r^3}, -\omega^2 y' + \frac{My'}{r^3}, \frac{Mz'}{r^3}\right) \tag{12.6.9}$$

and

$$\Gamma^{i'}{}_{0'j'} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{12.6.10}$$

(3) We omit the prime notation for new coordinates in this part for simplicity. The geodesic of laboratory in the new coordinates is $t = t$, $x = 0$, $y = 0$, $z = 0$. In order to get the equation of motion of the garbage in the new coordinates, we only need to know the geodesic deviation of the garbage relative to the laboratory. The geodesic deviation equation is

$$\frac{d}{dt}\left(\frac{dn^\alpha}{dt} + \Gamma^\alpha{}_{\nu\beta}n^\nu u^\beta\right) + \Gamma^\alpha{}_{\mu\gamma}u^\gamma\left(\frac{dn^\mu}{dt} + \Gamma^\mu{}_{\nu\beta}n^\nu u^\beta\right) + R^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta = 0. \tag{12.6.11}$$

Since $u = (1, 0, 0, 0)$, $n^0 = 0$, we have

$$\frac{d}{dt}\left(\frac{dn^i}{dt} + \Gamma^i{}_{j0}n^j\right) + \Gamma^i{}_{j0}\left(\frac{dn^j}{dt} + \Gamma^j{}_{k0}n^k\right) + R^i{}_{0j0}n^j = 0. \tag{12.6.12}$$

or more explicitly,

$$\begin{aligned} &\frac{d^2n^x}{dt^2} - 2\omega\frac{dn^y}{dt} - \omega^2 n^x - \frac{2M}{r_0^3}n^x = 0; \\ &\frac{d^2n^y}{dt^2} + 2\omega\frac{dn^x}{dt} - \omega^2 n^y + \frac{M}{r_0^3}n^y = 0; \\ &\frac{d^2n^z}{dt^2} + \frac{M}{r_0^3}n^z = 0. \end{aligned} \tag{12.6.13}$$

As $\omega = (M/r_0^3)^{1/2}$, we have

$$\begin{aligned} &\frac{d^2n^x}{dt^2} - 2\sqrt{\frac{M}{r_0^3}}\frac{dn^y}{dt} - \frac{3M}{r_0^3}n^x = 0; \\ &\frac{d^2n^y}{dt^2} + 2\sqrt{\frac{M}{r_0^3}}\frac{dn^x}{dt} = 0; \\ &\frac{d^2n^z}{dt^2} + \frac{M}{r_0^3}n^z = 0. \end{aligned} \tag{12.6.14}$$

12.7 FROM NEWTON TO CARTAN

From the standard axioms of Newtonian theory (last part of Box 12.4) derive the geometric axioms (first part of Box 12.4). Suggested procedure: Verify each of the geometric axioms by a calculation in the Galilean coordinate system. Make free use of the calculations and results in Section 12.1.

Solution:

The first axiom of standard formulation states that there exist a universal time t, a set of Cartesian space coordinates x^j (called "Galilean coordinates"), and a Newtonian gravitational potential Φ. This is equivalent to the first axiom of geometric formulation: There exists a function t called "universal time", and a symmetric covariant derivative $\boldsymbol{\nabla}$ (with associated geodesics, parallel transport law, curvature operator, etc.). We now verify each of the geometric axioms by a calculation in the Galilean coordinate system.

The third axiom of standard formulation states that the equation of motion for a freely falling particle is

$$\frac{d^2x^j}{dt^2}+\frac{\partial\Phi}{\partial x^j}=0. \tag{12.7.1}$$

So in Galilean frame, we have

$$\Gamma^i{}_{00}=\Phi_{,i};\quad \text{all other terms of } \Gamma^\alpha{}_{\beta\gamma} \text{ all vanish.} \tag{12.7.2}$$

For any vector $\boldsymbol{u}$,

$$(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{d}t)_\alpha=-\Gamma^\mu{}_{\alpha\beta}(\boldsymbol{d}t)_\mu u^\beta=-\Gamma^0{}_{\alpha\beta}u^\beta=0, \tag{12.7.3}$$

i.e. the second axiom of geometric formulation holds in Galilean frame. As a result, if $\boldsymbol{w}$ is a spatial vector field, we have

$$\langle \boldsymbol{d}t,\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle=\boldsymbol{\nabla}_{\boldsymbol{u}}\langle \boldsymbol{d}t,\boldsymbol{w}\rangle-\langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{d}t,\boldsymbol{w}\rangle=0. \tag{12.7.4}$$

The curvature tensor in Galilean frame takes the form as

$$R^i{}_{0j0}=-R^i{}_{00j}=\Phi_{,ij};\quad \text{all other terms of } R^\alpha{}_{\beta\gamma\delta} \text{ all vanish.} \tag{12.7.5}$$

So, if $\boldsymbol{w}$ is spatial, for any $\boldsymbol{u}$ and $\boldsymbol{n}$, we have

$$R^\alpha{}_{\beta\gamma\delta}w^\beta u^\gamma n^\delta=R^\alpha{}_{i\gamma\delta}w^i u^\gamma n^\delta=0. \tag{12.7.6}$$

Meanwhile, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are spatial,

$$R^\alpha{}_{\beta\gamma\delta}v^\gamma w^\delta=R^\alpha{}_{\beta ij}v^i w^j=0. \tag{12.7.7}$$

As a consequence, the third and forth axiom of geometric formulation hold in Galilean frame.

The second axiom of standard formulation states that the density of mass ρ generates the Newtonian potential by Poisson's equation,

$$\boldsymbol{\nabla}^2\Phi=\Phi_{,ii}=4\pi\rho. \tag{12.7.8}$$

So $R_{00}\equiv R^i{}_{0i0}=\Phi_{ii}=4\pi\rho$ and all other terms of Ricci tensor vanish. So the fifth axiom of geometric formulation holds in Galilean frame.

In Galilean frame, we have $\Gamma^i{}_{jk}=0$, so the metric $g_{ij}=\delta_{ij}$ cefined for spatial vector is compatible with the covariant derivative. So the sixth axiom of geometric holds in Galilean frame.

The Jacobi curvature tensor is defined as

$$J^\alpha{}_{\beta\gamma\delta}=\frac{1}{2}(R^\alpha{}_{\gamma\beta\delta}+R^\alpha{}_{\delta\beta\gamma}). \tag{12.7.9}$$

So in Galilean frame, we have

$$J^i{}_{j00} = \Phi_{,ij}; \quad J^i{}_{00j} = J^i{}_{0j0} = -\frac{1}{2}\Phi_{,ij}. \tag{12.7.10}$$

And for spatial $\boldsymbol{v}$, $\boldsymbol{w}$ and any $\boldsymbol{n}$, $\boldsymbol{u}$,

$$J^\alpha{}_{\beta\gamma\delta} v_\alpha w^\beta u^\gamma n^\delta = J^i{}_{j00} v_i w^j u^0 n^0 = \Phi_{,ij} v^i w^j u^0 n^0 \tag{12.7.11}$$

and

$$J^\alpha{}_{\beta\gamma\delta} w_\alpha v^\beta u^\gamma n^\delta = J^i{}_{j00} w_i v^j u^0 n^0 = \Phi_{,ij} w^i v^j u^0 n^0 = \Phi_{,ji} w^j v^i u^0 n^0 = \Phi_{,ij} v^i w^j u^0 n^0. \tag{12.7.12}$$

So the seventh axiom of geometric formulation holds in Galilean frame.

The forth axiom of standard formulation states that "Ideal rods" measure the Galilean coordinate lengths; "ideal clocks" measure universal time. This is equivalent to the eighth axiom of geometric formulation.

12.8 FROM CARTAN TO NEWTON

From the geometric axioms of Newtonian theory (first part of Box 12.4) derive the standard axioms (last part of Box 12.4).

Solution:

(1) Pick three orthonormal, spatial basis vectors ($\boldsymbol{e}_i$ with $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$) at some event $\mathcal{P}_0$. Parallel transport each of them by arbitrary routes to all other events in spacetime.

(2) The third axiom of geometric formulation states that spatial vectors are unchanged by parallel transport around infinitesimal closed curves; i.e.

$$\mathcal{R}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{w} = 0 \text{ if } \boldsymbol{w} \text{ is spatial, for every } \boldsymbol{u} \text{ and } \boldsymbol{n}. \tag{12.8.1}$$

So the resultant vector fields $\boldsymbol{e}_j$ are independent of the arbitrary transport routes and $\boldsymbol{\nabla e}_j = 0$. And so

$$[\boldsymbol{e}_i, \boldsymbol{e}_j] = \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j - \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_i = 0. \tag{12.8.2}$$

(3) Pick an arbitrary "time line", which passes through each space slice (slice of constant t) once and only once. Parametrize it by t and select its tangent vector as the basis vector $\boldsymbol{e}_0$ at each event along it. Parallel transport each of these $\boldsymbol{e}_0$'s throughout its respective space slice by arbitrary routes.

(4) The forth axiom of geometric formulation states that all vectors are unchanged by parallel transport around infinitesimal, spatial, closed curves; i.e.,

$$\mathcal{R}(\boldsymbol{v}, \boldsymbol{w}) = 0 \text{ for every spatial} \boldsymbol{v} \text{ and } \boldsymbol{w}. \tag{12.8.3}$$

So the resultant field $\boldsymbol{e}_0$ is independent of the transport routes and $\boldsymbol{\nabla}_{\boldsymbol{e}_i} \boldsymbol{e}_0 = 0$. And so

$$[\boldsymbol{e}_i, \boldsymbol{e}_0] = \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_0 - \nabla_{\boldsymbol{e}_0} \boldsymbol{e}_i = 0. \tag{12.8.4}$$

(5) Because $[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = 0$ for all pairs of the four basis-vector fields, there exists a coordinate system ("Galilean coordinates") in which $\boldsymbol{e}_\alpha = \partial/\partial x^\alpha$.

(6) At the point $\mathcal{P}_0$, we have $\boldsymbol{e}_i$ with $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. As

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{e}_i, \boldsymbol{e}_j) = (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{e}_i) \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \cdot (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{e}_j) = 0, \tag{12.8.5}$$

$\boldsymbol{e}_i, \boldsymbol{e}_j = \delta_{ij}$ holds at every point. And because $\boldsymbol{\nabla}_\gamma \boldsymbol{e}_\beta$ vanishes unless $\beta = \gamma = 0$, $\Gamma^\alpha{}_{\beta\gamma} = 0$ unless $\beta = \gamma = 0$. The second axiom of geometric formulation states that $\boldsymbol{\nabla} \boldsymbol{d}t = 0$, so we know $\Gamma^0{}_{00} = 0$ holds. In conclusion, the only nonzero components of the connection coefficient are $\Gamma^i{}_{00}$.

(7) The seventh axiom of geometric formulation states that the Jacobi curvature operator is "self-adjoint" when operating on spatial vectors;i.e.

$$\boldsymbol{v} \cdot [\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{w}] = \boldsymbol{w} \cdot [\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{v}] \text{ for all spatial } \boldsymbol{v}, \boldsymbol{w}; \text{ and for any} \boldsymbol{u}, \boldsymbol{n}. \tag{12.8.6}$$

So we have $J^i{}_{j\alpha\beta} = J^j{}_{i\alpha\beta}$. As $J^i{}_{j00} = R^i{}_{0j0}$, $J^j{}_{i00} = R^j{}_{0i0}$, we have $R^i{}_{0j0} = R^j{}_{0i0}$, and so $\Gamma^i{}_{00,j} = \Gamma^j{}_{00,i}$. From this we can conclude that there exists a potential Φ such that $\Gamma^i{}_{00} = \Phi_{,i}$.

(8) The fifth axiom of geometric formulation states that the Ricci curvature tensor has the form

$$\boldsymbol{Ricci} = 4\pi\rho \boldsymbol{d}t \otimes \boldsymbol{d}t, \tag{12.8.7}$$

i.e. $R_{00} = 4\pi\rho$. It is easy to get $R_{00} = R^i{}_{0i0} = \Phi_{,ii}$ from previous analysis, so we have

$$\Phi_{,ii} = 4\pi\rho. \tag{12.8.8}$$

(9) The eighth axiom of geometric formulation states that "Ideal rods" measure the lengths that are calculated with the spatial metric; "ideal clocks" measure universal time t (or some multiple thereof); and "freely falling particles" move along geodesics of $\boldsymbol{\nabla}$. In the coordinates we build, it is equivalent to that "Ideal rods" measure the coordinate lengths; "ideal clocks" measure coordinate time. And the geodesics equation are

$$\frac{d^2t}{d\lambda^2} + \Gamma^0{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \frac{d^2t}{d\lambda^2} = 0 \tag{12.8.9}$$

and

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \frac{d^2x^i}{d\lambda^2} + \Phi_{,i}\frac{dt}{d\lambda}\frac{dt}{d\lambda} = 0. \tag{12.8.10}$$

It is equivalent to $\lambda = t$ and

$$\frac{d^2x^i}{dt^2} + \Phi_{,i} = 0. \tag{12.8.11}$$

12.9 SPATIAL METRIC ALLOWED BY OTHER AXIOMS

Show that the geometric axioms (1), (2), and (3) of Box 12.4 permit one to introduce a spatial metric satisfying axiom (6).

Solution:

Pick three spatial basis vectors $\boldsymbol{e}_i$ at some event $\mathcal{P}_0$. The first axiom ensures that we can parallel transport each of them by arbitrary routes to all other events in spacetime. The second axiom ensures that the resultant vector fields are spatial everywhere. The third axiom ensures that the resultant vector fields $\boldsymbol{e}_j$ are independent of the arbitrary transport routes and $\boldsymbol{\nabla}\boldsymbol{e}_j = 0$ holds automatically by construction.

We define the spatial metric by $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. For any spatial $\boldsymbol{w} = w^i\boldsymbol{e}_i$ and $\boldsymbol{v} = v^i\boldsymbol{e}_i$, and for any $\boldsymbol{u}$ whatsoever, we have

$$\nabla_{\boldsymbol{u}}(\boldsymbol{w} \cdot \boldsymbol{v}) = \nabla_{\boldsymbol{u}}(\sum_i w^i v^i) = \sum_i (w^i{}_{,j} v^i + v^i{}_{,j} w^i)u^j. \tag{12.9.1}$$

On the other hand, we have

$$(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}) \cdot \boldsymbol{v} = \boldsymbol{\nabla}_{\boldsymbol{u}}(w^i\boldsymbol{e}_i) \cdot \boldsymbol{v} = (\boldsymbol{\nabla}_{\boldsymbol{u}} w^i)\boldsymbol{e}_i \cdot \boldsymbol{v} = \sum_i w^i{}_{,j} v^i u^j \tag{12.9.2}$$

and similarly

$$(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v})\cdot\boldsymbol{w}=\sum_i v^i{}_{,j}w^iu^j. \tag{12.9.3}$$

Now we obtain

$$\nabla_{\boldsymbol{u}}(\boldsymbol{w}\cdot\boldsymbol{v})=(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w})\cdot\boldsymbol{v}+(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v})\cdot\boldsymbol{w}, \tag{12.9.4}$$

i.e. spatial metric satisfying axiom (6).

12.10 SPACETIME METRIC FORBIDDEN BY OTHER AXIOMS

Show that in Newtonian spacetime it is impossible to construct a nondegenerate spacetime metric $\boldsymbol{g}$, defined on all vectors, that is compatible with the covariant derivative in the sense that

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{g}(\boldsymbol{u},\boldsymbol{p})=\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n},\boldsymbol{p})+\boldsymbol{g}(\boldsymbol{n},\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{p}). \tag{12.10.1}$$

Solution:

We first assume the existence of such a metric $\boldsymbol{g}$ which is compatible with the covariant derivative. The compatible condition is equivalent to $\boldsymbol{\nabla g}=0$, and in components form we have

$$g_{\alpha\beta,\gamma}-\Gamma^\mu{}_{\alpha\gamma}g_{\mu\beta}-\Gamma^\mu{}_{\beta\gamma}g_{\alpha\mu}=0. \tag{12.10.2}$$

In the frame we build in exercise 12.8, the only nonzero components of the connection coefficient are $\Gamma^i{}_{00}$. It is easy to verify that $g_{ij,\alpha}=0$ so that g_{ij} is a constant. There must exist a rotation $x^i\to\Lambda^i{}_jx^j$ that can bring $g_{ij}=\delta_{ij}$ and the only nonzero components of the connection coefficient are still $\Gamma^i{}_{00}=\Phi_{,i}$. The only non vanishing components of $g_{\alpha\beta,\gamma}$ are

$$g_{00,0}=2\Phi_{,i}g_{0i},\quad g_{0i,0}=\Phi_{,i}. \tag{12.10.3}$$

So $0=g_{0i,j0}=g_{0i,0j}=\Phi_{,ij}$. This implies that $\rho=0$ everywhere in this spacetime. So for a non-empty Newtonian spacetime, the metric compatible with the covariant derivative can not exist.

Chapter 13

RIEMANNIAN GEOMETRY: METRIC AS FOUNDATION OF ALL

13.1 TEST WHETHER SPACETIME IS LOCAL LORENTZ

Prove that the above two-step procedure for testing whether spacetime is locally Lorentz is valid: i.e., prove that if the procedure says "yes," then there exists an orthonormal basis with $g_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$ at the event in question; if it says "no," then no such basis exists.

Solution:

(1) If there exit a timelike vector $\boldsymbol{u}$, then we choose $\boldsymbol{e}_{\hat{0}} = \boldsymbol{u}/\sqrt{-\boldsymbol{u}\cdot\boldsymbol{u}}$ and so we have $\boldsymbol{e}_{\hat{0}}\cdot\boldsymbol{e}_{\hat{0}} = -1$. Choose an arbitrary vector $\boldsymbol{v}_{\hat{1}}$ perpendicular to $\boldsymbol{u}$. After that, we choose an vector $\boldsymbol{v}_{\hat{2}}$ perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}_{\hat{1}}$. Finally, we choose an vector $\boldsymbol{v}_{\hat{3}}$ perpendicular to $\boldsymbol{u}$, $\boldsymbol{v}_{\hat{1}}$ and $\boldsymbol{v}_{\hat{3}}$. As all non-zero vectors perpendicular to $\boldsymbol{u}$ are spacelike, if we define $\boldsymbol{e}_{\hat{i}} = \boldsymbol{v}_{\hat{i}}/\sqrt{-\boldsymbol{v}_{\hat{i}}\cdot\boldsymbol{v}_{\hat{i}}}$, we have $\boldsymbol{e}_{\hat{i}}\cdot\boldsymbol{e}_{\hat{i}} = 1$. In conclusion, we have

$$g_{\hat{\alpha}\hat{\beta}} = \boldsymbol{e}_{\hat{\alpha}}\cdot\boldsymbol{e}_{\hat{\beta}} = \eta_{\alpha\beta}. \tag{13.1.1}$$

So the spacetime is locally Lorentz.

(2) If the spacetime is locally Lorentz, then $\boldsymbol{u} = \boldsymbol{e}_{\hat{0}}$ is a timelike vector. The vector perpendicular to $\boldsymbol{e}_{\hat{0}}$ must take the form as

$$\boldsymbol{v} = x_{\hat{1}}\boldsymbol{e}_{\hat{1}} + x_{\hat{2}}\boldsymbol{e}_{\hat{2}} + x_{\hat{3}}\boldsymbol{e}_{\hat{3}}. \tag{13.1.2}$$

So we have

$$\boldsymbol{v}\cdot\boldsymbol{v} = x_{\hat{1}}^2 + x_{\hat{2}}^2 + x_{\hat{3}}^2 > 0, \tag{13.1.3}$$

i.e. all non-zero vectors $\boldsymbol{v}$ perpendicular to $\boldsymbol{u}$ are spacelike.

13.2 PRACTICE WITH METRIC

The calculation in the exercise is straightforward and answers to this exercise have been given in the textbook.

13.3 MATHEMATICAL REPRESENTATION OF LOCAL LORENTZ FRAME

By definition, a local Lorentz frame at a given event $\mathcal{P}_0$ is the closest thing there to a global Lorentz frame. Thus, it should be a coordinate system with $g_{\mu\nu}(\mathcal{P}_0) = \eta_{\mu\nu}$ and with as many derivatives of $g_{\mu\nu}$ as possible vanishing at $\mathcal{P}_0$. Prove that there exist coordinates in which $g_{\mu\nu}(\mathcal{P}_0) = \eta_{\mu\nu}$ and $g_{\mu\nu,\rho}(\mathcal{P}_0) = 0$, but that $g_{\mu\nu,\rho\sigma}(\mathcal{P}_0) = 0$ cannot vanish in general. Hence, such coordinates are the mathematical representation of a local Lorentz frame.

Solution:

For an arbitrary coordinate frame, the metric is $g_{\mu\nu}$. In the new coordinate frame $x^{\mu'}$, we have

$$g_{\alpha'\beta'} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu}. \tag{13.3.1}$$

As a result, at the point $\mathcal{P}_0$, we have

$$g_{\alpha'\beta'} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu}; \tag{13.3.2}$$

$$g_{\alpha'\beta',\gamma'} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \frac{\partial x^\rho}{\partial x^{\gamma'}} g_{\mu\nu,\rho} + \frac{\partial^2 x^\mu}{\partial x^{\alpha'} \partial x^{\gamma'}} \frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu} + \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial^2 x^\nu}{\partial x^{\beta'} \partial x^{\gamma'}} g_{\mu\nu}; \tag{13.3.3}$$

and

$$\begin{aligned} g_{\alpha'\beta',\gamma'\delta'} = {} & \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \frac{\partial x^\rho}{\partial x^{\gamma'}} \frac{\partial x^\sigma}{\partial x^{\delta'}} g_{\mu\nu,\rho\sigma} + \frac{\partial^2 x^\mu}{\partial x^{\alpha'} \partial x^{\delta'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \frac{\partial x^\rho}{\partial x^{\gamma'}} g_{\mu\nu,\rho} + \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial^2 x^\nu}{\partial x^{\beta'} \partial x^{\delta'}} \frac{\partial x^\rho}{\partial x^{\gamma'}} g_{\mu\nu,\rho} \\ & + \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \frac{\partial^2 x^\rho}{\partial x^{\gamma'} \partial x^{\delta'}} g_{\mu\nu,\rho} + \frac{\partial^2 x^\mu}{\partial x^{\alpha'} \partial x^{\gamma'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \frac{\partial x^\rho}{\partial x^{\delta'}} g_{\mu\nu,\rho} + \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial^2 x^\nu}{\partial x^{\beta'} \partial x^{\gamma'}} \frac{\partial x^\rho}{\partial x^{\delta'}} g_{\mu\nu,\rho} \\ & + \frac{\partial^2 x^\mu}{\partial x^{\alpha'} \partial x^{\gamma'}} \frac{\partial^2 x^\nu}{\partial x^{\beta'} \partial x^{\delta'}} g_{\mu\nu} + \frac{\partial^3 x^\mu}{\partial x^{\alpha'} \partial x^{\gamma'} \partial x^{\delta'}} \frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu} + \frac{\partial^2 x^\mu}{\partial x^{\alpha'} \partial x^{\delta'}} \frac{\partial^2 x^\nu}{\partial x^{\beta'} \partial x^{\gamma'}} g_{\mu\nu} + \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial^3 x^\nu}{\partial x^{\beta'} \partial x^{\gamma'} \partial x^{\delta'}} g_{\mu\nu}. \end{aligned} \tag{13.3.4}$$

As for the equations

$$g_{\alpha'\beta'}(\mathcal{P}_0) = \eta_{\alpha'\beta'}, \quad g_{\alpha'\beta',\gamma'}(\mathcal{P}_0) = 0, \tag{13.3.5}$$

we have $10 + 10 \times 4 = 50$ independent equations. And we also have $4 \times 4 + 4 \times 10 = 56$ unknown constants $\partial x^\mu / \partial x^{\alpha'}(\mathcal{P}_0)$ and $\partial^2 x^\mu / \partial x^{\alpha'} \partial x^{\beta'}(\mathcal{P}_0)$. So the equations must have solutions. However, if we further demand that

$$g_{\alpha'\beta',\gamma'\delta'}(\mathcal{P}_0) = 0, \tag{13.3.6}$$

we have $10 \times 10 = 100$ more equations. But we only have $4 \times 20 = 80$ more unknown constants $\partial^3 x^\mu / \partial x^{\alpha'} \partial x^{\beta'} \partial x^{\gamma'}(\mathcal{P}_0)$. So the total number of equations is more than the the total number of unknown constants and equations have no solution generally.

13.4 CONSEQUENCES OF COMPATIBILITY BETWEEN g AND ∇

(a) From the condition of compatibility $\boldsymbol{\nabla g} = 0$, derive the chain rule (13.20). (b) From the condition of compatibility $\boldsymbol{\nabla g} = 0$ and definitions (13.21) and (13.22), derive equation (13.23) for the connection coefficients.

The detailed solution has been given in the "answer" part attached to exercise 8.15 of the textbook.

13.5 ONCE TIMELIKE, ALWAYS TIMELIKE

Show that a geodesic of spacetime which is timelike at one event is everywhere timelike. Similarly, show that a geodesic initially spacelike is everywhere spacelike, and a geodesic initially null is everywhere null.

Solution:

The tangent vector of a geodesic satisfies the geodesic equation

$$\frac{D\boldsymbol{u}}{ds} = 0. \tag{13.5.1}$$

So we have

$$\frac{d(\boldsymbol{u}\cdot\boldsymbol{u})}{ds} = \frac{D(\boldsymbol{u}\cdot\boldsymbol{u})}{ds} = 2\boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{ds} = 0, \tag{13.5.2}$$

i.e. a geodesic of spacetime which is timelike at one event is everywhere timelike, a geodesic initially spacelike is everywhere spacelike, and a geodesic initially null is everywhere null.

13.6 SPACELIKE GEODESICS HAVE EXTREMAL LENGTH

Show that any spacehke curve linking two events $\mathcal{A}$ and $\mathcal{B}$ is a geodesic if and only if it extremizes the proper length

$$s = \int_{\mathcal{A}}^{\mathcal{B}} (g_{\mu\nu}dx^\mu dx^\nu)^{1/2}. \tag{13.6.1}$$

Solution:

For a variation $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$, we have

$$\begin{aligned}
\delta s &= \int_{\mathcal{A}}^{\mathcal{B}} \frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2}\left[g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta x^\sigma(\lambda) + 2g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right]d\lambda \\
&= \int_{\mathcal{A}}^{\mathcal{B}} \left\{\frac{1}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2} g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d}{d\lambda}\left[\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2} g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\right]\right\}\delta x^\sigma(\lambda)d\lambda \\
&= \int_{\mathcal{A}}^{\mathcal{B}} \left[\frac{1}{2}\frac{d\lambda}{ds}g_{\mu\nu,\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d}{d\lambda}\left(\frac{d\lambda}{ds}g_{\mu\sigma}\frac{dx^\mu}{d\lambda}\right)\right]\delta x^\sigma(\lambda)d\lambda \\
&= \int_{\mathcal{A}}^{\mathcal{B}} \left[\frac{1}{2}g_{\mu\nu,\sigma}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - \frac{d}{ds}\left(g_{\mu\sigma}\frac{dx^\mu}{ds}\right)\right]\delta x^\sigma(\lambda)ds \\
&= \int_{\mathcal{A}}^{\mathcal{B}} -g_{\rho\sigma}\left[\Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{d^2x^\rho}{ds^2}\right]\delta x^\sigma(\lambda)ds
\end{aligned} \tag{13.6.2}$$

So $\delta s = 0$ is equivalent to

$$\frac{d^2x^\rho}{ds^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{13.6.3}$$

13.7 METRIC TENSOR MEASURED BY LIGHT SIGNALS AND FREE PARTICLES

(a) Instead of parametrizing a timelike geodesic by the proper time τ, parametrize it by an arbitrary parameter μ,

$$\tau = F(\mu). \tag{13.7.1}$$

Write the geodesic equation in the μ-parametrization.

(b) Use now the coordinate time t as a parameter. Throw out a cloud of free particles with different "velocities" $v^i = dx^i/dt$ and observe their "accelerations" $a^i = d^2x^i/dt^2$. Discuss what combinations of the components of the affine connection $\Gamma^\iota{}_{\kappa\lambda}$ one can measure in this way.

(c) Show that one can measure the conformal metric $\bar{g}_{\iota\kappa}$ i.e., the ratios of the components of the metric tensor $g_{\iota\kappa}$ to a given component

$$\bar{g}_{\iota\kappa} = \Lambda g_{\iota\kappa}, \quad \Lambda \equiv (-g_{00})^{-1}, \tag{13.7.2}$$

using only the light signals moving along the null geodesics $g_{\iota\kappa}dx^\iota dx^\kappa = 0$.

(d) Combine now the results of (b) and (c). Assume that $\Gamma^\iota{}_{\kappa\lambda}$ is generated by the metric tensor by (13.22), (13.23), in the coordinate frame x^ι. Show that one can determine Λ everywhere, if one prescribes it at one event (equivalent to fixing the unit of time).

Solution:

(a) As

$$\begin{aligned}\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= \frac{d}{d\mu}\left(\frac{dx^\rho}{d\mu}\frac{d\mu}{d\tau}\right)\frac{d\mu}{d\tau} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\mu}\frac{dx^\nu}{d\mu}\frac{d\mu}{d\tau}\frac{d\mu}{d\tau} \\ &= \left(\frac{d\mu}{d\tau}\right)^2\left(\frac{d^2x^\rho}{d\mu^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\mu}\frac{dx^\nu}{d\mu}\right) + \frac{d^2\mu}{d\tau^2},\end{aligned} \tag{13.7.3}$$

the geodesic equation can be written as

$$\frac{d\mu}{d\tau}\left(\frac{d^2x^\rho}{d\mu^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\mu}\frac{dx^\nu}{d\mu}\right) + \frac{d}{d\mu}\left(\frac{d\mu}{d\tau}\right) = 0. \tag{13.7.4}$$

(b) Let $\mu = t$ and define $dt/d\tau \equiv \gamma$, then we have

$$a^i + \Gamma^i{}_{00} + 2\Gamma^i{}_{0j}v^j + \Gamma^i{}_{jk}v^jv^k + \gamma^{-1}\dot{\gamma} = 0 \tag{13.7.5}$$

and

$$\Gamma^0{}_{00} + 2\Gamma^0{}_{0j}v^j + \Gamma^0{}_{jk}v^jv^k + \gamma^{-1}\dot{\gamma} = 0. \tag{13.7.6}$$

(c) (d) I have not figured out how to set up the measurements yet. Any help is appreciated.

13.8 RIEMANN ANTISYMMETRIC IN FIRST TWO INDICES

(a) Derive the antisymmetry condition (13.43).

(b) Explain in geometric terms the meaning of this antisymmetry.

Solution:

(a) For a scalar field f, we have

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{v}}f - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\nabla}_{\boldsymbol{u}}f = \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}}f - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}f = \partial_{[\boldsymbol{u},\boldsymbol{v}]}f = \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}f \tag{13.8.1}$$

by the defination of $[\boldsymbol{u},\boldsymbol{v}]$. So we have $\mathcal{R}(\boldsymbol{u},\boldsymbol{v})f = 0$. As $\nabla\boldsymbol{g} = 0$, we have

$$0 = \mathcal{R}(\boldsymbol{u},\boldsymbol{v})(\boldsymbol{s}\cdot\boldsymbol{w}) = \boldsymbol{s}\cdot\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} + \boldsymbol{w}\cdot\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{s}. \tag{13.8.2}$$

In terms of components, we would arrive at that $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta]\gamma\delta}$.

(b) Parallel transport a vector $\boldsymbol{A}$ around a path. When $\boldsymbol{A}$ gets back to the starting point, the difference with original $\boldsymbol{A}$ is $\delta\boldsymbol{A}$. Now we parallel transport the vector $\boldsymbol{A}$ around the same path in inverse direction. When $\boldsymbol{A}$ gets back to the starting point, the difference with original $\boldsymbol{A}$ must be $-\delta\boldsymbol{A}$.

13.9 NUMBER OF INDEPENDENT COMPONENTS OF RIEMANN

(a) In the absence of metric, a complete set of symmetry conditions for Riemann is $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}$ and $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{[\beta\gamma\delta]}$. Show that in four-dimensional spacetime these reduce the number of independent components from 256 to 80.

(b) Show that in a manifold of n dimensions without metric, the number of independent components is $n^2(n^2-1)/3$.

(c) In the presence of metric, a complete set of symmetries is $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}$ and $R_{\alpha\beta\gamma\delta} = R_{\alpha[\beta\gamma\delta]}$. Show that in four-dimensional spacetime, these reduce the number of independent components to 20.

(d) Show that in a manifold of n dimensions with metric, the number of independent com ponents is $n^2(n^2-1)/12$.

Solution:

(a) The condition $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}$ contains $4 \times 3/2 + 4 = 10$ independent constraints for each α and β. The condition $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{[\beta\gamma\delta]}$ imposes C_4^3 independent constraints for each α. So we have

$$256 - 4 \times 4 \times 10 - 4 \times 4 = 80 \tag{13.9.1}$$

independent components.

(b) In a manifold of n dimensions without metric, the number of independent components is

$$n^4 - n^2(C_n^2 + C_n^1) - nC_n^3 = \frac{n^2(n^2-1)}{3}. \tag{13.9.2}$$

(c) The condition $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}$ indicates that the pair (α, β) and (γ, δ) are unordered and unequivalent. So the number of remaining independent components are $6 \times 6 = 36$. The condition $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{[\beta\gamma\delta]}$ imposes C_4^3 more independent constraints for each α. So we have $36 - 16 = 20$ independent components.

(d) In a manifold of n dimensions with metric, the number of independent components is

$$C_n^2 C_n^2 - nC_n^3 = \frac{n^2(n^2-1)}{12}. \tag{13.9.3}$$

13.10 RIEMANN SYMMETRIC IN EXCHANGE OF PAIRS; COMPLETELY ANTISYMMETRIC PART VANISHES

From the complete set of symmetries in the presence of a metric, $R_{\alpha\beta\gamma\beta} = R_{[\alpha\beta][\gamma\delta]}$ and $R_{\alpha[\beta\gamma\delta]} = 0$, derive (a) symmetry under pair exchange, $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$, and (b) vanishing of completely antisymmetric part, $R_{[\alpha\beta\gamma\delta]} = 0$. Then (c) show that the following form a complete set of symmetries:

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta}, \quad R_{[\alpha\beta\gamma\delta]} = 0. \tag{13.10.1}$$

Solution:

(a) Since $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$ and $R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$, we have

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\alpha\gamma\delta\beta} - R_{\alpha\delta\beta\gamma} \\ &= R_{\gamma\alpha\delta\beta} + R_{\delta\alpha\beta\gamma} \\ &= -R_{\gamma\beta\alpha\delta} - R_{\gamma\delta\beta\alpha} - R_{\delta\gamma\alpha\beta} - R_{\delta\beta\gamma\alpha} \\ &= (R_{\beta\gamma\alpha\delta} + R_{\beta\delta\gamma\alpha}) + 2R_{\gamma\delta\alpha\beta} \\ &= -R_{\beta\alpha\delta\gamma} + 2R_{\gamma\delta\alpha\beta} \\ &= R_{\alpha\beta\gamma\delta} + 2R_{\gamma\delta\alpha\beta}, \end{aligned} \tag{13.10.2}$$

i.e. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$.

(b)

$$4R_{[\alpha\beta\gamma\delta]} = R_{\alpha[\beta\gamma\delta]} - R_{\beta[\gamma\delta\alpha]} + R_{\gamma[\delta\alpha\beta]} - R_{\delta[\alpha\beta\gamma]} = 0. \tag{13.10.3}$$

(c) If $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta}$, we have

$$\begin{aligned} 4R_{[\alpha\beta\gamma\delta]} &= R_{\alpha[\beta\gamma\delta]} - R_{\beta[\gamma\delta\alpha]} + R_{\gamma[\delta\alpha\beta]} - R_{\delta[\alpha\beta\gamma]} \\ &= 2R_{\alpha[\beta\gamma\delta]} - 2R_{\beta[\gamma\delta\alpha]} \\ &= 2R_{\alpha[\beta\gamma\delta]} - 2R_{\beta[\alpha\gamma\delta]} \\ &= 4R_{\alpha[\beta\gamma\delta]}. \end{aligned} \tag{13.10.4}$$

If $R_{[\alpha\beta\gamma\delta]} = 0$, we have

$$R_{\alpha[\beta\gamma\delta]} = 0. \tag{13.10.5}$$

13.11 DOUBLE DUAL OF RIEMANN: EINSTEIN

(a) Show that $\boldsymbol{\mathcal{G}} = {}^*\boldsymbol{Riemann}^*$ contains precisely the same amount of information as $\boldsymbol{Riemann}$, and satisfies precisely the same set of symmetries.

(b) From the symmetries of $\boldsymbol{\mathcal{G}}$, show that $\boldsymbol{Einstein}$ is symmetric.

(c) Show that the Bianchi identities (13.42), when written in terms of $\boldsymbol{\mathcal{G}}$, take the form $\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{G}} = 0$.

(d) By contractIng the Bianchi identities $\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{G}} = 0$, show that $\boldsymbol{G} = \boldsymbol{Einstein}$ has vanishing divergence.

Solution:

(a) The defination of $\mathcal{G}$ is

$$\mathcal{G}^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\rho\sigma} \frac{1}{2}\epsilon_{\rho\sigma\gamma\delta}. \tag{13.11.1}$$

So we have

$$\mathcal{G}^{\alpha\beta}{}_{\gamma\delta}\epsilon_{\alpha\beta\kappa\lambda} = \frac{1}{4}\delta^{\alpha\beta\mu\nu}_{\alpha\beta\kappa\lambda} R_{\mu\nu}{}^{\rho\sigma}\epsilon_{\rho\sigma\gamma\delta} = -\frac{1}{2}\delta^{\mu\nu}_{\kappa\lambda} R_{\mu\nu}{}^{\rho\sigma}\epsilon_{\rho\sigma\gamma\delta} = -R_{\kappa\lambda}{}^{\rho\sigma}\epsilon_{\rho\sigma\gamma\delta}. \tag{13.11.2}$$

Further more, we have

$$\mathcal{G}^{\alpha\beta}{}_{\gamma\delta}\epsilon_{\alpha\beta\kappa\lambda}\epsilon^{\mu\nu\gamma\delta} = -R_{\kappa\lambda}{}^{\rho\sigma}\epsilon_{\rho\sigma\gamma\delta}\epsilon^{\mu\nu\gamma\delta} = 2R_{\kappa\lambda}{}^{\rho\sigma}\delta^{\mu\nu}_{\rho\sigma} = 4R_{\kappa\lambda}{}^{\mu\nu}. \tag{13.11.3}$$

Now we can see that $\boldsymbol{\mathcal{G}}$ contains precisely the same amount of information as $\boldsymbol{Riemann}$. From the defination of $\boldsymbol{\mathcal{G}}$, it is easy to see that $\mathcal{G}^{\alpha\beta}{}_{\gamma\delta} = \mathcal{G}^{[\alpha\beta]}{}_{[\gamma\delta]}$. And we also have

$$\mathcal{G}_{\alpha\beta\gamma\delta} = \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon_{\rho\sigma\gamma\delta}R^{\mu\nu\rho\sigma} = \frac{1}{4}\epsilon_{\alpha\beta\rho\sigma}\epsilon_{\mu\nu\gamma\delta}R^{\rho\sigma\mu\nu} = \frac{1}{4}\epsilon_{\rho\sigma\alpha\beta}\epsilon_{\gamma\delta\mu\nu}R^{\mu\nu\rho\sigma} = \mathcal{G}_{\gamma\delta\alpha\beta}. \tag{13.11.4}$$

If α, β, γ and δ are all different, then it is easy to see that

$$\cancel{G}_{\alpha\beta\gamma\delta} = \frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon_{\rho\sigma\gamma\delta}R^{\mu\nu\rho\sigma} = \epsilon_{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}R^{\gamma\delta\alpha\beta} \text{ (no summation) } = R^{\gamma\delta\alpha\beta} = R^{\alpha\beta\gamma\delta}. \tag{13.11.5}$$

$R^{\alpha[\beta\gamma\delta]} = 0$ would imply that $\cancel{G}_{\alpha[\beta\gamma\delta]} = 0$.

If β, γ and δ are not all different, then $\cancel{G}_{\alpha[\beta\gamma\delta]} = 0$ can be obtained directly.

If α is the same as one of β, γ and δ, say $\alpha = \beta$, then we have

$$\cancel{G}_{\alpha\beta\gamma\delta} + \cancel{G}_{\alpha\gamma\delta\beta} + \cancel{G}_{\alpha\delta\beta\gamma} = \cancel{G}_{\alpha\gamma\delta\beta} + \cancel{G}_{\beta\delta\alpha\gamma} = 0. \tag{13.11.6}$$

In conclusion, $\cancel{\boldsymbol{G}} = {}^*\boldsymbol{Riemann}^*$ satisfies precisely the same set of symmetries as $\boldsymbol{Riemann}$.

(b)

$$G_{\beta\delta} = \cancel{G}^{\alpha}{}_{\beta\alpha\delta} = g^{\alpha\mu}\cancel{G}_{\mu\beta\alpha\delta} = g^{\alpha\mu}\cancel{G}_{\alpha\delta\mu\beta} = \cancel{G}^{\alpha}{}_{\delta\alpha\beta} = G_{\delta\beta}. \tag{13.11.7}$$

(c)

$$\cancel{G}^{\alpha\beta}{}_{\gamma\delta}{}^{;\delta} = \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}R_{\mu\nu}{}^{\rho\sigma;\delta}\epsilon_{\rho\sigma\gamma\delta} = \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}R_{\mu\nu}{}^{[\rho\sigma;\delta]}\epsilon_{\rho\sigma\gamma\delta} = 0. \tag{13.11.8}$$

(d)

$$G_{\beta\delta}{}^{;\delta} = \cancel{G}^{\alpha}{}_{\beta\alpha\delta}{}^{;\delta} = 0. \tag{13.11.9}$$

13.12 RICCI AND EINSTEIN RELATED

(a) From the symmetries of Riemann, show that Ricci is symmetric.

(b) Show that Ricci is related to Einstein by equation (13.49).

Solution:

(a)

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} = g^{\alpha\mu}R_{\mu\beta\alpha\delta} = g^{\alpha\mu}R_{\alpha\delta\mu\beta} = R^{\alpha}{}_{\delta\alpha\beta} = R_{\delta\beta}. \tag{13.12.1}$$

(b)

$$\begin{aligned} G^{\beta}{}_{\delta} &= -\frac{1}{4}\delta^{\alpha\beta\mu\nu}_{\alpha\delta\rho\sigma}R_{\mu\nu}{}^{\rho\sigma} = -\frac{1}{4}\delta^{\beta\mu\nu}_{\delta\rho\sigma}R_{\mu\nu}{}^{\rho\sigma} = -\frac{1}{4}(\delta^{\beta}_{\delta}\delta^{\mu\nu}_{\rho\sigma} + \delta^{\beta}_{\rho}\delta^{\mu\nu}_{\sigma\delta} + \delta^{\beta}_{\sigma}\delta^{\mu\nu}_{\gamma\rho})R_{\mu\nu}{}^{\rho\sigma} \\ &= -\frac{1}{2}(\delta^{\beta}_{\delta}R_{\rho\sigma}{}^{\rho\sigma} + \delta^{\beta}_{\rho}R_{\sigma\delta}{}^{\rho\sigma} + \delta^{\beta}_{\sigma}R_{\beta\rho}{}^{\rho\sigma}) \\ &= -\frac{1}{2}\delta^{\beta}_{\delta}R + R^{\beta}{}_{\delta} \end{aligned} \tag{13.12.2}$$

13.13 THE WEYL CONFORMAL TENSOR

(a) Show that the Weyl conformal tensor (13.50) possesses the same symmetries [(13.40), (13.41), (13.43) to (13.45)] as the Riemann tensor.

(b) Show that the Weyl tensor is completely "trace-free"; i.e., that contraction of $C^{\alpha}{}_{\beta\gamma\delta}$ on any pair of slots vanishes. Thus, $C^{\alpha}{}_{\beta\gamma\delta}$ can be regarded as the trace-free part of Riemann, and $R^{\alpha}{}_{\beta}$ can be regarded as the trace of Riemann. Riemann is determined entirely by its trace-free part $C^{\alpha}{}_{\beta\gamma\delta}$ and its trace $R^{\alpha}{}_{\beta}$.

(c) Show that in spacetime the Weyl tensor has 10 independent components.

(d) Show that in an n-dimensional manifold the number of independent components of Weyl is

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} \text{ for } n \geq 3, \quad 0 \text{ for } n \leq 3. \tag{13.13.1}$$

Thus, in manifolds of 1, 2, or 3 dimensions, the Weyl tensor is identically zero, and the Ricci tensor completely determines the Riemann tensor.

Solution:

(a) The defination of Weyl tensor is

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma}R^{\beta]}{}_{\delta]} + \frac{1}{3}\delta^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]}R. \tag{13.13.2}$$

So it is easy to see that $C_{\alpha\beta\gamma\delta} = C_{[\alpha\beta][\gamma\delta]}$. We can expand the Wely tensor to

$$C^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} - \frac{1}{2}(\delta^{\alpha}_{\gamma}R_{\beta\delta} - g_{\beta\gamma}R^{\alpha}_{\delta} - \delta^{\alpha}_{\delta}R_{\beta\gamma} + g_{\beta\delta}R^{\alpha}_{\gamma}) + \frac{1}{6}(\delta^{\alpha}_{\gamma}g_{\beta\delta} - g_{\beta\gamma}\delta^{\alpha}_{\delta})R. \tag{13.13.3}$$

So

$$C^{\alpha}{}_{[\beta\gamma\delta]} = R^{\alpha}{}_{[\beta\gamma\delta]} - (\delta^{\alpha}_{[\gamma}R_{\beta\delta]} - g_{[\beta\gamma}R^{\alpha}_{\delta]}) + \frac{1}{6}(\delta^{\alpha}_{[\gamma}g_{\beta\delta]})R = 0. \tag{13.13.4}$$

(b)

$$C^{\alpha}{}_{\beta\alpha\delta} = R_{\beta\delta} - \frac{1}{2}(4R_{\beta\delta} - R_{\beta\delta} - R_{\beta\delta} + g_{\beta\delta}R) + \frac{1}{6}(4g_{\beta\delta} - g_{\beta\delta})R = 0. \tag{13.13.5}$$

(c) The symmetries of $C^{\alpha\beta}{}_{\gamma\delta}$ left Wely tensor with 20 independent components. The vanishing of $C^{\alpha}{}_{\beta\alpha\delta}$ imposes 10 more constraints. So in spacetime the Weyl tensor has 10 independent components.

(d) The symmetries of $C^{\alpha\beta}{}_{\gamma\delta}$ left Wely tensor with $n^2(n^2-1)/12$ independent components, as indicated in exercise 13.9. The trace part of $R^{\alpha\beta}{}_{\gamma\delta}$ has $n(n+1)/2$ independent components when $n \geq 3$, 1 independent components when $n = 2$ and vanishes when $n = 1$. So in an n-dimensional manifold the number of independent components of Weyl is

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} \text{ for } n \geq 3, \quad 0 \text{ for } n \leq 3. \tag{13.13.6}$$

13.14 INERTIAL AND CORIOLIS FORCES

An accelerated observer studies the path of a freely falling particle as it passes through the origin of his proper reference frame. If

$$\boldsymbol{v} = (dx^{\hat{j}}/dx^{\hat{0}})\boldsymbol{e}_{\hat{j}} \tag{13.14.1}$$

is the particle's ordinary velocity, show that its ordinary acceleration relative to the observer's proper reference frame is

$$\frac{d^2x^{\hat{j}}}{dx^{\hat{0}^2}}\boldsymbol{e}_{\hat{j}} = -\boldsymbol{a} - 2\boldsymbol{\omega} \times \boldsymbol{v} + 2(\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{v}. \tag{13.14.2}$$

Here $\boldsymbol{a}$ is the observer's own 4-acceleration, and $\boldsymbol{\omega}$ is the angular velocity with which his spatial basis vectors $\boldsymbol{e}_{\hat{j}}$ are rotating.

Solution:

The geodesic equation in observer's frame is

$$\frac{d^2x^{\hat{0}}}{d\tau^2} + \Gamma^{\hat{0}}{}_{\hat{\alpha}\hat{\beta}} \frac{dx^{\hat{\alpha}}}{d\tau}\frac{dx^{\hat{\beta}}}{d\tau} = 0 \tag{13.14.3}$$

and

$$\frac{d^2x^{\hat{i}}}{d\tau^2} + \Gamma^{\hat{i}}{}_{\hat{\alpha}\hat{\beta}} \frac{dx^{\hat{\alpha}}}{d\tau}\frac{dx^{\hat{\beta}}}{d\tau} = 0. \tag{13.14.4}$$

Recall equation (13.69) in the textbook, we have

$$\frac{d^2x^{\hat{0}}}{d\tau^2} + 2a^{\hat{i}} \frac{dx^{\hat{i}}}{d\tau}\frac{dx^{\hat{0}}}{d\tau} = 0 \tag{13.14.5}$$

and

$$\frac{d^2x^{\hat{i}}}{d\tau^2} + a^{\hat{i}} \frac{dx^{\hat{0}}}{d\tau}\frac{dx^{\hat{0}}}{d\tau} - 2\epsilon_{\hat{i}\hat{j}\hat{k}}\omega^{\hat{k}} \frac{dx^{\hat{j}}}{d\tau}\frac{dx^{\hat{0}}}{d\tau} = 0. \tag{13.14.6}$$

Now we can get

$$\frac{d^2x^{\hat{0}}}{d\tau^2} + 2a^{\hat{i}}v^{\hat{i}}(\frac{dx^{\hat{0}}}{d\tau})^2 = 0 \tag{13.14.7}$$

and

$$\left(\frac{d^2x^{\hat{i}}}{dx^{\hat{0}2}} + a^{\hat{i}} - 2\epsilon_{\hat{i}\hat{j}\hat{k}}v^{\hat{j}}\omega^{\hat{k}}\right)\left(\frac{dx^{\hat{0}}}{d\tau}\right)^2 + v^{\hat{i}}\frac{d^2x^{\hat{0}}}{d\tau^2} = 0. \tag{13.14.8}$$

Finally, we would arrive at

$$\frac{d^2x^{\hat{j}}}{dx^{\hat{0}2}}\boldsymbol{e}_{\hat{j}} = -\boldsymbol{a} - 2\boldsymbol{\omega}\times\boldsymbol{v} + 2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v}. \tag{13.14.9}$$

13.15 ROTATION GROUP: METRIC

Show that for the manifold SO(3) of the rotation group, there exists a metric $\boldsymbol{g}$ that is compatible with the covariant derivative $\boldsymbol{\nabla}$. Prove existence by exhibiting the metric components explicitly in the noncoordinate basis of generators $\{\boldsymbol{e}_\alpha\}$.

Solution:

The compatible metric is $g_{\alpha\beta} \equiv \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = \delta_{\alpha\beta}$. We have

$$g_{\alpha\beta;\gamma} = \delta_{\alpha\beta,\gamma} - \Gamma^{\mu}{}_{\beta\gamma}\delta_{\alpha\mu} - \Gamma^{\mu}{}_{\alpha\gamma}\delta_{\mu\beta} = \frac{1}{2}(\epsilon_{\alpha\beta\gamma} + \epsilon_{\beta\alpha\gamma}) = 0. \tag{13.15.1}$$

So the metric is compatible with the connection.

Chapter 14

CALCULATION OF CURVATURE

14.1 CURVATURE OF A TWO-DIMENSIONAL HYPERBOLOID

Compute the curvature of the hyperboloid $t^2 - x^2 - y^2 = T^2 =$ const in $2+1$ Minkowski spacetime with $ds_3^2 = -dt^2 + dx^2 + dy^2$. First show that intervals within this two-dimensional surface can be expressed in the form $ds^2 = T^2(d\alpha^2 + \sinh^2\alpha d\phi^2)$ by a suitable choice of coordinates α, ϕ, on the hyperboloid.

Solution:

If we define

$$x = T\sinh\alpha\sin\phi; \quad y = T\sinh\alpha\cos\phi, \tag{14.1.1}$$

then we have $t = T\cosh\alpha$. In this coordinates, we have

$$ds^2 = T^2(d\alpha^2 + \sinh^2\alpha d\phi^2), \tag{14.1.2}$$

i.e.

$$g_{\mu\nu} = \begin{pmatrix} T^2 & 0 \\ 0 & T^2\sinh^2\alpha \end{pmatrix}. \tag{14.1.3}$$

The nonvanishing components of connection are

$$\Gamma^{\alpha}{}_{\phi\phi} = -\sinh\alpha\cosh\alpha; \quad \Gamma^{\phi}{}_{\alpha\phi} = \Gamma^{\phi}{}_{\phi\alpha} = \frac{\cosh\alpha}{\sinh\alpha}. \tag{14.1.4}$$

Now, we can get

$$R^{\alpha}{}_{\phi\alpha\phi} = \Gamma^{\alpha}{}_{\phi\phi,\alpha} - \Gamma^{\alpha}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\alpha} = -\sinh^2\alpha. \tag{14.1.5}$$

All other components of $\boldsymbol{Riemann}$ can be got by using symmetry of $\boldsymbol{Riemann}$. And so we can get

$$R^{\mu}{}_{\nu} = -\frac{1}{T^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14.1.6}$$

The Ricci scaler is $R = -2/T^2$.

14.2 RIEMANNIAN CURVATURE EXPRESSIBLE IN TERMS OF RICCI CURVATURE IN TWO AND THREE DIMENSIONS

In two dimensions, there is only one independent curvature component, R_{1212}. Evidently the single scalar quantity R must carry the same information. The two-dimensional identity $R_{\mu\nu\alpha\beta} = \frac{1}{2}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$

is established by noting that it is the only tensor formula giving $R_{\mu\nu\alpha\beta}$ as a linear function of R, constructed from R and the metric alone, and with the correct contracted value $R^{\mu\nu}{}_{\mu\nu} = R$. Establish a corresponding three-dimensional identity expressing R_{ijkl} in terms of the Ricci tensor R_{jk} and the metric.

Solution:

The corresponding three-dimensional identity expressing R_{ijkl} in terms of the Ricci tensor R_{jk} and the metric is

$$R_{ijkl} = g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik} + \frac{1}{2}R(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{14.2.1}$$

It is easy to verify that

$$R_{ijkl} = R_{[ij][kl]}, \quad R_{i[jkl]} = 0, \tag{14.2.2}$$

and

$$R^{i}{}_{jil} = 3R_{jl} - R_{jl} - R_{jl} + g_{jl}R + \frac{1}{2}R(3g_{jl} - g_{jl}) = R_{jl}. \tag{14.2.3}$$

14.3 CURVATURE OF 3-SPHERE IN ORTHONORMAL FRAME

Compute the curvature tensor for a 3-sphere

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{14.3.1}$$

or for a 3-hyperboloid

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{14.3.2}$$

Convert the coordinate-based components $R^{i}{}_{jkl}$ to a corresponding orthonormal basis, $R^{\hat{i}}{}_{\hat{j}\hat{k}\hat{l}}$. Display $R^{[\hat{i}\hat{j}]}{}_{[\hat{k}\hat{l}]}$ as a 3×3 matrix with appropriately labeled rows and columns.

Solution:

The nonvanishing components of $R^{i}{}_{jkl}$ for a 3-sphere are

$$\begin{aligned} &R^{1}{}_{212} = -R^{1}{}_{221} = \sin^2\chi; \quad R^{1}{}_{313} = -R^{1}{}_{331} = \sin^2\chi\sin^2\theta; \quad R^{2}{}_{323} = -R^{2}{}_{332} = \sin^2\chi\sin^2\theta; \\ &R^{2}{}_{121} = -R^{2}{}_{112} = 1; \quad R^{3}{}_{131} = -R^{3}{}_{113} = 1; \quad R^{3}{}_{232} = -R^{3}{}_{223} = \sin^2\chi. \end{aligned} \tag{14.3.3}$$

We can convert it to a corresponding orthonormal basis and we have

$$R^{[\hat{i}\hat{j}]}{}_{[\hat{k}\hat{l}]} = \frac{1}{a^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14.3.4}$$

The labels for each row and column are [12], [23], [31].

In the case of a 3-hyperboloid, we have

$$R^{[\hat{i}\hat{j}]}{}_{[\hat{k}\hat{l}]} = -\frac{1}{a^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{14.3.5}$$

14.4 EINSTEIN EQUATIONS FOR THE CLOSED FRIEDMANN UNIVERSE CALCULATED BY USING THE GEODESIC LAGRANGIAN METHOD

The line element of interest here is (see Chapter 27)

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{14.4.1}$$

(a) Set up the variational integral (14.10) for a geodesic in this metric, then successively vary t, χ, θ, and ϕ to obtain, after some rearrangement, four equations $0 = \ddot{t} + \cdots$, $0 = \ddot{\chi} + \cdots$, etc. displaying the Γ's in the form of equation (14.8).

(b) Use this display as a table of Γ's to compute $R^t{}_{\chi\mu\nu}$ and $R^\chi{}_{\theta\mu\nu}$ which only $R^t{}_{\chi t\chi}$ and $R^\chi{}_{\theta\chi\theta}$ are non-zero (consequence of the complete equivalence of all directions tangent to the $\chi\theta\phi$ sphere).

(c) Convert to an orthonormal frame with $\boldsymbol{\omega}^{\hat{t}} = \boldsymbol{d}t$, $\boldsymbol{\omega}^{\hat{\chi}} = a\boldsymbol{d}\chi$, $\boldsymbol{\omega}^{\hat{\theta}} = ?$, $\boldsymbol{\omega}^{\hat{\phi}} = ?$, and list $R^{\hat{t}\hat{\chi}}{}_{\hat{t}\hat{\chi}}$ and $R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}}$. Explain why all other components are known by symmetry in terms of these two.

(d) Calculate, using equations (14.7), all independent components of the EinsteIn tensor $G^{\hat{\mu}}{}_{\hat{\nu}}$.

Solution:

(a) The action integral is

$$I = \frac{1}{2}\int(-\dot{t}^2 + a^2(t)\dot{\chi}^2 + a^2(t)\sin^2\chi\dot{\theta}^2 + a^2(t)\sin^2\chi\sin^2\theta\dot{\phi}^2)d\tau. \tag{14.4.2}$$

The variation of $\delta t(\tau)$ gives

$$\ddot{t} + \dot{a}\dot{\chi}^2 + a\dot{a}\sin^2\chi\dot{\theta}^2 + a\dot{a}\sin^2\chi\sin^2\theta\dot{\phi}^2 = 0. \tag{14.4.3}$$

So we have

$$\Gamma^t{}_{\chi\chi} = aa'; \quad \Gamma^t{}_{\theta\theta} = aa'\sin^2\chi; \quad \Gamma^t{}_{\phi\phi} = aa'\sin^2\chi\sin^2\theta. \tag{14.4.4}$$

The variation of $\delta\chi(\tau)$ gives

$$a^2\ddot{\chi} + 2aa'\dot{t}\dot{\chi} - a^2\sin\chi\cos\chi\dot{\theta}^2 - a^2\sin\chi\cos\chi\sin^2\theta\dot{\theta}^2 = 0. \tag{14.4.5}$$

So we have

$$\Gamma^\chi{}_{t\chi} = \Gamma^\chi{}_{\chi t} = \frac{a'}{a}; \quad \Gamma^\chi{}_{\theta\theta} = -\sin\chi\cos\chi; \quad \Gamma^\chi{}_{\phi\phi} = -\sin^2\chi\sin\theta\cos\theta. \tag{14.4.6}$$

The variation of $\theta(\tau)$ gives

$$a^2\sin^2\chi\ddot{\theta} + 2aa'\sin^2\chi\dot{t}\dot{\theta} + 2a^2\sin\chi\cos\chi\dot{\chi}\dot{\theta} - a^2\sin^2\chi\sin\theta\cos\theta\dot{\phi}^2 = 0. \tag{14.4.7}$$

So we have

$$\Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \frac{a'}{a}; \quad \Gamma^\theta{}_{\chi\theta} = \Gamma^\theta{}_{\theta\chi} = \frac{\cos\chi}{\sin\chi}; \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta. \tag{14.4.8}$$

The variation of $\phi(\tau)$ gives

$$a^2\sin^2\chi\sin^2\theta\ddot{\phi} + 2aa'\sin^2\chi\sin^2\theta\dot{t}\dot{\phi} + 2a^2\sin\chi\cos\chi\sin^2\theta\dot{\chi}\dot{\phi} + 2a^2\sin^2\chi\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0. \tag{14.4.9}$$

So we have

$$\Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \frac{a'}{a}; \quad \Gamma^\phi{}_{\chi\phi} = \Gamma^\phi{}_{\phi\chi} = \frac{\cos\chi}{\sin\chi}; \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}. \tag{14.4.10}$$

(b)

$$R^t{}_{\chi t\chi} = -R^t{}_{\chi\chi t} = aa''; \quad R^\chi{}_{\theta\chi\theta} = -R^\chi{}_{\theta\theta\chi} = \sin^2\chi(1 + a'^2). \tag{14.4.11}$$

(c)

$$R^{\hat{t}\hat{\chi}}{}_{\hat{t}\hat{\chi}} = \frac{a''}{a}; \quad R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}} = \frac{1+a'^2}{a^2}. \tag{14.4.12}$$

By complete equivalence of all directions tangent to the $\chi\theta\phi$ sphere, we have

$$R^{\hat{t}\hat{\theta}}{}_{\hat{t}\hat{\theta}} = R^{\hat{t}\hat{\phi}}{}_{\hat{t}\hat{\phi}} = R^{\hat{t}\hat{\chi}}{}_{\hat{t}\hat{\chi}} = \frac{a''}{a} \tag{14.4.13}$$

and all other $R^{\hat{t}\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$s vanish. By complete equivalence of all directions tangent to the $\chi\theta\phi$ sphere, we have

$$R^{\hat{\chi}\hat{\phi}}{}_{\hat{\chi}\hat{\phi}} = R^{\hat{\theta}\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} = R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}} = \frac{1+a'^2}{a^2} \tag{14.4.14}$$

and all other $R^{\hat{\chi}\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$ and $R^{\hat{\theta}\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$s vanish.

(d)

$$G^{\hat{t}}{}_{\hat{t}} = -3R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}} = -\frac{3+3a'^2}{a^2}; \quad G^{\hat{\chi}}{}_{\hat{\chi}} = G^{\hat{\theta}}{}_{\hat{\theta}} = G^{\hat{\phi}}{}_{\hat{\phi}} = -2R^{\hat{t}\hat{\chi}}{}_{\hat{t}\hat{\chi}} - R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}} = -\frac{2aa''+a'^2+1}{a^2} \tag{14.4.15}$$

14.5 EXTERIOR DERIVATIVE OF A PRODUCT OF FORMS

Establish equation (14.13a) by working up recursively from forms of lower order to forms of higher order.

Solution:

Suppose

$$\boldsymbol{\alpha} = \alpha_{|\mu_1\cdots\mu_p|}\boldsymbol{d}x^{\mu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\mu_p} \tag{14.5.1}$$

and

$$\boldsymbol{\beta} = \beta_{|\nu_1\cdots\nu_q|}\boldsymbol{d}x^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\nu_q}. \tag{14.5.2}$$

Then we have

$$\boldsymbol{\alpha}\wedge\boldsymbol{\beta} = \alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|}\boldsymbol{d}x^{\mu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\mu_p}\wedge\boldsymbol{d}x^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\nu_q}. \tag{14.5.3}$$

So

$$\begin{aligned}\boldsymbol{d}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) &= (\alpha_{|\mu_1\cdots\mu_p|,\rho}\beta_{|\nu_1\cdots\nu_q|}+\alpha_{|\mu_1\cdots\mu_p|}\beta_{|\nu_1\cdots\nu_q|,\rho})\boldsymbol{d}x^{\rho}\wedge\boldsymbol{d}x^{\mu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\mu_p}\wedge\boldsymbol{d}x^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\nu_q}\\ &= \alpha_{|\mu_1\cdots\mu_p|,\rho}\boldsymbol{d}x^{\rho}\wedge\boldsymbol{d}x^{\mu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\mu_p}\wedge\beta_{|\nu_1\cdots\nu_q|}\boldsymbol{d}x^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\nu_q}\\ &+ (-1)^p\alpha_{|\mu_1\cdots\mu_p|}\boldsymbol{d}x^{\mu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\mu_p}\wedge\beta_{|\nu_1\cdots\nu_q|,\rho}\boldsymbol{d}x^{\rho}\wedge\boldsymbol{d}x^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}x^{\nu_q}\\ &= \boldsymbol{d}\alpha\wedge\boldsymbol{\beta}+(-1)^p\boldsymbol{\alpha}\wedge\boldsymbol{d}\beta.\end{aligned} \tag{14.5.4}$$

14.6 RELATIONSHIP BETWEEN EXTERIOR DERIVATIVE AND COMMUTATOR

Establish formula (14.21).

Solution:

Since both sides in (14.21) are linear with respect to $\boldsymbol{\alpha}$, we may assume that $\boldsymbol{\alpha}$ a monomial:

$$\boldsymbol{\alpha} = g\boldsymbol{d}f, \tag{14.6.1}$$

where f, g are smooth functions on M. Therefore

$$\boldsymbol{d}\boldsymbol{\alpha} = \boldsymbol{d}g \wedge \boldsymbol{d}f. \tag{14.6.2}$$

So we have

$$\langle \boldsymbol{d}\alpha, \boldsymbol{u} \wedge \boldsymbol{v}\rangle = \langle \boldsymbol{d}g \wedge \boldsymbol{d}f, \boldsymbol{u} \wedge \boldsymbol{v}\rangle = \begin{vmatrix} \langle \boldsymbol{d}g, \boldsymbol{u}\rangle & \langle \boldsymbol{d}g, \boldsymbol{v}\rangle \\ \langle \boldsymbol{d}f, \boldsymbol{u}\rangle & \langle \boldsymbol{d}f, \boldsymbol{v}\rangle \end{vmatrix} = \partial_{\boldsymbol{u}} g \partial_{\boldsymbol{v}} f - \partial_{\boldsymbol{u}} f \partial_{\boldsymbol{v}} g. \tag{14.6.3}$$

On the other hand,

$$\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle = \langle g\boldsymbol{d}f, \boldsymbol{v}\rangle = g\partial_{\boldsymbol{v}} f; \quad \langle \boldsymbol{\alpha}, \boldsymbol{u}\rangle = \langle g\boldsymbol{d}f, \boldsymbol{u}\rangle = g\partial_{\boldsymbol{u}} f. \tag{14.6.4}$$

So we have

$$\partial_{\boldsymbol{u}}\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle - \partial_{\boldsymbol{v}}\langle \boldsymbol{\alpha}, \boldsymbol{u}\rangle = \partial_{\boldsymbol{u}} g \partial_{\boldsymbol{v}} f - \partial_{\boldsymbol{u}} f \partial_{\boldsymbol{v}} g + g(\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}} - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}})f. \tag{14.6.5}$$

Note that

$$\langle \boldsymbol{\alpha}, [\boldsymbol{u}, \boldsymbol{v}]\rangle = g\langle \boldsymbol{d}f, [\boldsymbol{u}, \boldsymbol{v}]\rangle = g\partial_{[\boldsymbol{u},\boldsymbol{v}]} f = g(\partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}} - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}})f. \tag{14.6.6}$$

We can conclude that

$$\langle \boldsymbol{d}\alpha, \boldsymbol{u} \wedge \boldsymbol{v}\rangle = \partial_{\boldsymbol{u}}\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle - \partial_{\boldsymbol{v}}\langle \boldsymbol{\alpha}, \boldsymbol{u}\rangle + \langle \boldsymbol{\alpha}, [\boldsymbol{u}, \boldsymbol{v}]\rangle. \tag{14.6.7}$$

14.7 CHRISTOFFEL FORMULA DERIVED FROM CONNECTION FORMS

In a coordinate frame $\boldsymbol{\omega}^\mu = \boldsymbol{d}x^\mu$, show that equation (14.31a) requires $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ and that, with this symmetry established, equation (14.31 b) gives an expression for $\partial g_{\mu\nu}/\partial x^\alpha$ which can be solved to give the Christoffel formula

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{14.7.1}$$

Solution:

In the coordinate frame $\boldsymbol{\omega}^\mu = \boldsymbol{d}x^\mu$, we have

$$\boldsymbol{\omega}^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\boldsymbol{d}x^\rho. \tag{14.7.2}$$

So

$$0 = \boldsymbol{d}\boldsymbol{\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu = \Gamma^\mu{}_{\nu\rho}\boldsymbol{d}x^\rho \wedge \boldsymbol{d}x^\nu, \tag{14.7.3}$$

which implies $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$. We also have

$$g_{\mu\nu,\rho}\boldsymbol{d}x^\rho = \boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu} = (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho})\boldsymbol{d}x^\rho, \tag{14.7.4}$$

i.e.

$$g_{\mu\nu,\rho} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}, \tag{14.7.5}$$

which can lead to

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{14.7.6}$$

14.8 RIEMANN-CHRISTOFFEL CURVATURE FORMULA RELATED TO CURVATURE FORMS

Substitute $\boldsymbol{\omega}^{\mu}{}_{\nu} = \Gamma^{\mu}{}_{\nu\rho}\boldsymbol{d}x^{\rho}$ into equation (14.18), and from the result read out, according to equation (14.25), the classical formula (3) of Box 14.2 for the components $R^{\mu}{}_{\nu\alpha\beta}$.

Solution:

$$\begin{aligned}\mathcal{R}^{\mu}{}_{\nu} &= \boldsymbol{d}\boldsymbol{\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\lambda} \wedge \boldsymbol{\omega}^{\lambda}{}_{\nu} \\ &= \Gamma^{\mu}{}_{\nu\beta,\alpha}\boldsymbol{d}x^{\alpha} \wedge \boldsymbol{d}x^{\beta} + \Gamma^{\mu}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\nu\beta}\boldsymbol{d}x^{\alpha} \wedge \boldsymbol{d}x^{\beta} \\ &= R^{\mu}{}_{\nu|\alpha\beta|}\boldsymbol{d}x^{\alpha} \wedge \boldsymbol{d}x^{\beta}. \end{aligned} \tag{14.8.1}$$

14.9 MATRIX NOTATION FOR REVIEW OF CARTAN STRUCTURE EQUATIONS

Let $e = (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n)$ be a row matrix whose entries are the basis vectors, and let ω be a column of basis 1-forms $\boldsymbol{\omega}^{\mu}$. Similarly let $\Omega = ||\boldsymbol{\omega}^{\mu}{}_{\nu}||$ and $\mathcal{R} = ||\mathcal{R}^{\mu}{}_{\nu}||$ be square matrices with 1-form and 2-form entries. This gives a compact notation in which $\boldsymbol{d}\boldsymbol{e}_{\mu} = \boldsymbol{e}_{\nu}\boldsymbol{\omega}^{\nu}{}_{\mu}$ and $\boldsymbol{d}\mathcal{P} = \boldsymbol{e}_{\mu}\boldsymbol{\omega}^{\mu}$ read

$$\boldsymbol{d}e = e\Omega \text{ and } \boldsymbol{d}\mathcal{P} = e\omega, \tag{14.9.1}$$

respectively.

(a) From equations 14.9.1 and $\boldsymbol{d}^2\mathcal{P} = 0$, derive equation (14.31a) in the form

$$0 = \boldsymbol{d}\omega + \Omega \wedge \omega. \tag{14.9.2}$$

(b) Compute $\boldsymbol{d}^2 e$ as motivation for definition (14.18), which reads

$$\mathcal{R} = \boldsymbol{d}\Omega + \Omega \wedge \Omega. \tag{14.9.3}$$

(c) From $\boldsymbol{d}^2\omega$, deduce $\mathcal{R} \wedge \omega = 0$ and then decompress the notation to get the antisymmetry relation $R^{\mu}{}_{[\alpha\beta\gamma]} = 0$.

(d) Compute $\boldsymbol{d}\mathcal{R}$ from equation 14.9.3, and relate it to the Bianchi identity $R^{\mu}{}_{\nu[\alpha\beta;\gamma]} = 0$.

(f) Let $v = \{v^{\mu}\}$ be a column of functions; so $\boldsymbol{v} = ev = \boldsymbol{e}_{\mu}v^{\mu}$ is a vector field. Compute, in compact notation, $\boldsymbol{d}\boldsymbol{v}$ and $\boldsymbol{d}^2\boldsymbol{v}$ to show $\boldsymbol{d}^2\boldsymbol{v} = e\mathcal{R}v$ (which is equation 14.17).

Solution:

(a)

$$0 = \boldsymbol{d}^2\mathcal{P} = \boldsymbol{d}(e\omega) = \boldsymbol{d}e \wedge \omega + e\boldsymbol{d}\omega = e(\Omega \wedge \omega + \boldsymbol{d}\omega), \tag{14.9.4}$$

i.e.

$$0 = \Omega \wedge \omega + \boldsymbol{d}\omega. \tag{14.9.5}$$

(b)

$$\boldsymbol{d}^2 e = \boldsymbol{d}(e\Omega) = \boldsymbol{d}e \wedge \Omega + e\boldsymbol{d}\Omega = \boldsymbol{e}(\Omega \wedge \Omega + \boldsymbol{d}\Omega). \tag{14.9.6}$$

As $\boldsymbol{d}^2\boldsymbol{e}_v = \boldsymbol{e}_\mu \mathcal{R}^\mu{}_\nu$, we have

$$\mathcal{R} = \boldsymbol{d}\Omega + \Omega \wedge \Omega. \tag{14.9.7}$$

(c)

$$0 = \boldsymbol{d}(\boldsymbol{d}\omega + \Omega \wedge \omega) = \boldsymbol{d}\Omega \wedge \omega - \Omega \wedge \boldsymbol{d}\omega = (\boldsymbol{d}\Omega + \Omega \wedge \Omega) \wedge \omega = \mathcal{R} \wedge \omega. \tag{14.9.8}$$

As

$$\mathcal{R}^\mu{}_\alpha = \frac{1}{2} R^\mu{}_{\alpha\beta\gamma} \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma, \tag{14.9.9}$$

so we have

$$\mathcal{R}^\mu{}_\alpha \wedge \boldsymbol{\omega}^\alpha = \frac{1}{2} R^\mu{}_{\alpha\beta\gamma} \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha = 0, \tag{14.9.10}$$

i.e.

$$R^\mu{}_{[\alpha\beta\gamma]} = 0. \tag{14.9.11}$$

(d)

$$\begin{aligned} \boldsymbol{d}\mathcal{R} &= \boldsymbol{d}(\boldsymbol{d}\Omega + \Omega \wedge \Omega) \\ &= \boldsymbol{d}\Omega \wedge \Omega - \Omega \wedge \boldsymbol{d}\Omega \\ &= (\mathcal{R} - \Omega \wedge \Omega) \wedge \Omega - \Omega \wedge (\mathcal{R} - \Omega \wedge \Omega) \\ &= \mathcal{R} \wedge \Omega - \Omega \wedge \mathcal{R}. \end{aligned} \tag{14.9.12}$$

It can be expanded as

$$\begin{aligned} & R^\mu{}_{\nu\alpha\beta,\gamma} \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta + R^\mu{}_{\nu\alpha\beta} \boldsymbol{d}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta - R^\mu{}_{\nu\alpha\beta} \boldsymbol{\omega}^\alpha \wedge \boldsymbol{d}\boldsymbol{\omega}^\beta \\ &= (R^\mu{}_{\rho\gamma\alpha} \Gamma^\rho{}_{\nu\beta} - \Gamma^\mu{}_{\rho\gamma} R^\rho{}_{\nu\alpha\beta}) \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta. \end{aligned} \tag{14.9.13}$$

On the other hand, we have

$$R^\mu{}_{\nu\alpha\beta;\gamma} = R^\mu{}_{\nu\alpha\beta,\gamma} + \Gamma^\mu{}_{\rho\gamma} R^\rho{}_{\nu\alpha\beta} - \Gamma^\rho{}_{\nu\gamma} R^\mu{}_{\rho\alpha\beta} - \Gamma^\rho{}_{\alpha\gamma} R^\mu{}_{\nu\rho\beta} - \Gamma^\rho{}_{\beta\gamma} R^\mu{}_{\nu\alpha\rho}. \tag{14.9.14}$$

From (a) we know

$$\Gamma^\alpha{}_{\beta\gamma} \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\beta + \boldsymbol{d}\boldsymbol{\omega}^\alpha = 0. \tag{14.9.15}$$

So we have

$$\begin{aligned} & R^\mu{}_{\nu\alpha\beta;\gamma} \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \\ &= (R^\mu{}_{\nu\alpha\beta,\gamma} + \Gamma^\mu{}_{\rho\gamma} R^\rho{}_{\nu\alpha\beta} - \Gamma^\rho{}_{\nu\gamma} R^\mu{}_{\rho\alpha\beta}) \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \\ &- R^\mu{}_{\nu\rho\beta} \boldsymbol{\omega}^\beta \wedge \Gamma^\rho{}_{\alpha\gamma} \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha - R^\mu{}_{\nu\alpha\rho} \boldsymbol{\omega}^\alpha \wedge \Gamma^\rho{}_{\beta\gamma} \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma \\ &= (R^\mu{}_{\nu\alpha\beta,\gamma} + \Gamma^\mu{}_{\rho\gamma} R^\rho{}_{\nu\alpha\beta} - \Gamma^\rho{}_{\nu\beta} R^\mu{}_{\rho\gamma\alpha}) \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta + R^\mu{}_{\nu\rho\beta} \boldsymbol{\omega}^\beta \wedge \boldsymbol{d}\boldsymbol{\omega}^\rho - R^\mu{}_{\nu\alpha\rho} \boldsymbol{\omega}^\alpha \wedge \boldsymbol{d}\boldsymbol{\omega}^\rho \\ &= (R^\mu{}_{\nu\alpha\beta,\gamma} + \Gamma^\mu{}_{\rho\gamma} R^\rho{}_{\nu\alpha\beta} - \Gamma^\rho{}_{\nu\beta} R^\mu{}_{\rho\gamma\alpha}) \boldsymbol{\omega}^\gamma \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta + R^\mu{}_{\nu\alpha\beta} \boldsymbol{d}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta - R^\mu{}_{\nu\alpha\beta} \boldsymbol{\omega}^\alpha \wedge \boldsymbol{d}\boldsymbol{\omega}^\beta \\ &= 0, \end{aligned} \tag{14.9.16}$$

i.e $R^\mu{}_{\nu[\alpha\beta;\gamma]} = 0$.

(f)Firstly, we have

$$\boldsymbol{d}\boldsymbol{v} = \boldsymbol{d}e v + e\boldsymbol{d}v. \tag{14.9.17}$$

Then we can get

$$\boldsymbol{d}^2\boldsymbol{v} = \boldsymbol{d}^2 e v - \boldsymbol{d}e \wedge \boldsymbol{d}v + \boldsymbol{d}e \wedge \boldsymbol{d}v + e\boldsymbol{d}^2 v = \boldsymbol{d}^2 e v = e\mathcal{R}v. \tag{14.9.18}$$

14.10 TRANSFORMATION RULES FOR CONNECTION FORMS IN COMPACT NOTATION

Using the notation of the previous exercise, write $e' = eA$ in place of $\boldsymbol{e}_{\mu'} = \boldsymbol{e}_\nu A^\nu{}_{\mu'}$, and similarly $\omega' = A^{-1}\omega$, to represent a change of frame. Show that $\boldsymbol{d}\mathcal{P} \equiv e\omega = e'\omega'$. Substitute $e' = eA$ in $\boldsymbol{d}e' = e'\Omega'$ to deduce the transformation law

$$\Omega' = A^{-1}\Omega A + A^{-1}\boldsymbol{d}A. \tag{14.10.1}$$

Rewrite this in decompressed notation for coordinate frames with $A^\nu{}_{\mu'} = \partial x^\nu/\partial x^{\mu'}$ as a formula of the form $\Gamma^{\mu'}{}_{\alpha'\beta'} =$(?).

Solution:

As

$$\boldsymbol{d}e' = \boldsymbol{d}(eA) = \boldsymbol{d}eA + e\boldsymbol{d}A = e(\Omega A + \boldsymbol{d}A) = e'(A^{-1}\Omega A + A^{-1}\boldsymbol{d}A), \tag{14.10.2}$$

so we have

$$\Omega' = A^{-1}\Omega A + A^{-1}\boldsymbol{d}A. \tag{14.10.3}$$

The equation above can be written as

$$\begin{aligned}\Gamma^{\mu'}{}_{\alpha'\beta'}\boldsymbol{d}x^{\beta'} &= \frac{\partial x^{\mu'}}{\partial x^\mu}\Gamma^\mu{}_{\alpha\beta}\boldsymbol{d}x^\beta\frac{\partial x^\alpha}{\partial x^{\alpha'}} + \frac{\partial x^{\mu'}}{\partial x^\mu}\boldsymbol{d}\left(\frac{\partial x^\mu}{\partial x^{\alpha'}}\right)\\ &= \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\mu{}_{\alpha\beta} + \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\alpha'}\partial x^{\beta'}}\right)\boldsymbol{d}x^{\beta'}.\end{aligned} \tag{14.10.4}$$

So we have

$$\Gamma^{\mu'}{}_{\alpha'\beta'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\mu{}_{\alpha\beta} + \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\alpha'}\partial x^{\beta'}}. \tag{14.10.5}$$

14.11 SPACE IS FLAT IF THE CURVATURE VANISHES

If coordinates exist in which all straight lines ($d^2x^\mu/d\lambda^2$) are geodesics, then one says the space is flat. Evidently all $\Gamma^\mu{}_{\alpha\beta}$ and $R^\mu{}_{\nu\alpha\beta}$ vanish in this case, by equation (14.8) and equation (3) in Box 14.2. Show conversely that, if $\mathcal{R} = 0$, then such coordinates exist. Use the results of the previous problem to find differential equations for a transformation A to a basis e' where $\Omega' = 0$. What are the conditions for complete integrability of these equations? Why will the basis forms $\boldsymbol{\omega}^{\mu'}$ in this new frame be coordinate differentials $\boldsymbol{\omega}^{\mu'} = \boldsymbol{d}x^{\mu'}$?

Solution:

If $\Omega' = 0$ in new bases, the transformation matrix would satisfy that

$$\boldsymbol{d}A = -\Omega A. \tag{14.11.1}$$

Because

$$\mathcal{R} = \boldsymbol{d}\Omega + \Omega\wedge\Omega = 0, \tag{14.11.2}$$

we have

$$\boldsymbol{d}^2A = -\boldsymbol{d}\Omega A + \Omega\wedge\boldsymbol{d}A = \Omega\wedge(\Omega A + \boldsymbol{d}A) = 0. \tag{14.11.3}$$

As a result, the solution of 14.14.1 exists. In the new bases, we have

$$\boldsymbol{d}\omega' = -\Omega'\wedge\omega' = 0. \tag{14.11.4}$$

So the basis forms $\boldsymbol{\omega}^{\mu'}$ are coordinate differentials $\boldsymbol{\omega}^{\mu'} = \boldsymbol{d}x^{\mu'}$.

14.12 SYSTEMATIC COMPUTATION OF CONNECTION FORMS IN ORTHONORMAL FRAMES

Deduce equation (14.32) by applying equation (14.21) to basis vectors, using equations (8.14) to define $c_{\mu\nu}{}^{\alpha}$. Then show that, in an orthonormal frame (or any frames with $g_{\mu\nu}$ = const), equation (14.33) provides a solution of equations (14.31), which define $\boldsymbol{\omega}^{\mu}{}_{\nu}$ [Compare also equation (8.24b).]

Solution:

From equation (14.21), we have

$$\langle \boldsymbol{d\omega}^{\alpha}, \boldsymbol{e}_{\mu} \wedge \boldsymbol{e}_{\nu} \rangle = \partial_{\mu} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\nu} \rangle - \partial_{\nu} \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\mu} \rangle - \langle \boldsymbol{\omega}^{\alpha}, [\boldsymbol{e}_{\mu}, \boldsymbol{e}_{\nu}] \rangle = -\langle \boldsymbol{\omega}^{\alpha}, c_{\mu\nu}{}^{\beta} \boldsymbol{e}_{\beta} \rangle = -c_{\mu\nu}{}^{\alpha}. \tag{14.12.1}$$

So we have

$$\boldsymbol{d\omega}^{\alpha} = -c_{\mu\nu}{}^{\alpha} \boldsymbol{\omega}^{\mu} \otimes \boldsymbol{\omega}^{\nu} = -c_{|\mu\nu|}{}^{\alpha} \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu}. \tag{14.12.2}$$

If

$$\boldsymbol{\omega}_{\mu\nu} = \frac{1}{2}(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu}) \boldsymbol{\omega}^{\alpha}, \tag{14.12.3}$$

then we can derive that

$$\begin{aligned} & \boldsymbol{d\omega}^{\alpha} + \boldsymbol{\omega}^{\alpha}{}_{\nu} \wedge \boldsymbol{\omega}^{\nu} \\ &= -c_{|\mu\nu|}{}^{\alpha} \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu} + \frac{1}{2}(c^{\alpha}{}_{\nu\mu} + c^{\alpha}{}_{\mu\nu} - c_{\nu\mu}{}^{\alpha}) \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu} \\ &= -c_{|\mu\nu|}{}^{\alpha} \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu} - c_{|\nu\mu|}{}^{\alpha} \boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu} \\ &= 0 \end{aligned} \tag{14.12.4}$$

and

$$\boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu} = \frac{1}{2}(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu} + c_{\nu\mu\alpha} + c_{\nu\alpha\mu} - c_{\mu\alpha\nu}) \boldsymbol{\omega}^{\alpha} = 0. \tag{14.12.5}$$

So equation (14.33) provides a solution of equations (14.31).

14.13 SCHWARZSCHILD CURVATURE FORMS

Use the obvious orthonormal frame $\boldsymbol{\omega}^{\hat{t}} = e^{\Phi} \boldsymbol{dt}$, $\boldsymbol{\omega}^{\hat{r}} = e^{\Lambda} \boldsymbol{dr}$, $\boldsymbol{\omega}^{\hat{\theta}} = r \boldsymbol{d\theta}$, $\boldsymbol{\omega}^{\hat{\phi}} = r \sin\theta \boldsymbol{d\phi}$ for the Schwarzschild metric

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dt^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{14.13.1}$$

in which Φ and Λ are functions of r only; and compute the curvature forms $\mathcal{R}^{\hat{\mu}}{}_{\hat{\nu}}$ and the Einstein tensor $G^{\hat{\mu}}{}_{\hat{\nu}}$ by the methods of Box 14.5.

Solution:

(a) From the defination of $\boldsymbol{\omega}^{\hat{\mu}}$, we have

$$\begin{aligned} & \boldsymbol{d\omega}^{\hat{t}} = e^{\Phi} \Phi' \boldsymbol{dr} \wedge \boldsymbol{dt} = e^{-\Lambda} \Phi' \boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{t}}; \\ & \boldsymbol{d\omega}^{\hat{r}} = e^{\Lambda} \Lambda' \boldsymbol{dr} \wedge \boldsymbol{dr} = 0; \\ & \boldsymbol{d\omega}^{\hat{\theta}} = \boldsymbol{dr} \wedge \boldsymbol{d\theta} = e^{-\Lambda} r^{-1} \boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}}; \\ & \boldsymbol{d\omega}^{\hat{\phi}} = \sin\theta \boldsymbol{dr} \wedge \boldsymbol{d\phi} + r \cos\theta \boldsymbol{d\theta} \wedge \boldsymbol{d\phi} = e^{-\Lambda} r^{-1} \boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + r^{-1} \cot\theta \boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}. \end{aligned} \tag{14.13.2}$$

(b) From $\boldsymbol{d\omega}^{\mu} + \boldsymbol{\omega}^{\mu}{}_{\nu} \wedge \boldsymbol{\omega}^{\nu} = 0$ and $\boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu} = 0$, we can infer that

$$\boldsymbol{\omega}^{\hat{t}}{}_{\hat{r}} = e^{-\Lambda} \Phi' \boldsymbol{\omega}^{\hat{t}}; \quad \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{r}} = e^{-\Lambda} r^{-1} \boldsymbol{\omega}^{\hat{\theta}}; \quad \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{r}} = e^{-\Lambda} r^{-1} \boldsymbol{\omega}^{\hat{\phi}}; \quad \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\theta}} = r^{-1} \cot\theta \boldsymbol{\omega}^{\hat{\phi}}. \tag{14.13.3}$$

(c) From $\mathcal{R}^{\mu}{}_{\nu} = \boldsymbol{d\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\lambda} \wedge \boldsymbol{\omega}^{\lambda}{}_{\nu}$, we have

$$\begin{aligned}
\mathcal{R}^{\hat{t}}{}_{\hat{r}} &= \boldsymbol{d\omega}^{\hat{t}}{}_{\hat{r}} = -e^{-2\Lambda}(-\Lambda'\Phi' + \Phi'' + \Phi'^2)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{r}} \equiv E\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{r}} \\
\mathcal{R}^{\hat{t}}{}_{\hat{\theta}} &= \boldsymbol{\omega}^{\hat{t}}{}_{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{r}}{}_{\hat{\theta}} = -r^{-1}e^{-2\Lambda}\Phi'\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} \equiv \bar{E}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} \\
\mathcal{R}^{\hat{t}}{}_{\hat{\phi}} &= \boldsymbol{\omega}^{\hat{t}}{}_{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{r}}{}_{\hat{\phi}} = -r^{-1}e^{-2\Lambda}\Phi'\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} = \bar{E}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} \\
\mathcal{R}^{\hat{r}}{}_{\hat{\theta}} &= \boldsymbol{d\omega}^{\hat{r}}{}_{\hat{\theta}} + \boldsymbol{\omega}^{\hat{r}}{}_{\hat{\phi}} \wedge \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\theta}} = r^{-1}e^{-2\Lambda}\Lambda'\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}} \equiv \bar{F}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\theta}} \\
\mathcal{R}^{\hat{r}}{}_{\hat{\phi}} &= \boldsymbol{d\omega}^{\hat{r}}{}_{\hat{\phi}} + \boldsymbol{\omega}^{\hat{r}}{}_{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\phi}} = r^{-1}e^{-2\Lambda}\Lambda'\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} = \bar{F}\boldsymbol{\omega}^{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{\phi}} \\
\mathcal{R}^{\hat{\theta}}{}_{\hat{\phi}} &= \boldsymbol{d\omega}^{\hat{\theta}}{}_{\hat{\phi}} + \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{r}} \wedge \boldsymbol{\omega}^{\hat{r}}{}_{\hat{\phi}} = r^{-2}(1 - e^{-2\Lambda})\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}} = F\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}.
\end{aligned} \tag{14.13.4}$$

(4) From equation (14.7), we have

$$G^{\hat{t}}_{\hat{t}} = -(F + 2\bar{F}); \quad G^{\hat{r}}_{\hat{r}} = -(F + 2\bar{E}); \quad G^{\hat{\theta}}_{\hat{\theta}} = G^{\hat{\phi}}_{\hat{\phi}} = -(E + \bar{E} + \bar{F}). \tag{14.13.5}$$

All other components vanish.

14.14 MATRIX DISPLAY OF THE RIEMANN-TENSOR COMPONENTS

Use the symmetries of the Riemann tensor to justify displaying its components in an orthonormal frame in the form

$$R^{\hat{\mu}\hat{\nu}}{}_{\hat{\alpha}\hat{\beta}} = \begin{array}{c} \\ 01 \\ 02 \\ 03 \\ \\ 23 \\ 31 \\ 12 \end{array} \begin{pmatrix} & & & | & & & \\ & E & & | & & H & \\ & & & | & & & \\ - & - & - & | & - & - & - \\ & & & | & & & \\ & -H^{\mathrm{T}} & & | & & F & \\ & & & | & & & \end{pmatrix}, \tag{14.14.1}$$

where the rows are labeled by index pairs $\hat{\mu}\hat{\nu} = 01$, 02, etc., as shown; and the columns $\hat{\alpha}\hat{\beta}$, similarly. Here E, F, and H are each 3×3 matrices with (why?)

$$E = E^{\mathrm{T}}, \quad F = F^{\mathrm{T}}, \quad \mathrm{Tr}H = 0, \tag{14.14.2}$$

where E^{T} means the transpose of E.

Solution:

$$\begin{aligned}
R^{\hat{0}\hat{i}}{}_{\hat{0}\hat{j}} &= -R_{\hat{0}\hat{i}\hat{0}\hat{j}} = -R_{\hat{0}\hat{j}\hat{0}\hat{i}} = R^{\hat{0}\hat{j}}{}_{\hat{0}\hat{i}}; \\
R^{\hat{i}\hat{j}}{}_{\hat{k}\hat{l}} &= R_{\hat{i}\hat{j}\hat{k}\hat{l}} = R_{\hat{k}\hat{l}\hat{i}\hat{j}} = R^{\hat{k}\hat{l}}{}_{\hat{i}\hat{j}}; \\
R^{\hat{0}\hat{1}}{}_{\hat{2}\hat{3}} &+ R^{\hat{0}\hat{2}}{}_{\hat{3}\hat{1}} + R^{\hat{0}\hat{3}}{}_{\hat{1}\hat{2}} = 0.
\end{aligned} \tag{14.14.3}$$

So we have

$$E = E^{\mathrm{T}}, \quad F = F^{\mathrm{T}}, \quad \mathrm{Tr}H = 0. \tag{14.14.4}$$

14.15 RIEMANN MATRIX WITH VANISHING EINSTEIN TENSOR

Show that the empty-space Einstein equations $G^{\hat{\mu}}{}_{\hat{\nu}}$ allow the matrix in equation 14.14.1 to be simplified to the form

$$R^{\hat{\mu}\hat{\nu}}{}_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & H \\ -H & E \end{pmatrix}, \tag{14.15.1}$$

where now, in additIon to the equality $E = F$ that this form implies, the further conditions

$$\mathrm{Tr}E = 0, \quad H = H^{\mathrm{T}} \tag{14.15.2}$$

hold.

Solution:

If $G^{\hat{\mu}}{}_{\hat{\nu}} = 0$, we have

$$R^{\hat{\mu}}{}_{\hat{\nu}} = R^{\hat{\rho}\hat{\mu}}{}_{\hat{\rho}\hat{\nu}} = 0, \tag{14.15.3}$$

i.e.

$$\begin{aligned}
R^{\hat{1}\hat{0}}{}_{\hat{1}\hat{0}} + R^{\hat{2}\hat{0}}{}_{\hat{2}\hat{0}} + R^{\hat{3}\hat{0}}{}_{\hat{3}\hat{0}} &= \mathrm{Tr}E = 0, \\
R^{\hat{2}\hat{0}}{}_{\hat{2}\hat{1}} + R^{\hat{3}\hat{0}}{}_{\hat{3}\hat{1}} = R^{\hat{0}\hat{2}}{}_{\hat{1}\hat{2}} - R^{\hat{0}\hat{3}}{}_{\hat{3}\hat{1}} &= H_{23} - H_{32} = 0, \\
R^{\hat{1}\hat{0}}{}_{\hat{1}\hat{2}} + R^{\hat{3}\hat{0}}{}_{\hat{3}\hat{2}} = -R^{\hat{0}\hat{1}}{}_{\hat{1}\hat{2}} + R^{\hat{0}\hat{3}}{}_{\hat{2}\hat{3}} &= -H_{13} + H_{31} = 0, \\
R^{\hat{1}\hat{0}}{}_{\hat{1}\hat{3}} + R^{\hat{2}\hat{0}}{}_{\hat{2}\hat{3}} = R^{\hat{0}\hat{1}}{}_{\hat{3}\hat{1}} - R^{\hat{0}\hat{2}}{}_{\hat{2}\hat{3}} &= H_{12} - H_{21} = 0, \\
R^{\hat{0}\hat{1}}{}_{\hat{0}\hat{1}} + R^{\hat{2}\hat{1}}{}_{\hat{2}\hat{1}} + R^{\hat{3}\hat{1}}{}_{\hat{3}\hat{1}} &= E_{11} + F_{22} + F_{33} = 0, \\
R^{\hat{0}\hat{2}}{}_{\hat{0}\hat{2}} + R^{\hat{1}\hat{2}}{}_{\hat{1}\hat{2}} + R^{\hat{3}\hat{2}}{}_{\hat{3}\hat{2}} &= E_{22} + F_{33} + F_{11} = 0, \\
R^{\hat{0}\hat{3}}{}_{\hat{0}\hat{3}} + R^{\hat{1}\hat{3}}{}_{\hat{1}\hat{3}} + R^{\hat{2}\hat{3}}{}_{\hat{3}\hat{3}} &= E_{33} + F_{22} + F_{11} = 0, \\
R^{\hat{0}\hat{1}}{}_{\hat{0}\hat{2}} + R^{\hat{3}\hat{1}}{}_{\hat{3}\hat{2}} = R^{\hat{0}\hat{1}}{}_{\hat{0}\hat{2}} - R^{\hat{3}\hat{1}}{}_{\hat{2}\hat{3}} &= E_{12} - F_{21} = 0, \\
R^{\hat{0}\hat{2}}{}_{\hat{0}\hat{3}} + R^{\hat{1}\hat{2}}{}_{\hat{1}\hat{3}} = R^{\hat{0}\hat{2}}{}_{\hat{0}\hat{3}} - R^{\hat{1}\hat{2}}{}_{\hat{3}\hat{1}} &= E_{23} - F_{32} = 0, \\
R^{\hat{0}\hat{3}}{}_{\hat{0}\hat{1}} + R^{\hat{2}\hat{3}}{}_{\hat{2}\hat{1}} = R^{\hat{0}\hat{3}}{}_{\hat{0}\hat{1}} - R^{\hat{2}\hat{3}}{}_{\hat{1}\hat{2}} &= E_{31} - F_{13} = 0.
\end{aligned} \tag{14.15.4}$$

We can conclude that

$$E = F, \quad H = H^{\mathrm{T}}, \quad \mathrm{Tr}E = 0. \tag{14.15.5}$$

14.16 COMPUTATION OF CURVATURE FOR A PULSATING OR COLLAPSING STAR

Spherically symmetric motions of self-gravitating bodies are discussed in Chapters 26 and 32. A metric form often adopted in this situation is

$$ds^2 = -e^{2\Phi}dT^2 + e^{2\Lambda}dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{14.16.1}$$

where now Φ, Λ, and r are each functions of the two coordinates R and T. Compute the curvature 2-forms and the Einstein tensor for this metric, using the methods of Box 14.5. In the guessing of the $\boldsymbol{\omega}^{\mu}{}_{\nu}$ most of the terms will already be evident from the corresponding calculation in exercise 14.13.

The exercise can be solved straightforwardly following the procedure in Exercise 14.13. And the answer is already provided in the textbook.

14.17 BIANCHI IDENTITY IN $d\mathcal{R} = 0$ FORM

Define the Riemann tensor as a bivector-valued 2-form,

$$\mathcal{R} = \frac{1}{2}\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu \mathcal{R}^{\mu\nu}, \tag{14.17.1}$$

and evaluate $\boldsymbol{d}\mathcal{R}$ to make it manifest that $\boldsymbol{d}\mathcal{R} = 0$. Use

$$\mathcal{R}^{\mu\nu} = \boldsymbol{d}\boldsymbol{\omega}^{\mu\nu} - \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^{\nu\alpha}, \tag{14.17.2}$$

which is derived easily in an orthonormal frame (adequate for proving $\boldsymbol{d}\mathcal{R} = 0$), or (as a test of skill) in a general frame where $\mathcal{R}^{\mu\nu} = \mathcal{R}^\mu{}_\alpha g^{\alpha\nu}$ and (why?) $\boldsymbol{d}g^{\mu\nu} = -g^{\mu\alpha}(\boldsymbol{d}g_{\alpha\beta})g^{\beta\nu}$.

Solution:

As $g^{\mu\alpha}g_{\alpha\beta} = \delta^\mu_\beta$, we have

$$0 = \boldsymbol{d}(g^{\mu\alpha}g_{\alpha\beta}) = (\boldsymbol{d}g^{\mu\alpha})g_{\alpha\beta} + g^{\mu\alpha}\boldsymbol{d}g_{\alpha\beta}. \tag{14.17.3}$$

Thus,

$$\boldsymbol{d}g^{\mu\nu} = (\boldsymbol{d}g^{\mu\alpha})g_{\alpha\beta}g^{\beta\nu} = -g^{\mu\alpha}(\boldsymbol{d}g_{\alpha\beta})g^{\beta\nu}. \tag{14.17.4}$$

Recall that

$$\boldsymbol{d}\boldsymbol{\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu = 0, \quad \boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu}, \tag{14.17.5}$$

and

$$\mathcal{R}^\mu{}_\nu \equiv \boldsymbol{d}\boldsymbol{\omega}^\mu{}_\nu + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^\alpha{}_\nu. \tag{14.17.6}$$

Thus,

$$\boldsymbol{d}g^{\mu\nu} = -g^{\mu\alpha}(\boldsymbol{\omega}_{\alpha\beta} + \boldsymbol{\omega}_{\beta\alpha})g^{\beta\nu} = -(\boldsymbol{\omega}^{\mu\nu} + \boldsymbol{\omega}^{\nu\mu}), \tag{14.17.7}$$

$$\begin{aligned}
\mathcal{R}^{\mu\nu} &= \mathcal{R}^\mu{}_\alpha g^{\alpha\nu} \\
&= (\boldsymbol{d}\boldsymbol{\omega}^\mu{}_\alpha)g^{\alpha\nu} + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^{\alpha\nu} \\
&= (\boldsymbol{d}\boldsymbol{\omega}^\mu{}_\alpha)g^{\alpha\nu} - \boldsymbol{\omega}^\mu{}_\alpha \wedge (\boldsymbol{d}g^{\alpha\nu} + \boldsymbol{\omega}^{\nu\alpha}) \\
&= \boldsymbol{d}\boldsymbol{\omega}^{\mu\nu} - \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^{\nu\alpha}.
\end{aligned} \tag{14.17.8}$$

Now we have

$$\begin{aligned}
2\boldsymbol{d}\mathcal{R} &= \boldsymbol{d}(\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu)\mathcal{R}^{\mu\nu} + \boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu \boldsymbol{d}\mathcal{R}^{\mu\nu} \\
&= (\boldsymbol{e}_\alpha \boldsymbol{\omega}^\alpha{}_\mu \wedge \boldsymbol{e}_\nu + \boldsymbol{e}_\mu \wedge \boldsymbol{e}_\alpha \boldsymbol{\omega}^\alpha{}_\nu)(\boldsymbol{d}\boldsymbol{\omega}^{\mu\nu} - \boldsymbol{\omega}^\mu{}_\beta \wedge \boldsymbol{\omega}^{\nu\beta}) + (\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu)(-\boldsymbol{d}\boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^{\nu\alpha} + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{d}\boldsymbol{\omega}^{\nu\alpha}).
\end{aligned} \tag{14.17.9}$$

Note that

$$\boldsymbol{d}\boldsymbol{\omega}^\mu{}_\alpha = \boldsymbol{d}(\boldsymbol{\omega}^{\mu\beta}g_{\beta\alpha}) = \boldsymbol{d}(\boldsymbol{\omega}^{\mu\beta})g_{\beta\alpha} - \boldsymbol{\omega}^{\mu\beta} \wedge (\boldsymbol{\omega}_{\alpha\beta} + \boldsymbol{\omega}_{\beta\alpha}) \tag{14.17.10}$$

and

$$\boldsymbol{d}\omega^{\nu\alpha} + \boldsymbol{d}\omega^{\alpha\nu} = -\boldsymbol{d}^2 g^{\alpha\nu} = 0. \tag{14.17.11}$$

So we have

$$\begin{aligned}
2\boldsymbol{d}\mathcal{R} &= (\boldsymbol{e}_\alpha \boldsymbol{\omega}^\alpha{}_\mu \wedge \boldsymbol{e}_\nu + \boldsymbol{e}_\mu \wedge \boldsymbol{e}_\alpha \boldsymbol{\omega}^\alpha{}_\nu)(-\boldsymbol{\omega}^\mu{}_\beta \wedge \boldsymbol{\omega}^{\nu\beta}) + (\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu)(\boldsymbol{\omega}^{\mu\beta} \wedge (\boldsymbol{\omega}_{\alpha\beta} + \boldsymbol{\omega}_{\beta\alpha}) \wedge \boldsymbol{\omega}^{\nu\alpha}) \\
&= 0.
\end{aligned} \tag{14.17.12}$$

14.18 LOCAL CONSERVATION OF ENERGY AND MOMENTUM: $d^*T = 0$ MEANS $\nabla \cdot T = 0$

Let the duality operator $*$, as defined for exterior differential forms in Box 4.1, act on the forms, but not on the contravariant vectors, which appear when the stress-energy tensor $\boldsymbol{T}$ or the Einstein tensor $\boldsymbol{G}$ is written as a mixed $(1,1)$ tensor:

$$\boldsymbol{T} = \boldsymbol{e}_\mu T^\mu{}_\nu \boldsymbol{\omega}_\nu \tag{14.18.1}$$

or

$$\boldsymbol{G} = \boldsymbol{e}_\mu G^\mu{}_\nu \boldsymbol{\omega}_\nu. \tag{14.18.2}$$

(a) Give an expression for $^*\boldsymbol{T}$ (or $^*\boldsymbol{G}$) expanded in terms of basis vectors and forms.

(b) Show that

$$^*\boldsymbol{T} = \boldsymbol{e}_\mu T^{\mu\nu} d^3\Sigma_\nu, \tag{14.18.3}$$

where $d^3\Sigma_\nu = \epsilon_{\nu|\alpha\beta\gamma|}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma$.

(c) Compute $\boldsymbol{d}^*\boldsymbol{T}$ using the generalized exterior derivative $\boldsymbol{d}$; find that

$$\boldsymbol{d}^*\boldsymbol{T} = \boldsymbol{e}_\mu T^{\mu\nu}{}_{;\nu} \sqrt{|g|}\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3. \tag{14.18.4}$$

Solution:

(a)

$$^*\boldsymbol{T} = \frac{1}{3!}\sqrt{|g|}\epsilon_{\nu\alpha\beta\gamma}\boldsymbol{e}_\mu T^{\mu\nu}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma. \tag{14.18.5}$$

(b) As $1/3!\epsilon_{\nu\alpha\beta\gamma}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma = \epsilon_{\nu|\alpha\beta\gamma|}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma = d^3\Sigma_\nu$, we have

$$^*\boldsymbol{T} = \boldsymbol{e}_\mu T^{\mu\nu} d^3\Sigma_\nu. \tag{14.18.6}$$

(c) For simplicity, we perform the calculation in the frame where $\boldsymbol{e}_\mu$ are coordinate basis.

$$\begin{aligned}
\boldsymbol{d}^*\boldsymbol{T} &= [\boldsymbol{e}_\mu(\sqrt{|g|}T^{\mu\nu})_{,\rho} + \boldsymbol{e}_\sigma\Gamma^\sigma{}_{\mu\rho}\sqrt{|g|}T^{\mu\nu}]\epsilon_{\nu|\alpha\beta\gamma|}\boldsymbol{\omega}^\rho \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma \\
&= \boldsymbol{e}_\mu[(\sqrt{|g|}T^{\mu\nu})_{,\rho} + \Gamma^\mu{}_{\sigma\rho}\sqrt{|g|}T^{\sigma\nu}]\epsilon_{\nu|\alpha\beta\gamma|}\boldsymbol{\omega}^\rho \wedge \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma \\
&= -\frac{1}{3!}\boldsymbol{e}_\mu[(\sqrt{|g|}T^{\mu\nu})_{,\rho} + \Gamma^\mu{}_{\sigma\rho}\sqrt{|g|}T^{\sigma\nu}]\epsilon_{\nu\alpha\beta\gamma}\epsilon^{\rho\alpha\beta\gamma}\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \\
&= \boldsymbol{e}_\mu[(\sqrt{|g|}T^{\mu\nu})_{,\nu} + \Gamma^\mu{}_{\sigma\nu}\sqrt{|g|}T^{\sigma\nu}]\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \\
&= \boldsymbol{e}_\mu[T^{\mu\nu}{}_{,\nu} + \Gamma^\sigma{}_{\nu\sigma}T^{\mu\nu} + \Gamma^\mu{}_{\sigma\nu}T^{\sigma\nu}]\sqrt{|g|}\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \\
&= \boldsymbol{e}_\mu[T^{\mu\nu}{}_{,\nu} + \Gamma^\nu{}_{\sigma\nu}T^{\mu\sigma} + \Gamma^\mu{}_{\sigma\nu}T^{\sigma\nu}]\sqrt{|g|}\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \\
&= \boldsymbol{e}_\mu T^{\mu\nu}{}_{;\nu}\sqrt{|g|}\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3.
\end{aligned} \tag{14.18.7}$$

Chapter 15

BIANCHI IDENTITIES AND THE BOUNDARY OF A BOUNDARY

15.1 THE BOUNDARY OF THE BOUNDARY OF A 4-SIMPLEX

In the analysis of the development in time of a geometry lacking all symmetry, when one is compelled to resort to a computer, one can, as one option, break up the 4-geometry into simplexes [four-dimensional analog of two-dimensional triangle, three-dimensional tetrahedron; vertices of "central simplex" conveniently considered to be at $(t, x, y, z) = (0, 1, 1, 1)$, $(0, 1, -1, -1)$, $(0, -1, 1, -1)$, $(0, -1, -1, 1)$, $(\sqrt{5}, 0, 0, 0)$, for example], sufficiently numerous, and each sufficiently small, that the geometry inside each can be idealized as flat (Lorentzian), with all the curvature concentrated at the join between simplices (see discussion of dynamics of geometry via Regge calculus in Chapter 42). Determine ("give a mathematical description of") the boundary (three-dimensional) of such a simplex. Take one piece of this boundary and determine its boundary (two-dimensional). For one piece of this two-dimensional boundary, verify that there is at exactly one other place, and no more, in the bookkeeping on the boundary of a boundary, another two-dimensional piece that cancels it ("facelessness" of the 3-boundary of the simplex).

Solution:

Suppose the vertices of the simplex are A_1, A_2, A_3, A_4 and A_5. The boundaries are 1234, 1253, 1245, 1354 and 2345.

- The boundaries of 1234 are 123, 142, 134 and 243.
- The boundaries of 1253 are 125, 132, 153 and 235.
- The boundaries of 1245 are 124, 152, 145 and 254.
- The boundaries of 1354 are 135, 143, 154 and 345.
- The boundaries of 2345 are 234, 253, 245 and 354.

Then we can see that the boundaries of boundaries are zero.

15.2 THE BEL-ROBINSON TENSOR

Define the Bel-Robinson tensor by

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta\ \delta}^{\ \rho\ \sigma} + {}^*R_{\alpha\rho\gamma\sigma}{}^*R_{\beta\ \delta}^{\ \rho\ \sigma}. \tag{15.2.1}$$

Show that in empty spacetime this tensor can be rewritten as

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta\ \delta}^{\ \rho\ \sigma} + R_{\alpha\rho\delta\sigma}R_{\beta\ \gamma}^{\ \rho\ \sigma} - \frac{1}{8}g_{\alpha\beta}g_{\gamma\delta}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda\mu}. \tag{15.2.2}$$

Show also that in empty spacetime

$$T^{\alpha}{}_{\beta\gamma\delta;\alpha} = 0, \tag{15.2.3}$$

$$T_{\alpha\beta\gamma\delta} \text{ is symmetric and traceless on all pairs of indices.} \tag{15.2.4}$$

Solution:

The solution is base on the paper *A simple tensorial proof for the completely symmetric property of the Bel-Robinson tensor (Lau Loi So, 2010)*.

(1) Firstly, we can derive that

$$\begin{aligned}
12{}^*R_{\alpha[\beta\gamma\delta]} &= 2\delta^{\lambda\sigma\rho}_{\beta\gamma\delta}{}^*R_{\alpha\lambda\sigma\rho} \\
&= \delta^{\mu\lambda\sigma\rho}_{\mu\beta\gamma\delta}\epsilon_{\alpha\lambda\kappa\xi}R^{\kappa\xi}{}_{\sigma\rho} \\
&= \epsilon_{\mu\beta\gamma\delta}\delta^{\mu\lambda\sigma\rho}_{\alpha\lambda\kappa\xi}R^{\kappa\xi}{}_{\sigma\rho} \\
&= \epsilon_{\mu\beta\gamma\delta}\delta^{\mu\sigma\rho}_{\alpha\kappa\xi}R^{\kappa\xi}{}_{\sigma\rho} \\
&= \epsilon_{\mu\beta\gamma\delta}(\delta^{\mu}_{\alpha}\delta^{\sigma}_{\kappa}\delta^{\rho}_{\xi} - \delta^{\mu}_{\alpha}\delta^{\sigma}_{\xi}\delta^{\rho}_{\kappa} + \delta^{\mu}_{\kappa}\delta^{\sigma}_{\xi}\delta^{\rho}_{\alpha} - \delta^{\mu}_{\kappa}\delta^{\sigma}_{\alpha}\delta^{\rho}_{\xi} + \delta^{\mu}_{\xi}\delta^{\sigma}_{\alpha}\delta^{\rho}_{\kappa} - \delta^{\mu}_{\xi}\delta^{\sigma}_{\kappa}\delta^{\rho}_{\alpha})R^{\kappa\xi}{}_{\sigma\rho} \\
&= \epsilon_{\mu\beta\gamma\delta}(2\delta^{\mu}_{\alpha}R - 4R^{\mu}{}_{\alpha}) \\
&= -4\epsilon_{\mu\beta\gamma\delta}G^{\mu}{}_{\alpha}.
\end{aligned} \tag{15.2.5}$$

Thus in empty space, where $\boldsymbol{G} = 0$, we have ${}^*R_{\alpha[\beta\gamma\delta]} = 0$. Note that ${}^*R_{\alpha\beta\gamma\delta} = {}^*R_{[\alpha\beta][\gamma\delta]}$. We can further derive that

$$\begin{aligned}
{}^*R_{\alpha\beta\gamma\delta} &= -{}^*R_{\alpha\gamma\delta\beta} - {}^*R_{\alpha\delta\beta\gamma} \\
&= {}^*R_{\gamma\alpha\delta\beta} + {}^*R_{\delta\alpha\beta\gamma} \\
&= -{}^*R_{\gamma\beta\alpha\delta} - {}^*R_{\gamma\delta\beta\alpha} - {}^*R_{\delta\gamma\alpha\beta} - {}^*R_{\delta\beta\gamma\alpha} \\
&= ({}^*R_{\beta\gamma\alpha\delta} + {}^*R_{\beta\delta\gamma\alpha}) + 2{}^*R_{\gamma\delta\alpha\beta} \\
&= -{}^*R_{\beta\alpha\delta\gamma} + 2{}^*R_{\gamma\delta\alpha\beta} \\
&= {}^*R_{\alpha\beta\gamma\delta} + 2{}^*R_{\gamma\delta\alpha\beta}.
\end{aligned} \tag{15.2.6}$$

So in empty space, we also have ${}^*R_{\alpha\beta\gamma\delta} = {}^*R_{\gamma\delta\alpha\beta}$.

(2) From

$$\begin{aligned}
{}^*R_{\alpha\rho\gamma\sigma}{}^*R_{\beta\ \delta}^{\ \rho\ \sigma} &= \frac{1}{4}\epsilon_{\alpha\rho}{}^{\mu\nu}\epsilon_{\beta}{}^{\rho\kappa}{}_{\lambda}R_{\mu\nu\gamma\sigma}R_{\kappa\ \delta}^{\ \lambda\ \sigma} \\
&= -\frac{1}{4}g_{\beta\iota}\delta^{\iota\kappa\lambda}_{\alpha\mu\nu}R^{\mu\nu}{}_{\gamma\sigma}R_{\kappa\lambda\delta}{}^{\sigma} \\
&= -\frac{1}{4}g_{\beta\iota}(\delta^{\iota}_{\alpha}\delta^{\kappa\lambda}_{\mu\nu} + \delta^{\iota}_{\mu}\delta^{\kappa\lambda}_{\nu\alpha} + \delta^{\iota}_{\nu}\delta^{\kappa\lambda}_{\alpha\mu})R^{\mu\nu}{}_{\gamma\sigma}R_{\kappa\lambda\delta}{}^{\sigma} \\
&= -\frac{1}{2}g_{\beta\alpha}R^{\mu\nu}{}_{\gamma\sigma}R_{\mu\nu\delta}{}^{\sigma} - \frac{1}{2}g_{\beta\mu}R^{\mu\nu}{}_{\gamma\sigma}R_{\nu\alpha\delta}{}^{\sigma} - \frac{1}{2}g_{\beta\nu}R^{\mu\nu}{}_{\gamma\sigma}R_{\alpha\mu\delta}{}^{\sigma} \\
&= -\frac{1}{2}g_{\alpha\beta}R^{\mu\nu\sigma}{}_{\gamma}R_{\mu\nu\sigma\delta} + R_{\alpha\rho\delta\sigma}R_{\beta\ \gamma}^{\ \rho\ \sigma},
\end{aligned} \tag{15.2.7}$$

we can get

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + R_{\alpha\rho\delta\sigma}R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{2}g_{\alpha\beta}R^{\mu\nu\sigma}{}_{\gamma}R_{\mu\nu\sigma\delta}. \tag{15.2.8}$$

It is easy to see that $T_{\alpha\beta\gamma\delta} = T_{(\alpha\beta)(\gamma\delta)}$ from equation 15.2.8.

(3) From the definition of Bel-Robinson tensor, we have

$$T_{\alpha\beta\gamma\delta} - T_{\alpha\delta\gamma\beta} = R_{\alpha\rho\gamma\sigma}(R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} - R_{\delta}{}^{\rho}{}_{\beta}{}^{\sigma}) + {}^{*}R_{\alpha\rho\gamma\sigma}({}^{*}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} - {}^{*}R_{\delta}{}^{\rho}{}_{\beta}{}^{\sigma}). \tag{15.2.9}$$

Employing identity $R_{\alpha[\beta\gamma\delta]} = 0$ and $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$, we have

$$R_{\alpha\rho\gamma\sigma}(R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} - R_{\delta}{}^{\rho}{}_{\beta}{}^{\sigma}) = R_{\alpha\rho\gamma\sigma}R_{\beta\delta}{}^{\rho\sigma} = \frac{1}{2}(R_{\alpha\rho\gamma\sigma} - R_{\alpha\sigma\gamma\rho})R_{\beta\delta}{}^{\rho\sigma} = \frac{1}{2}R_{\alpha\gamma\rho\sigma}R_{\beta\delta}{}^{\rho\sigma}. \tag{15.2.10}$$

In empty space, we also have the identity ${}^{*}R_{\alpha[\beta\gamma\delta]} = 0$ and ${}^{*}R_{\alpha\beta\gamma\delta} = {}^{*}R_{\gamma\delta\alpha\beta}$. Similarly, we can get

$${}^{*}R_{\alpha\rho\gamma\sigma}({}^{*}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} - {}^{*}R_{\delta}{}^{\rho}{}_{\beta}{}^{\sigma}) = \frac{1}{2}{}^{*}R_{\alpha\gamma\rho\sigma}{}^{*}R_{\beta\delta}{}^{\rho\sigma}. \tag{15.2.11}$$

Furthermore, we can deduce that

$$\begin{aligned} {}^{*}R_{\alpha\gamma\rho\sigma}{}^{*}R_{\beta\delta}{}^{\rho\sigma} &= {}^{*}R_{\rho\sigma\alpha\gamma}{}^{*}R^{\rho\sigma}{}_{\beta\delta} = \frac{1}{4}\epsilon_{\rho\sigma\mu\nu}R^{\mu\nu}{}_{\alpha\gamma}\epsilon^{\rho\sigma\xi\kappa}R_{\xi\kappa\beta\delta} = -\frac{1}{4}\delta^{\rho\sigma\xi\kappa}_{\rho\sigma\mu\nu}R^{\mu\nu}{}_{\alpha\gamma}R_{\xi\kappa\beta\delta} \\ &= -\frac{1}{2}\delta^{\xi\kappa}_{\mu\nu}R^{\mu\nu}{}_{\alpha\gamma}R_{\xi\kappa\beta\delta} = -R^{\mu\nu}{}_{\alpha\gamma}R_{\mu\nu\beta\delta} = -R_{\alpha\gamma\rho\sigma}R_{\beta\delta}{}^{\rho\sigma}. \end{aligned} \tag{15.2.12}$$

Go back to 15.2.9, we can obtain

$$T_{\alpha\beta\gamma\delta} = T_{\alpha\delta\gamma\beta}\,. \tag{15.2.13}$$

Recall that $T_{\alpha\beta\gamma\delta} = T_{(\alpha\beta)(\gamma\delta)}$. We now prove that $T_{\alpha\beta\gamma\delta}$ is symmetric on all pairs of indices.

(4) From equation 15.2.8, we have

$$\begin{aligned} T_{\gamma\delta\alpha\beta} &= R_{\gamma\rho\alpha\sigma}R_{\delta}{}^{\rho}{}_{\beta}{}^{\sigma} + R_{\gamma\rho\beta\sigma}R_{\delta}{}^{\rho}{}_{\alpha}{}^{\sigma} - \frac{1}{2}g_{\gamma\delta}R^{\mu\nu\sigma}{}_{\alpha}R_{\mu\nu\sigma\beta} \\ &= R_{\gamma\sigma\alpha\rho}R_{\delta}{}^{\sigma}{}_{\beta}{}^{\rho} + R_{\gamma\sigma\beta\rho}R_{\delta}{}^{\sigma}{}_{\alpha}{}^{\rho} - \frac{1}{2}g_{\gamma\delta}R^{\mu\nu\sigma}{}_{\alpha}R_{\mu\nu\sigma\beta} \\ &= R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + R_{\alpha\rho\delta\sigma}R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{2}g_{\gamma\delta}R^{\mu\nu\sigma}{}_{\alpha}R_{\mu\nu\sigma\beta}. \end{aligned} \tag{15.2.14}$$

In empty space, we have $T_{\alpha\beta\gamma\delta} = T_{\gamma\delta\alpha\beta}$. So we can get

$$g_{\gamma\delta}R^{\mu\nu\sigma}{}_{\alpha}R_{\mu\nu\sigma\beta} = g_{\alpha\beta}R^{\mu\nu\sigma}{}_{\gamma}R_{\mu\nu\sigma\delta}. \tag{15.2.15}$$

Contracting index γ and δ, we have

$$4R^{\mu\nu\sigma}{}_{\alpha}R_{\mu\nu\sigma\beta} = g_{\gamma\delta}R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}. \tag{15.2.16}$$

Bring it into 15.2.8, we have

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + R_{\alpha\rho\delta\sigma}R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\gamma\delta}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda\mu}. \tag{15.2.17}$$

(5) From the Bianchi identitiy $R^{\alpha}{}_{\beta[\rho\sigma;\lambda]} = 0$, we have

$$R^{\alpha}{}_{\beta\rho\sigma;\alpha} + R^{\alpha}{}_{\beta\alpha\rho;\sigma} + R^{\alpha}{}_{\beta\sigma\alpha;\rho} = 0 \tag{15.2.18}$$

by contracting α and λ. In empty space, we have $R^{\alpha}{}_{\beta\rho\sigma;\alpha} = 0$. Thus, we have

$$\begin{aligned}
T^{\alpha}{}_{\beta\gamma\delta;\alpha} &= \left(R^{\alpha}{}_{\rho\gamma\sigma} R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + R^{\alpha}{}_{\rho\delta\sigma} R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{2}\delta^{\alpha}_{\beta} R^{\mu\nu\sigma}{}_{\gamma} R_{\mu\nu\sigma\delta} \right)_{;\alpha} \\
&= R^{\alpha}{}_{\rho\gamma\sigma} R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma}{}_{;\alpha} + R^{\alpha}{}_{\rho\delta\sigma} R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma}{}_{;\alpha} - \frac{1}{2} R^{\mu\nu\sigma}{}_{\gamma;\beta} R_{\mu\nu\sigma\delta} - \frac{1}{2} R^{\mu\nu\sigma}{}_{\gamma} R_{\mu\nu\sigma\delta;\beta} \\
&= R^{\alpha\rho\sigma}{}_{\gamma} R_{\beta\rho\sigma\delta;\alpha} + R^{\alpha\rho\sigma}{}_{\delta} R_{\beta\rho\sigma\gamma;\alpha} - \frac{1}{2} R^{\alpha\rho\sigma}{}_{\gamma;\beta} R_{\alpha\rho\sigma\delta} - \frac{1}{2} R^{\alpha\rho\sigma}{}_{\gamma} R_{\alpha\rho\sigma\delta;\beta} \\
&= R^{\alpha\rho\sigma}{}_{\gamma} \left(R_{\beta\rho\sigma\delta;\alpha} - \frac{1}{2} R_{\alpha\rho\sigma\delta;\beta} \right) + R^{\alpha\rho\sigma}{}_{\delta} \left(R_{\beta\rho\sigma\gamma;\alpha} - \frac{1}{2} R_{\alpha\rho\sigma\delta;\beta} \right) \\
&= \frac{1}{2} R^{\alpha\rho\sigma}{}_{\gamma} \left(2R_{\sigma\delta\beta\rho;\alpha} + R_{\sigma\delta\rho\alpha;\beta} \right) + \frac{1}{2} R^{\alpha\rho\sigma}{}_{\delta} \left(2R_{\sigma\gamma\beta\rho;\alpha} + R_{\sigma\delta\rho\alpha;\beta} \right) \\
&= \frac{1}{2} R^{\alpha\rho\sigma}{}_{\gamma} \left(R_{\sigma\delta\beta\rho;\alpha} + R_{\sigma\delta\alpha\beta;\rho} + R_{\sigma\delta\rho\alpha;\beta} \right) + \frac{1}{2} R^{\alpha\rho\sigma}{}_{\delta} \left(R_{\sigma\gamma\beta\rho;\alpha} + R_{\sigma\gamma\alpha\beta;\rho} + R_{\sigma\delta\rho\alpha;\beta} \right) \\
&= 0.
\end{aligned} \tag{15.2.19}$$

(6) Contracting index α and β in $T_{\alpha\beta\gamma\delta}$, we have

$$\begin{aligned}
T^{\alpha}{}_{\alpha\gamma\delta} &= R_{\alpha\rho\gamma\sigma} R^{\alpha\rho}{}_{\delta}{}^{\sigma} + R_{\alpha\rho\delta\sigma} R^{\alpha\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{2} g_{\gamma\delta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \\
&= R_{\gamma\sigma\alpha\rho} R_{\delta}{}^{\sigma\alpha\rho} + R_{\delta\sigma\alpha\rho} R_{\gamma}{}^{\sigma\alpha\rho} - \frac{1}{2} g_{\gamma\delta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \\
&= \frac{1}{4} g_{\gamma\delta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} + \frac{1}{4} g_{\gamma\delta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} - \frac{1}{2} g_{\gamma\delta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \\
&= 0.
\end{aligned} \tag{15.2.20}$$

Since $T_{\alpha\beta\gamma\delta}$ is symmetric on all pairs of indices, $T_{\alpha\beta\gamma\delta}$ must be traceless on all pairs of indices.

Chapter 16

EQUIVALENCE PRINCIPLE AND MEASUREMENT OF GRAVITATIONAL FIELD

16.1 HYDRODYNAMICS IN A WEAK GRAVITATIONAL FIELD

(a) In section 18.4 it will be shown that for a nearly Newtonian system, analyzed in an appropriate nearly global Lorentz coordinate system, the metric has the form

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) \tag{16.1.1}$$

where Φ is the Newtonian potential ($-1 \ll \Phi < 0$). Consider a nearly Newtonian perfect fluid [stress-energy tensor

$$T^{\alpha\beta} = (\rho+p)u^\alpha u^\beta + pg^{\alpha\beta}, \quad p \ll \rho; \tag{16.1.2}$$

see Box 5.1 and section 5.10] moving in such a spacetime with ordinary velocity

$$v^j \equiv dx^j/dt \ll 1. \tag{16.1.3}$$

Show that the equations $T^{\mu\nu}{}_{;\nu} = 0$ for this system reduce to the familiar Newtonian law of mass conservation, and the Newtonian equation of motion for a fluid in a gravitational field:

$$\frac{d\rho}{dt} = -\rho\frac{\partial v^j}{\partial x^j}, \quad \rho\frac{dv^j}{dt} = -\rho\frac{\partial\Phi}{\partial x^j} - \frac{\partial p}{\partial x^j}, \tag{16.1.4}$$

where d/dt is the time derivative comoving with the matter

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v^j\frac{\partial}{\partial x^j}. \tag{16.1.5}$$

(b) Use these equations to calculate the pressure gradient in the Earth's atmosphere as a function of temperature and pressure. In the calculation, use the nonrelativistic relation $\rho = n_M\mu_M$, where n_M is the number density of molecules and μ_M is the mean rest mass per molecule; use the ideal-gas equation of state

$$p = n_M kT \quad (k = \text{ Boltzmann's constant }); \tag{16.1.6}$$

and use the spherically symmetric form, $\Phi = -M/r$, for the Earth's Newtonian potential. If the pressure at sea level is 1.01×10^6 dynes/cm^2, what, approximately, is the pressure on top of Mount Everest (altitude 8,840 meters)? (Make a reasonable assumption about the temperature distribution of the atmosphere.)

Solution:

Keep the first order of the small value, we have

$$g^{\alpha\beta} = \begin{pmatrix} -1+2\Phi & 0 & 0 & 0 \\ 0 & 1+2\Phi & 0 & 0 \\ 0 & 0 & 1+2\Phi & 0 \\ 0 & 0 & 0 & 1+2\Phi \end{pmatrix} \tag{16.1.7}$$

and

$$\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Phi_{,i}, \quad \Gamma^i{}_{00} = \Phi_{,i}, \quad \Gamma^i{}_{jk} = \Phi_{,i}\delta_{jk} - \Phi_{,j}\delta_{ik} - \Phi_{,k}\delta_{ij}. \tag{16.1.8}$$

The stress-energy tensor is

$$T^{\alpha\beta} = \begin{pmatrix} \rho & \rho v^i \\ \rho v^i & p\delta^{ij} + \rho v^i v^j \end{pmatrix} \tag{16.1.9}$$

Then we have

$$0 = T^{00}{}_{;0} + T^{0i}{}_{;i} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho v^i}{\partial x^i} = \frac{d\rho}{dt} + \rho \frac{\partial v^i}{\partial x^i}. \tag{16.1.10}$$

and

$$0 = T^{i0}{}_{;0} + T^{ij}{}_{;j} = \frac{\partial \rho v^i}{\partial t} + \rho \frac{\partial \Phi}{\partial x^i} + \frac{\partial p\delta^{ij} + \rho v^i v^j}{\partial x^j} = v^i \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v^j}{\partial x^j} \right) + \rho \frac{\partial v^i}{\partial t} + \rho v^j \frac{\partial v^i}{\partial x^j} + \rho \frac{\partial \Phi}{\partial x^i} + \frac{\partial p}{\partial x^j}, \tag{16.1.11}$$

i.e.

$$\frac{d\rho}{dt} = -\rho \frac{\partial v^j}{\partial x^j}, \quad \rho \frac{dv^j}{dt} = -\rho \frac{\partial \Phi}{\partial x^j} - \frac{\partial p}{\partial x^j}. \tag{16.1.12}$$

(b) Assume the atmosphere is static. We have

$$\rho \frac{\partial \Phi}{\partial r} + \frac{\partial p}{\partial r} = 0, \tag{16.1.13}$$

i.e.

$$\frac{\partial p}{\partial r} = -\rho \frac{\partial \Phi}{\partial r} = -\frac{GM\mu}{r^2} \frac{p}{kT}, \tag{16.1.14}$$

Assume temperature is a constant and the height of Mount Everest is much smaller than radius of the earth. We can get

$$\ln\left(\frac{p}{p_0}\right) = -\frac{GM\mu h}{kTR^2} \tag{16.1.15}$$

Suppose $T \approx 300K$, we have

$$p \approx 0.36 p_0 = 3.7 \times 10^5 \text{ dynes/cm}^2. \tag{16.1.16}$$

16.2 WORLD LINES OF PHOTONS

Show that in flat spacetime the conservation law for the 4-momentum of a freely moving photon can be written

$$\boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{p} = 0. \tag{16.2.1}$$

According to the equivalence principle, this equation must be true also in curved spacetime. Show that this means photons move along null geodesics of curved spacetime with affine parameter λ related to 4-momentum by

$$\boldsymbol{p} = d/d\lambda. \tag{16.2.2}$$

Solution:

Because $\boldsymbol{p}$ is conserved, we have $p^\mu =$ constant in flat spacetime, i.e.

$$\boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{p} = 0. \tag{16.2.3}$$

In curved spacetime, we have $\boldsymbol{p}\cdot\boldsymbol{p} = 0$ when photon is emitted. Then we have $\boldsymbol{p}\cdot\boldsymbol{p} = 0$ along the path of photon, i,e, photons move along null geodesics of curved spacetime with affine parameter λ related to 4-momentum by

$$\boldsymbol{p} = d/d\lambda. \tag{16.2.4}$$

16.3 NONCOM MUTATION OF COVARIANT DERIVATIVES

Let $\boldsymbol{B}$ be a vector field and $\boldsymbol{S}$ be a second-rank tensor field. Show that

$$\begin{aligned} B^{\mu}{}_{;\alpha\beta} &= B^{\mu}{}_{;\beta\alpha} + R^{\mu}{}_{\nu\beta\alpha}B^{\nu}; \\ S^{\mu\nu}{}_{;\alpha\beta} &= S^{\mu\nu}{}_{;\beta\alpha} + R^{\mu}{}_{\rho\beta\alpha}S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}. \end{aligned} \tag{16.3.1}$$

From equation 16.3.1, show that

$$B^{\mu;\alpha}{}_{\mu} = B^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}B^{\mu}. \tag{16.3.2}$$

Solution:

Since

$$\begin{aligned} B^{\mu}{}_{;\alpha\beta} &= B^{\mu}{}_{;\alpha,\beta} + \Gamma^{\mu}{}_{\nu\beta}B^{\nu}{}_{;\alpha} - \Gamma^{\nu}{}_{\alpha\beta}B^{\mu}{}_{;\nu} \\ &= B^{\mu}{}_{,\alpha\beta} + (\Gamma^{\mu}{}_{\nu\alpha}B^{\nu})_{,\beta} + \Gamma^{\mu}{}_{\nu\beta}(B^{\nu}{}_{,\alpha} + \Gamma^{\nu}{}_{\rho\alpha}B^{\rho}) - \Gamma^{\nu}{}_{\alpha\beta}B^{\mu}{}_{;\nu}, \end{aligned} \tag{16.3.3}$$

we have

$$B^{\mu}{}_{;\alpha\beta} - B^{\mu}{}_{;\beta\alpha} = (\Gamma^{\mu}{}_{\nu\alpha,\beta} + \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha} - \Gamma^{\mu}{}_{\nu\beta,\alpha} - \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta})B^{\nu} = R^{\mu}{}_{\nu\beta\alpha}B^{\nu}. \tag{16.3.4}$$

By contraction, we have

$$B^{\mu}{}_{;\mu\beta} - B^{\mu}{}_{;\beta\mu} = R^{\mu}{}_{\nu\beta\mu}B^{\nu} = -R_{\nu\beta}B^{\nu} = -R_{\beta\nu}B^{\nu}, \tag{16.3.5}$$

i.e.

$$B^{\mu;\alpha}{}_{\mu} = B^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}B^{\mu}. \tag{16.3.6}$$

Since

$$\begin{aligned} S^{\mu\nu}{}_{;\alpha\beta} &= S^{\mu\nu}{}_{;\alpha,\beta} + \Gamma^{\mu}{}_{\rho\beta}S^{\rho\nu}{}_{;\alpha} + \Gamma^{\nu}{}_{\rho\beta}S^{\mu\rho}{}_{;\alpha} - \Gamma^{\rho}{}_{\alpha\beta}S^{\mu\nu}{}_{;\rho} \\ &= S^{\mu\nu}{}_{,\alpha\beta} + (\Gamma^{\mu}{}_{\rho\alpha}S^{\rho\nu} + \Gamma^{\nu}{}_{\rho\alpha}S^{\mu\rho})_{,\beta} + \Gamma^{\mu}{}_{\rho\beta}(S^{\rho\nu}{}_{,\alpha} + \Gamma^{\rho}{}_{\sigma\alpha}S^{\sigma\nu} + \Gamma^{\nu}{}_{\sigma\alpha}S^{\rho\sigma}) \\ &+ \Gamma^{\nu}{}_{\rho\beta}(S^{\mu\rho}{}_{,\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}S^{\sigma\rho} + \Gamma^{\rho}{}_{\sigma\alpha}S^{\mu\sigma}) - \Gamma^{\rho}{}_{\alpha\beta}S^{\mu\nu}{}_{;\rho} \\ &= \Gamma^{\mu}{}_{\rho\alpha,\beta}S^{\rho\nu} + \Gamma^{\nu}{}_{\rho\alpha,\beta}S^{\mu\rho} + \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\sigma\alpha}S^{\sigma\nu} + \Gamma^{\nu}{}_{\rho\beta}\Gamma^{\mu}{}_{\sigma\alpha}S^{\sigma\nu} \\ &+ \text{ terms whose indexes are symmetric in } \alpha \text{ and } \beta, \end{aligned} \tag{16.3.7}$$

we have

$$S^{\mu\nu}{}_{;\alpha\beta} - S^{\mu\nu}{}_{;\beta\alpha} = R^{\mu}{}_{\rho\beta\alpha}S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}. \tag{16.3.8}$$

16.4 PRECESSION OF THE EQUINOXES

(a) Show that the transport law for the Earth's intrinsic angular momentum vector S^α in curved spacetime is

$$\frac{DS^\alpha}{D\tau} = \epsilon^{\alpha\beta\gamma\delta} I_{\beta\mu} R^\mu{}_{\nu\gamma\xi} u_\delta u^\nu u^\xi. \tag{16.4.1}$$

Here $d/d\tau = \boldsymbol{u}$ is 4-velocity along the Earth's world line; $I_{\beta\mu}$ is the Earth's "reduced quadrupole moment" (trace-free part of second moment of mass distribution), defined in the Earth's local Lorentz frame by

$$I_{\hat{0}\hat{0}} = I_{\hat{0}\hat{j}} = 0, \quad I_{\hat{j}\hat{k}} = \int \rho(x^{\hat{j}} x^{\hat{k}} - \frac{1}{3}\hat{r}^2 \delta_{jk}) d^3\hat{x}; \tag{16.4.2}$$

and $R^\mu{}_{\nu\gamma\xi}$ is the Riemann curvature produced at the Earth's location by the moon, sun, and planets.

(b) Rewrite equation 16.4.1 in the Earth's local Lorentz frame, using the equation

$$R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} = \partial^2\Phi/\partial x^{\hat{j}} \partial x^{\hat{k}} \tag{16.4.3}$$

for the components of Riemann in terms of the Newtonian gravitational potential.

(c) Calculate $dS^{\hat{j}}/d\hat{t}$ using Newton's theory of gravity from the beginning. The answer should be identical to that obtained in part (b) using Einstein's theory.

(d) Idealizing the moon and sun as point masses, calculate the long-term effect of the spacetime curvatures that they produce upon the Earth's rotation axis. Use the result of part (b), together with moderately accurate numerical values for the relevant solar-system parameters.

Solution:

(a) We would derive this result in the Earth's local Lorentz frame, ignoring the spacetime curvature due to the Earth. In this essentially Newtonian situation, curvature components due to the Earth, sun, moon, and planets superpose linearly; i.e. gravity too weak to be nonlinear. The acceleration at $x^{\hat{j}}$ relative to center of mass ($x^{\hat{j}} = 0$) produced by tidal gravitational forces is

$$\frac{d^2 x^{\hat{j}}}{d\hat{t}^2} = -R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}}. \tag{16.4.4}$$

The force per unit volume due to this acceleration, relative to center of mass is

$$\rho \frac{d^2 x^{\hat{j}}}{d\hat{t}^2} = -\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}}. \tag{16.4.5}$$

The torque per unit volume relative to center of mass is

$$\tau_{\hat{i}} = \epsilon_{\hat{0}\hat{i}\hat{j}\hat{k}} x^{\hat{j}} (-\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}}). \tag{16.4.6}$$

So the total torque about center of mass is

$$\mathcal{T}_{\hat{i}} = \int \epsilon_{\hat{0}\hat{i}\hat{j}\hat{k}} x^{\hat{j}} (-\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}}) d^3\hat{x}. \tag{16.4.7}$$

On the other hand, we have

$$\begin{aligned}
\epsilon^{\hat{i}\beta\gamma\delta} I_{\beta\mu} R^\mu{}_{\nu\gamma\xi} u_\delta u^\nu u^\xi &= -\epsilon^{\hat{i}\hat{j}\hat{k}\hat{0}} I_{\hat{j}\hat{l}} R^{\hat{l}}{}_{\hat{0}\hat{k}\hat{0}} \\
&= -\epsilon^{\hat{i}\hat{j}\hat{k}\hat{0}} \int \rho(x_{\hat{j}} x_{\hat{l}} - \frac{1}{3}\hat{r}^2 \delta_{jl}) R^{\hat{l}}{}_{\hat{0}\hat{k}\hat{0}} d^3\hat{x} \\
&= \mathcal{T}^{\hat{i}} + \frac{1}{3}\epsilon^{\hat{i}\hat{j}\hat{k}\hat{0}} \int \rho \hat{r}^2 R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} d^3\hat{x} \\
&= \mathcal{T}^{\hat{i}} \\
&= \frac{dS^{\hat{i}}}{d\hat{t}}
\end{aligned} \tag{16.4.8}$$

in the Earth's local Lorentz frame. Then 16.4.1 would hold in any frame.

(b) We have

$$\frac{dS^{\hat{i}}}{d\hat{t}} = -\epsilon_{\hat{i}\hat{j}\hat{k}} \int \rho \frac{\partial^2 \Phi}{\partial x^{\hat{k}} \partial x^{\hat{l}}} x^{\hat{j}} x^{\hat{l}} d^3\hat{x}. \tag{16.4.9}$$

(c) The gravity exerted on the point $x^{\hat{j}}$ relative to center of mass per unit volume are

$$f^{\hat{k}} = -\rho \left(\frac{\partial \Phi}{\partial x^{\hat{k}}}\bigg|_o - \frac{\partial^2 \Phi}{\partial x^{\hat{k}} \partial x^{\hat{l}}}\bigg|_o x^{\hat{l}} \right). \tag{16.4.10}$$

So the torque exerted on the earth is

$$\int \epsilon_{\hat{i}\hat{j}k} x^j f^k d^3\hat{x} = -\epsilon_{\hat{i}\hat{j}\hat{k}} \int \rho \left. \frac{\partial^2 \Phi}{\partial x^{\hat{k}} \partial x^{\hat{l}}} \right|_o x^{\hat{j}} x^{\hat{l}} d^3\hat{x}, \tag{16.4.11}$$

which is equivalent to 16.4.9.

(d) Please see the wikipedia for a detailed calculation.

16.5 GRAVITY GRADIOMETER

The gravity gradiometer of Box 16.5 moves through curved spacetime along an accelerated world line. Calculate the amplitude and phase of oscillation of one arm of the gradiometer relative to the other.

The detailed solution has been provided in "hint" attached to the exercise in the textbook.

Chapter 17

HOW MASS-ENERGY GENERATES CURVATURE

17.1 UNIQUENESS OF THE EINSTEIN TENSOR

(a) Show that the most general second-rank, symmetric tensor constructable from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$, and linear in $\boldsymbol{Riemann}$, is

$$aR_{\alpha\beta} + bRg_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{17.1.1}$$

where a, b and Λ are constants.

(b) Show that this tensor has an automatically vanishing divergence if and only if $b = -a/2$.

(c) Show that, in addition, this tensor vanishes in flat spacetime, if and only if $\Lambda = 0$, i.e., if and only if it is a multiple of the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$.

Solution:

(a) The contrcation of the first and second index of $\boldsymbol{Riemann}$ vanishes because $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$. So the only second rank tensor constructable from $\boldsymbol{Riemann}$ and linear in $\boldsymbol{Riemann}$ must be multiple of Ricci tensor and so the only scalar must be Ricci scalar. Then, the most general second-rank, symmetric tensor constructable from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$, and linear in $\boldsymbol{Riemann}$, is

$$aR_{\alpha\beta} + bRg_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{17.1.2}$$

where a, b and Λ are constants.

(b)

$$\begin{aligned}(aR_{\alpha\beta} + bRg_{\alpha\beta} + \Lambda g_{\alpha\beta})^{;\beta} &= a(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})^{;\beta} + (b - \frac{a}{2})R^{,\beta}g_{\alpha\beta} \\ &= (b - \frac{a}{2})R_{,\alpha}.\end{aligned} \tag{17.1.3}$$

So this tensor has an automatically vanishing divergence if and only if $b = -a/2$.

(c) In flat spacetime, we have

$$aR_{\alpha\beta} + bRg_{\alpha\beta} + \Lambda g_{\alpha\beta} = \Lambda\eta_{\alpha\beta}. \tag{17.1.4}$$

So this tensor vanishes in flat spacetime if and only if $\Lambda = 0$.

17.2 NO TENSOR CONSTRUCT ABLE FROM FIRST DERIVATIVES OF METRIC

Show that there exists no tensor with components constructable from the ten metric coefficients $g_{\alpha\beta}$ and their 40 first derivatives $g_{\alpha\beta,\mu}$, except the metric tensor $\boldsymbol{g}$, and products of it with itself; e.g., $\boldsymbol{g}\otimes\boldsymbol{g}$.

Solution:

In the local Lorentz frame, we have $g_{\alpha\beta,\mu}=0$. So it is easy to see that tensors with components constructable from the ten metric coefficients and their 40 first derivatives are metric tensor and products of it with itself.

17.3 RIEMANN AS THE ONLY TENSOR CONSTRUCTABLE FROM, AND LINEAR IN SECOND DERIVATIVES OF METRIC

Show that (1) $\boldsymbol{Riemann}$, (2) $\boldsymbol{g}$, and (3) tensors (e.g., $\boldsymbol{Ricci}$) formed from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$ but linear in $\boldsymbol{Riemann}$, are the only tensors that (a) are constructable from the ten $g_{\alpha\beta}$ the 40 $g_{\alpha\beta,\mu}$ and the 100 $g_{\alpha\beta,\mu\nu}$ and (b) are linear in the $g_{\alpha\beta,\mu\nu}$.

Solution:

In an orthonormal, Riemann-normal coordinate system, we have

$$g_{\alpha\beta}=\eta_{\alpha\beta},\quad g_{\alpha\beta,\mu}=0,\quad g_{\alpha\beta,\mu\nu}=-\frac{1}{3}(R_{\alpha\mu\beta\nu}+R_{\alpha\nu\beta\mu}). \tag{17.3.1}$$

The only tensors that are constructable from the ten $g_{\alpha\beta}$ the 40 $g_{\alpha\beta,\mu}$ and the 100 $g_{\alpha\beta,\mu\nu}$ and are linear in the $g_{\alpha\beta,\mu\nu}$ must be of the form

$$A\boldsymbol{g}+B\boldsymbol{Riemann}+C\text{ contraction of }\boldsymbol{Riemann}, \tag{17.3.2}$$

i.e. (1) $\boldsymbol{Riemann}$, (2) $\boldsymbol{g}$, and (3) tensors (e.g., $\boldsymbol{Ricci}$) formed from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$ but linear in $\boldsymbol{Riemann}$.

17.4 UNIQUENESS OF THE EINSTEIN TENSOR

(a) Show that the Einstein tensor is the only second-rank, symmetric tensor that (1) has components constructable solely from $g_{\alpha\beta}$, $g_{\alpha\beta,\mu}$, $g_{\alpha\beta,\mu\nu}$; (2) has components linear in $g_{\alpha\beta,\mu\nu}$; (3) has an automatically vanishing divergence, $\boldsymbol{\nabla}\cdot\boldsymbol{G}=0$; and (4) vanishes in flat spacetime. This provides added motivation for choosing the Einstein tensor as the left side of the field equation $\boldsymbol{G}=8\pi\boldsymbol{T}$. (b) Show that, when condition (4) is dropped, the most general tensor is $\boldsymbol{G}+\Lambda\boldsymbol{g}$, where Λ is a constant.

Solution:

(a) From Exercise 17.3, the only tensor that (1) has components constructable solely from $g_{\alpha\beta}$, $g_{\alpha\beta,\mu}$, $g_{\alpha\beta,\mu\nu}$; (2) has components linear in $g_{\alpha\beta,\mu\nu}$ must be (1) $\boldsymbol{Riemann}$, (2) $\boldsymbol{g}$, and (3) tensors formed from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$ but linear in $\boldsymbol{Riemann}$. From Exercise 17.1 (a), the most general second-rank, symmetric tensor constructable from $\boldsymbol{Riemann}$ and $\boldsymbol{g}$, and linear in $\boldsymbol{Riemann}$ is

$$aR_{\alpha\beta}+bRg_{\alpha\beta}+\Lambda g_{\alpha\beta}, \tag{17.4.1}$$

where a, b and Λ are constants. From Exercise 17.1 (b) and (c), if we demamd that the tensor (3) has an automatically vanishing divergence, $\boldsymbol{\nabla} \cdot \boldsymbol{G} = 0$; and (4) vanishes in flat spacetime, the tensor must be a multiple of the Einstein tensor.

(b) From Exercise 17.1 (b), when condition (4) is dropped, the most general tensor is $\boldsymbol{G} + \Lambda \boldsymbol{g}$, where Λ is a constant.

17.5 MAGNITUDE OF COSMOLOGICAL CONSTANT

(a) What is the order of magnitude of the influence of the cosmological constant on the celestial mechanics of the solar system if $\Lambda \sim 10^{-57}\mathrm{cm}^{-2}$?

(b) Show that the mass-energy density of the vacuum $\rho^{(\mathrm{VAC})} = \Lambda/8\pi \sim 10^{-29}\mathrm{g/cm}^3$, corresponding to the maximum possible value of Λ, agrees in very rough magnitude with

$$\frac{\text{rest mass of an elementary particle}}{(\text{Compton wavelength of particle})^3} \times (\text{ gravitational fine-structure constant}) = \frac{m}{(\hbar/m)^3}\frac{m^2}{\hbar} = \frac{m^6}{\hbar^4}. \tag{17.5.1}$$

This numerology is suggestive, but has not led to any believable derivation of a stress-energy tensor for the vacuum.

Solution:

(a) The magnitude of curvature generated by sun at earth is of the order

$$\boldsymbol{R} \sim \frac{GM_\odot}{c^2 r^3} \sim 10^{-35}\mathrm{cm}^{-2}. \tag{17.5.2}$$

The order of magnitude of the influence of the cosmological constant on the celestial mechanics of the solar system if $\Lambda \sim 10^{-57}\mathrm{cm}^{-2}$ is 10^{-22}.

(b) In cgs units, we have

$$\frac{m^6}{\hbar^4} = \frac{m}{(\hbar/mc)^3}\frac{Gm^2}{\hbar c} = \frac{Gm^6c^2}{\hbar^4} \tag{17.5.3}$$

The value is $10^{-41}\mathrm{g/cm}^3$ for electron and $10^{-21}\mathrm{g/cm}^3$ for proton.

17.6 RAMIFICATIONS OF CORRESPONDENCE FOR GRAVITY IN A PASSIVE ROLE

From the correspondence relation $g_{00} = -1 - 2\Phi$, and from conditions (17.15) and (17.17) for Newtonian physics, derive the correspondence relations

$$\Gamma^i{}_{00} = \partial\Phi/\partial x^i, \quad R^i{}_{0j0} = \partial^2\Phi/\partial x^i \partial x^j. \tag{17.6.1}$$

Solution:

Keep the first order of the small value, we have

$$g^{ij} = (1 + 2\Phi)\delta^{ij} \tag{17.6.2}$$

and

$$\Gamma^{i}{}_{00} = (1+2\Phi)\frac{-g_{00,i}}{2} = \partial\Phi/\partial x^{i}. \tag{17.6.3}$$

Because all velocities small compared to c implies time derivatives small compared to space derivatives, we have

$$R^{i}{}_{0j0} = \Gamma^{i}{}_{00,j} - \Gamma^{i}{}_{0j,0} = \Gamma^{i}{}_{00,j} = \partial^{2}\Phi/\partial x^{i}\partial x^{j}. \tag{17.6.4}$$

17.7 CORRESPONDENCE IN THE LANGUAGE OF CURVED SPACE-TIME

Cartan's curved-spacetime formulation of Newtonian theory can be fond in Box 12.4 of the textbook. The formulation of the Einstein theory can be found in section 12.5 of the textbook.

17.8 EINSTEIN-FOKKER REDUCES TO NORDSTROM

The vanishing of the Weyl tensor [equation (13.50)] for a spacetime metric $\boldsymbol{g}$ guarantees that the metric is conform ally flat-i.e., that there exists a scalar field ϕ such that $\boldsymbol{g} = \phi^{2}\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a flat-spacetime metric. [See, e.g., Schouten (1954) for proof.] Thus, the Einstein-Fokker equation (17.24), $C^{\alpha\beta}{}_{\mu\nu} = 0$, is equivalent to the Nordstrom equation (17.23b). With this fact in hand, show that the Einstein-Fokker field equation $R = 24\pi T$ reduces to the Nordstrom field equation (17.23a).

Solution:

If $\boldsymbol{g} = \phi^{2}\boldsymbol{\eta}$, we have

$$\Gamma_{\alpha\beta\gamma} = \phi(\eta_{\alpha\beta}\phi_{,\gamma} + \eta_{\alpha\gamma}\phi_{,\beta} - \eta_{\beta\gamma}\phi_{,\alpha}). \tag{17.8.1}$$

And so

$$\Gamma^{\alpha}{}_{\beta\gamma} = \phi^{-1}(\delta^{\alpha}_{\beta}\phi_{,\gamma} + \delta^{\alpha}_{\gamma}\phi_{,\beta} - \eta_{\beta\gamma}\eta^{\alpha\mu}\phi_{,\mu}). \tag{17.8.2}$$

We can obtain the curvature

$$\begin{aligned}
R^{\alpha}{}_{\beta\mu\nu} &= \phi^{-1}(\delta^{\alpha}_{\nu}\phi_{,\beta\mu} - \eta_{\beta\nu}\eta^{\alpha\rho}\phi_{,\rho\mu}) - \phi^{-2}\phi_{,\mu}(\delta^{\alpha}_{\nu}\phi_{,\beta} - \eta_{\beta\nu}\eta^{\alpha\rho}\phi_{,\rho}) \\
&\quad - \phi^{-1}(\delta^{\alpha}_{\mu}\phi_{,\beta\nu} - \eta_{\beta\mu}\eta^{\alpha\rho}\phi_{,\rho\nu}) + \phi^{-2}\phi_{,\nu}(\delta^{\alpha}_{\mu}\phi_{,\beta} - \eta_{\beta\mu}\eta^{\alpha\rho}\phi_{,\rho}) \\
&\quad + \phi^{-2}(\delta^{\alpha}_{\lambda}\phi_{,\mu} + \delta^{\alpha}_{\mu}\phi_{,\lambda} - \eta_{\lambda\mu}\eta^{\alpha\rho}\phi_{,\rho})(\delta^{\lambda}_{\beta}\phi_{,\nu} + \delta^{\lambda}_{\nu}\phi_{,\beta} - \eta_{\beta\nu}\eta^{\lambda\sigma}\phi_{,\sigma}) \\
&\quad - \phi^{-2}(\delta^{\alpha}_{\lambda}\phi_{,\nu} + \delta^{\alpha}_{\nu}\phi_{,\lambda} - \eta_{\lambda\nu}\eta^{\alpha\rho}\phi_{,\rho})(\delta^{\lambda}_{\beta}\phi_{,\mu} + \delta^{\lambda}_{\mu}\phi_{,\beta} - \eta_{\beta\mu}\eta^{\lambda\sigma}\phi_{,\sigma}).
\end{aligned} \tag{17.8.3}$$

By contraction of α and μ, we have

$$\begin{aligned}
R_{\beta\nu} &= \phi^{-1}(\phi_{,\beta\nu} - \eta_{\beta\nu}\eta^{\mu\rho}\phi_{,\rho\mu}) - \phi^{-2}(\phi_{,\nu}\phi_{,\beta} - \eta_{\beta\nu}\eta^{\mu\rho}\phi_{,\mu}\phi_{,\rho}) \\
&\quad - 3\phi^{-1}\phi_{,\beta\nu} + 3\phi^{-2}\phi_{,\nu}\phi_{,\beta} \\
&\quad + 4\phi^{-2}(2\phi_{,\beta}\phi_{,\nu} - \eta_{\beta\nu}\eta^{\lambda\sigma}\phi_{,\sigma}\phi_{,\lambda}) \\
&\quad - \phi^{-2}(6\phi_{,\beta}\phi_{,\nu} - 2\eta_{\beta\nu}\eta^{\lambda\sigma}\phi_{,\sigma}\phi_{,\lambda}) \\
&= -\phi^{-1}(2\phi_{,\beta\nu} + \eta_{\beta\nu}\eta^{\mu\rho}\phi_{,\rho\mu}) + \phi^{-2}(4\phi_{,\beta}\phi_{,\nu} - \eta_{\beta\nu}\eta^{\lambda\sigma}\phi_{,\sigma}\phi_{,\lambda}).
\end{aligned} \tag{17.8.4}$$

So

$$R = \phi^{-2}\eta^{\beta\nu}R_{\beta\nu} = -6\phi^{-3}\eta^{\alpha\beta}\phi_{,\alpha\beta}. \tag{17.8.5}$$

On the hand, we have $T = \phi^{-2}\eta^{\alpha\beta}T_{\alpha\beta}$. So $R = 24\pi T$ is equivalent to

$$\eta^{\alpha\beta}\phi_{,\alpha\beta} = -4\pi\phi\eta^{\alpha\beta}T_{\alpha\beta}. \tag{17.8.6}$$

Chapter 18

WEAK GRAVITATIONAL FIELDS

18.1 GAUGE INVARIANCE OF THE RIEMANN CURVATURE

Show that in linearized theory the components of the Riemann tensor are

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}). \tag{18.1.1}$$

Then show that these components are left unchanged by a gauge transformation of the form discussed in Box 18.2 [equation (4b)]. Since the Einstein tensor is a contraction of the Riemann tensor, this shows that it is also gauge-invariant.

Solution:

The connection coefficients are

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}(h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu}). \tag{18.1.2}$$

So the curvature tensor is

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}{}_{\mu\nu,\beta} - \Gamma^{\alpha}{}_{\mu\beta,\nu} = \frac{1}{2}(h_{\mu}{}^{\alpha}{}_{,\nu\beta} + h_{\nu}{}^{\alpha}{}_{,\mu\beta} - h_{\mu\nu}{}^{,\alpha}{}_{,\beta}) - \frac{1}{2}(h_{\mu}{}^{\alpha}{}_{,\beta\nu} + h_{\beta}{}^{\alpha}{}_{,\mu\nu} - h_{\mu\beta}{}^{,\alpha}{}_{,\nu}), \tag{18.1.3}$$

i.e.

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}). \tag{18.1.4}$$

By gauge transformation $h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, we have

$$\delta R_{\alpha\mu\beta\nu} = -\frac{1}{2}(\xi_{\alpha,\nu\mu\beta} + \xi_{\nu,\alpha\mu\beta} + \xi_{\mu,\beta\nu\alpha} + \xi_{\beta,\mu\nu\alpha} - \xi_{\mu,\nu\alpha\beta} - \xi_{\nu,\mu\alpha\beta} - \xi_{\alpha,\beta\mu\nu} - \xi_{\beta,\alpha\mu\nu}) = 0. \tag{18.1.5}$$

So these components are left unchanged by a gauge transformation.

18.2 JUSTIFICATION OF LORENTZ GAUGE

Let a particular solution to the field equations (18.7) of linearized theory be given, in an arbitrary gauge. Show that there necessarily exist four generating functions $\xi_\mu(t, x^j)$ whose gauge transformation [Box 18.2, eq. (4b)] makes

$$\bar{h}^{\mathrm{NEW}\,\mu\alpha}{}_{,\alpha} = 0. \tag{18.2.1}$$

Also show that a subsequent gauge transformation leaves this Lorentz gauge condition unaffected if and only if its generating functions satisfy the sourceless wave equation

$$\xi^{\alpha,\beta}{}_{\beta} = 0. \tag{18.2.2}$$

Solution:

$$\bar{h}^{\mathrm{NEW}\,\mu\alpha}{}_{,\alpha} = (h^{\mathrm{OLD}\,\mu\alpha}{}_{,\alpha} - \xi^{\mu,\alpha}{}_{\alpha} - \xi^{\alpha,\mu}{}_{\alpha}) - \frac{1}{2}\eta^{\mu\alpha}(h^{\mathrm{OLD}} - 2\xi^{\nu}{}_{,\nu})_{,\alpha} = \bar{h}^{\mathrm{OLD}\,\mu\alpha}{}_{,\alpha} - \xi^{\mu,\alpha}{}_{\alpha}. \tag{18.2.3}$$

If we demand that ξ^{μ} satisfy the equation $\xi^{\mu,\alpha}{}_{\alpha} = \bar{h}^{\mathrm{OLD}\,\mu\alpha}{}_{,\alpha}$, then we have

$$\bar{h}^{\mathrm{NEW}\,\mu\alpha}{}_{,\alpha} = 0. \tag{18.2.4}$$

And if $\xi^{\mu,\alpha}{}_{\alpha} = 0$, the gauge transformation would leave this Lorentz gauge condition unaffected.

18.3 EXTERNAL FIELD OF A STATIC, SPHERICAL BODY

Consider the external gravitational field of a static spherical body, as described in the body's (nearly) Lorentz frame-i.e., in a nearly rectangular coordinate system $|h_{\mu\nu}| \ll 1$, in which the body is located at $x = y = z = 0$ for all t. By fiat, adopt Lorentz gauge.

(a) Show that the field equations (18.8b) and gauge conditions (18.8a) imply

$$\begin{aligned} &\bar{h}_{00} = 4M/(x^2+y^2+z^2)^{1/2}, \quad \bar{h}_{0j} = \bar{h}_{jk} = 0, \\ &h_{00} = h_{xx} = h_{yy} = h_{zz} = 2M/(x^2+y^2+z^2)^{1/2}, \quad h_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta, \end{aligned} \tag{18.3.1}$$

where M is a constant.

(b) Adopt spherical polar coordinates,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{18.3.2}$$

By regarding $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ as components of tensors in fiat spacetime (see end of Box 18.2), and by using the usual tensor transformation laws, put the solution found in (a) into the form

$$\begin{aligned} &\bar{h}_{00} = \frac{4M}{r}, \quad \bar{h}_{0j} = \bar{h}_{jk} = 0, \\ &h_{00} = \frac{2M}{r}, \quad h_{0j} = 0, \quad h_{jk} = \frac{2M}{r} g_{jk_{\mathrm{flat}}} \end{aligned} \tag{18.3.3}$$

where $g_{\alpha\beta_{\mathrm{flat}}}$ are the components of the fiat-spacetime metric in the spherical coordinate system

$$\begin{aligned} &g_{00_{\mathrm{flat}}} = -1, \quad g_{rr_{\mathrm{flat}}} = 1, \quad g_{\theta\theta_{\mathrm{flat}}} = r^2, \\ &g_{\phi\phi_{\mathrm{flat}}} = r^2\sin^2\theta, \quad g_{\alpha\beta_{\mathrm{flat}}} = 0 \text{ when } \alpha \neq \beta. \end{aligned} \tag{18.3.4}$$

Thereby conclude that the general relativistic line element, accurate to linearized order, is

$$ds^2 = -(1-2M/r)dt^2 + (1+2M/r)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2). \tag{18.3.5}$$

(c) Derive this general, static, spherically symmetric, Lorentz-gauge, vacuum solution to the linearized field equations from scratch, working entirely in spherical coordinates.

(d) Calculate the Riemann curvature tensor for this gravitational field. The answer should agree with equation (1.14).

Solution:

(a) From (18.8b), we have

$$-\bar{h}_{00,\alpha}{}^{\alpha} = 16\pi M\delta(r), \quad -\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \text{ if } \mu \neq 0 \text{ or } \nu \neq 0. \tag{18.3.6}$$

So we have the solution

$$\bar{h}_{00} = 4M/(x^2+y^2+z^2)^{1/2}, \quad \bar{h}_{0j} = \bar{h}_{jk} = 0, \tag{18.3.7}$$

and it satisfy the equation (18.8a). Since $h_{\mu\nu} = \bar{h}_{\mu\nu} - 1/2\,\eta_{\mu\nu}\bar{h}$ and $\bar{h} = -4M/(x^2+y^2+z^2)^{1/2}$, we can get

$$h_{00} = h_{xx} = h_{yy} = h_{zz} = 2M/(x^2+y^2+z^2)^{1/2}, \quad h_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta. \tag{18.3.8}$$

(b) The transformation matrix is

$$T^{\alpha}{}_{\alpha'} \equiv \frac{\partial x^{\mathrm{M}}}{\partial x^{\mathrm{S}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{18.3.9}$$

So we have

$$h_{\alpha'\beta'} = T^{\alpha}{}_{\alpha'}\, h_{\alpha\beta} T^{\beta}{}_{\beta'} \tag{18.3.10}$$

The calculation is straight forward and we can get

$$h_{\alpha'\beta'} = \frac{2M}{r}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{18.3.11}$$

(c) In spherical coordinates, we have

$$\bar{h}^{\mu\alpha}{}_{;\alpha} = 0, \quad \bar{h}_{\mu\nu;\alpha}{}^{\alpha} = -16\pi T_{\mu\nu}. \tag{18.3.12}$$

The components of coefficients are calculated from the metric tensor in spherical coordinates:

$$\Gamma^{r}{}_{\theta\theta} = -r,\ \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta,\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r},\ \Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta,\ \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r},\ \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta. \tag{18.3.13}$$

Then we can get

$$\begin{aligned} \bar{h}_{00;\alpha}{}^{\alpha} &= \bar{h}_{00,\alpha}{}^{\alpha} - g^{\alpha\beta}\Gamma^{\gamma}{}_{\alpha\beta}\bar{h}_{\mu\nu,\gamma} \\ &= \bar{h}_{00,\alpha}{}^{\alpha} - (g^{\theta\theta}\Gamma^{r}{}_{\theta\theta} + g^{\phi\phi}\Gamma^{r}{}_{\phi\phi})\bar{h}_{00,r} - g^{\phi\phi}\Gamma^{\theta}{}_{\phi\phi}h_{00,\theta} \\ &= \bar{h}_{00,\alpha}{}^{\alpha} + \frac{2}{r}\bar{h}_{00,r} + \frac{\cos\theta}{r^2\sin\theta}h_{00,\theta} = -16\pi M\delta(r). \end{aligned} \tag{18.3.14}$$

The solution will be

$$\bar{h}_{00} = \frac{4M}{r}. \tag{18.3.15}$$

Similarly, we will get $\bar{h}_{0j} = \bar{h}_{jk} = 0$.

(d)We have

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}) \tag{18.3.16}$$

and $h_{\mu\nu} = 2M/r\delta_{\mu\nu}$. If we demand that $r = (0,0,z)$, we can get the equation (1.14).

18.4 SPACETIME CURVATURE FOR A PLANE GRAVITATIONAL WAVE

Calculate the components of the Riemann curvature tensor [equations (18.9)] for the gravitational plane wave (18.11).

Solution:

On the one hand, we have

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(\bar{h}_{\alpha\nu,\mu\beta} + \bar{h}_{\mu\beta,\nu\alpha} - \bar{h}_{\mu\nu,\alpha\beta} - \bar{h}_{\alpha\beta,\mu\nu}) - \frac{1}{4}(\eta_{\alpha\nu}\bar{h}_{,\mu\beta} + \eta_{\mu\beta}\bar{h}_{,\nu\alpha} - \eta_{\mu\nu}\bar{h}_{,\alpha\beta} - \eta_{\alpha\beta}\bar{h}_{,\mu\nu}). \quad (18.4.1)$$

On the other hand, we have

$$\bar{h}_{xx} = \bar{h}_{xx}(t-z), \quad \bar{h}_{yy} = \bar{h}_{yy}(t-z), \quad \bar{h}_{xy} = \bar{h}_{xy}(t-z) \quad (18.4.2)$$

and all other first order components of metric vanish. So we have

$$\begin{aligned}
R_{xtxt} &= -\frac{1}{2}\bar{h}_{xx,tt} + \frac{1}{4}\eta_{xx}\bar{h}_{,tt} = -\frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},\\
R_{xtyt} &= -\frac{1}{2}\bar{h}_{xy,tt},\\
R_{xtzt} &= 0,\\
R_{xtxy} &= 0,\\
R_{xtyz} &= -\frac{1}{2}\bar{h}_{xy,tz} = \frac{1}{2}\bar{h}_{xy,tt},\\
R_{xtzx} &= \frac{1}{2}h_{xx,tz} - \frac{1}{4}\eta_{xx}\bar{h}_{,tz} = -\frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},\\
R_{ytyt} &= -\frac{1}{2}\bar{h}_{yy,tt} + \frac{1}{4}\eta_{yy}\bar{h}_{,tt} = \frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},\\
R_{ytzt} &= 0,\\
R_{ytxy} &= 0,\\
R_{ytyz} &= -\frac{1}{2}h_{yy,tz} + \frac{1}{4}\eta_{yy}\bar{h}_{,tz} = -\frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},\\
R_{ytzx} &= \frac{1}{2}h_{xy,tz} = -\frac{1}{2}\bar{h}_{xy,tt},\\
R_{ztzt} &= \frac{1}{4}(\eta_{zz}\bar{h}_{,tt} + \eta_{tt}\bar{h}_{,zz}) = 0,\\
R_{ztxy} &= 0,\\
R_{ztyz} &= 0,\\
R_{ztzx} &= 0,\\
R_{xyxy} &= 0,\\
R_{xyyz} &= 0,\\
R_{xyzx} &= 0,\\
R_{yzyz} &= -\frac{1}{2}\bar{h}_{yy,zz} + \frac{1}{4}\eta_{yy}\bar{h}_{,zz} = \frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},\\
R_{yzzx} &= \frac{1}{2}\bar{h}_{xy,zz} = \frac{1}{2}\bar{h}_{xy,tt},\\
R_{xzxz} &= -\frac{1}{2}\bar{h}_{xx,zz} + \frac{1}{4}\eta_{xx}\bar{h}_{,zz} = -\frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt},
\end{aligned} \quad (18.4.3)$$

All other components vanish except those obtainable from the above by the symmetries of Riemann tensor.

18.5 A PRIMITIVE GRAVITATIONAL-WAVE DETECTOR

Two beads slide almost freely on a smooth stick; only slight friction impedes their sliding. The stick falls freely through spacetime, with its center moving along a geodesic and its ends attached to gyroscopes, so they do not rotate. The beads are positioned equidistant (distance $l/2$) from the stick's center. Plane gravitational waves [equation (18.11) and exercise 18.4], impinging on the stick, push the beads back and forth ("geodesic deviation"; "tidal gravitational forces"). The resultant friction of beads on stick heats the stick; and the passage of the waves is detected by measuring the rise in stick temperature. Neglecting the effect of friction on the beads' motion, calculate the proper distance separating them as a function of time.

Solution:

Let $\boldsymbol{\xi}$ be the separation between the beads; and let $\boldsymbol{n} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$ be a unit vector that points along the stick in the stick's own rest frame. Then their separation has magnitude $l = \boldsymbol{\xi} \cdot \boldsymbol{n}$. The fact that the stick is nonrotating is embodied in a parallel-transport law for $\boldsymbol{n}$, $\boldsymbol{\nabla_u n} = 0$. Thus,

$$\frac{dl}{d\tau} = \boldsymbol{\nabla_u}(\boldsymbol{\xi} \cdot \boldsymbol{n}) = (\boldsymbol{\nabla_u \xi}) \cdot \boldsymbol{n}$$
$$\frac{d^2l}{d\tau^2} = \boldsymbol{\nabla_u}((\boldsymbol{\nabla_u \xi}) \cdot \boldsymbol{n}) = (\boldsymbol{\nabla_u \nabla_u \xi}) \cdot \boldsymbol{n}. \tag{18.5.1}$$

where $\boldsymbol{\tau}$ is the stick's proper time. But geodesic deviation $\boldsymbol{\xi}$ satisfy that

$$\boldsymbol{\nabla_u \nabla_u \xi} = -\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{u})\boldsymbol{u}. \tag{18.5.2}$$

So we have

$$\frac{d^2l}{d\tau^2} = -\boldsymbol{Riemann}(\boldsymbol{n}, \boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{u}) \tag{18.5.3}$$

In the stick's local Lorentz frame, we have

$$\frac{d^2l}{dt^2} = -lR_{i0j0}n^i n^j. \tag{18.5.4}$$

Orient the coordinates so the waves propagate in the z-direction and the stick's direction has components $n^z = \cos\theta$, $n^x = \sin\theta\cos\phi$, $n^y = \sin\theta\sin\phi$. So we have

$$\frac{d^2l}{dt^2} = -l\left[-\frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt}\sin^2\theta\cos^2\phi + \frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,tt}\sin^2\theta\sin^2\phi - \bar{h}_{xy,tt}\sin^2\theta\sin\phi\cos\phi\right] \tag{18.5.5}$$

Assume $\delta l \ll l_0$. The solution is

$$l = l_0\left[1 + \frac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})\sin^2\theta\cos 2\phi + \frac{1}{2}\bar{h}_{xy}\sin^2\theta\sin 2\phi\right], \tag{18.5.6}$$

where $\bar{h}_{jk}$ are evaluated on the stick's world line ($x = y = z = 0$).

18.6 BENDING OF LIGHT BY THE SUN

To high precision, the sun is static and spherical, so its external line element is (18.15c) with $\Phi = -M/r$; i.e.

$$ds^2 = -(1 - 2M/r)dt^2 + (1 + 2M/r)(dx^2 + dy^2 + dz^2) \text{ everywhere outside sun.} \tag{18.6.1}$$

A photon moving in the equatorial plane ($z = 0$) of this curved spacetime gets deflected very slightly from the world line

$$x = t, \quad y = b \equiv \text{ "impact parameter," } \quad z = 0. \tag{18.6.2}$$

Calculate the amount of deflection as follows.

(a) Write down the geodesic equation (16.4a) for the photon's world line,

$$\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma = 0. \tag{18.6.3}$$

Here $\boldsymbol{p} = d/d\lambda =$ (4-momentum of photon) = (tangent vector to photon's null geodesic).

(b) By evaluating the connection coefficients in the equatorial plane, and by using the approximate values, $|p^y| \ll p^0 \approx p^x$, of the 4-momentum components corresponding to the approximate world line 18.6.2, show that

$$\frac{dp^y}{d\lambda} = \frac{-2Mb}{(x^2+b^2)^{3/2}} p^x \frac{dx}{d\lambda}, \quad p^x = p^0 \left[1 + O\left(\frac{M}{b}\right)\right] = \text{ const } \left[1 + O\left(\frac{M}{b}\right)\right]. \tag{18.6.4}$$

(c) Integrate this equation for p^y, assuming $p^y = 0$ at $x = -\infty$ (photon moving precisely in x-direction initially); thereby obtain

$$p^y(x = +\infty) = -\frac{4M}{b} p^x. \tag{18.6.5}$$

(d) Show that this corresponds to deflection of light through the angle

$$\Delta\phi = 4M/b = 1''.75(R_\odot/b), \tag{18.6.6}$$

where $R_\odot$ is the radius of the sun. For a comparison of this prediction with experiment, see Box 40.1.

Solution:

(a) The non vanishing components of connection coefficients are

$$\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Phi_{,i}, \quad \Gamma^i{}_{00} = \Phi_{,i}, \quad \Gamma^i{}_{jk} = -\delta_{ij}\Phi_{,k} - \delta_{ik}\Phi_{,j} + \delta_{jk}\Phi_{,i}. \tag{18.6.7}$$

The geodesic equation of the photon is

$$\begin{aligned} &\frac{dp^0}{d\lambda} + 2\Phi_{,i} p^0 p^i = 0, \\ &\frac{dp^i}{d\lambda} + \Phi_{,i} p^0 p^0 - 2\Phi_{,j} p^i p^j + \Phi_i p^j p^j = 0. \end{aligned} \tag{18.6.8}$$

(b) Using the approximate values, $|p^y| \ll p^0 \approx p^x$, we have

$$\frac{dp^y}{d\lambda} = -2\Phi_{,y}(p^x)^2 = \frac{-2Mb}{(x^2+b^2)^{3/2}} p^x \frac{dx}{d\lambda} \tag{18.6.9}$$

and

$$\frac{dp^x}{d\lambda} = -2\Phi_{,y} p^y p^x. \tag{18.6.10}$$

Integrate 18.6.10 and we can get

$$\Delta p^x = -2p^0 \Delta\Phi \sim p^0 \cdot O\left(\frac{M}{b}\right). \tag{18.6.11}$$

And so

$$p^x = p^0 \left[1 + O\left(\frac{M}{b}\right)\right] = \text{ const } \left[1 + O\left(\frac{M}{b}\right)\right]. \tag{18.6.12}$$

(c) Integrate 18.6.9 and we can get

$$p^y(x = +\infty) = p^x \int_{-\infty}^{\infty} \frac{-2Mb}{(x^2+b^2)^{3/2}} dx = -\frac{4M}{b} p^x. \tag{18.6.13}$$

(d)

$$\Delta\phi = \left|\frac{p^y(x=+\infty)}{p^x}\right| = \frac{4M}{b} = \frac{4M}{R_\odot}(R_\odot/b) = 8.49\times10^{-6}(R_\odot/b) = 1.75''(R_\odot/b) \tag{18.6.14}$$

18.7 GRAVITATIONAL REDSHIFT

(a) Use the geodesic equation for a photon, written in the form

$$\frac{dp_\mu}{d\lambda} - \Gamma^\alpha{}_{\mu\beta}p_\alpha p^\beta = 0, \tag{18.7.1}$$

to prove that any photon moving freely in the sun's gravitational field [line element (18.16)] has $dp_0/d\lambda = 0$; i.e.,

$$p_0 = \text{ constant along photon's world line.} \tag{18.7.2}$$

(b) An atom at rest on the sun's surface emits a photon of wavelength λ_e, as seen in its orthonormal frame. An atom at rest far from the sun receives the photon, and measures its wavelength to be λ_e. Show that the photon is redshifted by the amount

$$z \equiv \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{M_\odot}{R_\odot} = 2\times10^{-6}. \tag{18.7.3}$$

Solution:

(a) The time component of geodesic equation is

$$\frac{dp_0}{d\lambda} = \Gamma^\alpha{}_{0\beta}p_\alpha p^\beta. \tag{18.7.4}$$

From equation 18.6.7, we have

$$\frac{dp_0}{d\lambda} = \Gamma^0{}_{0i}p_0p^i + \Gamma^i{}_{00}p_ip^0 = \Phi_{,i}(p_0p^i + p_ip^0) = 0 \tag{18.7.5}$$

by keeping the term linear in Φ. So p_0 is constant along photon's world line.

(b) For observer at rest on the sun's surface, we have $\boldsymbol{u}_e = (1-\Phi,0,0,0)$ and

$$h/\lambda_e = \boldsymbol{p}\cdot\boldsymbol{u}_e = p_0(1-\Phi). \tag{18.7.6}$$

For observer at rest far from the sun receives the photon, we have $\boldsymbol{u}_r = (1,0,0,0)$ and

$$h/\lambda_r = \boldsymbol{p}\cdot\boldsymbol{u}_r = p_0. \tag{18.7.7}$$

So

$$\frac{\lambda_r}{\lambda_e} - 1 = -\Phi = \frac{M_\odot}{R_\odot} = 2\times10^{-6}. \tag{18.7.8}$$

Chapter 19

MASS AND ANGULAR MOMENTUM OF A GRAVITATING SYSTEM

19.1 DERIVATION OF METRIC FAR OUTSIDE A WEAKLY GRAVITATING BODY

(a) Derive equation (19.5).

(b) Prove that the system's mass and angular momentum are conserved.

Solution:

(a) The outline of the derivation is given as "Hint" in the textbook. However, the complete derivation is so tedious that I can not figure out the solution here. Any help is appreciated.

(b) For a weakly gravitating source, self-gravity has negligible influence. So

$$T^{\alpha\beta}{}_{,\beta} \approx T^{\alpha\beta}{}_{;\beta} = 0. \tag{19.1.1}$$

Then, we have

$$\frac{dM}{dt} = \int T^{00}{}_{,0}\, d^3x = -\int T^{0i}{}_{,i}\, d^3x = -\int_{\Sigma\to\infty} T^{0i} d\Sigma_i = 0 \tag{19.1.2}$$

and

$$\begin{aligned}
\frac{dS_k}{dt} &= \int \epsilon_{klm} x^l T^{m0}{}_{,0}\, d^3x \\
&= -\int \epsilon_{klm} x^l T^{mn}{}_{,n}\, d^3x \\
&= -\int (\epsilon_{klm} x^l T^{mn})_{,n} d^3x + \int \epsilon_{knm} T^{mn} d^3x \\
&= -\int_{\Sigma\to\infty} \epsilon_{klm} x^l T^{mn} d\Sigma_n + 0 \\
&= 0.
\end{aligned} \tag{19.1.3}$$

19.2 GYROSCOPE PRECESSION

Derive equation (19.10) for the angular velocity of gyroscope precession.

Solution:

For the axis of gyroscope, we have

$$\left(\frac{D\boldsymbol{L}}{D\tau}\right)^j = \frac{dL^j}{d\tau} + u^\nu \Gamma^j{}_{k\nu} L^k = \frac{dL^j}{dt} + \Gamma^j{}_{k0} L^k, \tag{19.2.1}$$

keeping the first order of velocity and gravitaional potential. So we have

$$\frac{dL^j}{dt} = -\Gamma^j{}_{k0} L^k \equiv \epsilon^{jik} \Omega_i L_k, \tag{19.2.2}$$

where Ω_k is the angular velocity of the axis of gyroscope relative to the coordinate axis, which is tied rigidly to the "fixed stars". We can get

$$\Omega^l = -\frac{1}{2}\epsilon^{jlk}\Gamma_{jk0} = \frac{1}{2}\epsilon^{ljk}\Gamma_{jk0} = \frac{1}{4}\epsilon^{ljk}(g_{jk,0} + g_{j0,k} - g_{k0,j}) = \frac{1}{2}\epsilon^{ljk} g_{j0,k}. \tag{19.2.3}$$

As

$$g_{j0} = -2\epsilon_{jmn} S^m \frac{x^n}{r^3}, \tag{19.2.4}$$

we have

$$g_{j0,k} = -2\epsilon_{jmn} S^m \left(\frac{\delta^n_k}{r^3} - \frac{3x^n x^k}{r^5}\right). \tag{19.2.5}$$

Now we can get

$$\Omega^l = -\epsilon^{jkl}\epsilon_{jmn} S^m \left(\frac{\delta^n_k}{r^3} - \frac{3x^n x^k}{r^5}\right) = \frac{1}{r^3}\left(-S^l + \frac{3S^k x_k x^l}{r^2}\right), \tag{19.2.6}$$

i.e.

$$\boldsymbol{\Omega} = \frac{1}{r^3}\left[-\boldsymbol{S} + \frac{3(\boldsymbol{S}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2}\right] \tag{19.2.7}$$

19.3 GRAVITATIONAL FIELD FAR FROM A STATIONARY, FUlIY RELATIVISTIC SOURCE

The outline of the derivation given in the textbook is very detailed. A few steps omitted in the textbook is not hard to follow. Sorry for not writing down the complete derivation here.

Chapter 20

CONSERVATION LAWS FOR 4-MOMENTUM AND ANGULAR MOMENTUM

20.1 FLUX INTEGRAL FOR TOTAL MASS-ENERGY IN LINEARIZED THEORY

Show that the flux integral (20.6) for P^0 reduces to (20.7). Then show that, when applied to a nearly Newtonian source [line element (18.15c)], it reduces further to the familiar Newtonian flux integral (20.2).

Solution:

By the definition of $H^{\alpha\beta\gamma\delta}$, we have

$$H^{0i0j} = -(\bar{h}^{00}\eta^{ij}+\eta^{00}\bar{h}^{ij}-\bar{h}^{i0}\eta^{0j}-\bar{h}^{0j}\eta^{i0}) = -\delta^{ij}\bar{h}^{00}+\bar{h}^{ij} = -\delta^{ij}h^{00}+h^{ij}-\delta^{ij}h = h^{ij}-\delta^{ij}h^{k}{}_{k}. \quad (20.1.1)$$

So

$$\begin{aligned} P^0 &= \frac{1}{16\pi}\int_S H^{0i0j}{}_{,i}\, d^2S_j \\ &= \frac{1}{16\pi}\int_S (h^{ij}{}_{,i} - h^{k}{}_{k,j}) d^2S_j \\ &= \frac{1}{16\pi}\int_S (g_{jk,k} - g_{kk,j}) d^2S_j. \end{aligned} \quad (20.1.2)$$

As $h_{ij} = -2\Phi\delta_{ij}$, we have

$$\begin{aligned} P^0 &= \frac{1}{16\pi}\int_S (-2\Phi_{,j} + 6\Phi_{,j}) d^2S_j \\ &= \frac{1}{4\pi}\int_S \Phi_{,j} d^2S_j. \end{aligned} \quad (20.1.3)$$

20.2 FLUX INTEGRAL FOR ANGULAR MOMENTUM IN LINEARIZED THEORY

Derive the Gaussian flux integral (20.8) for $J^{\mu\nu}$.

Solution:

From the $H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi T^{\mu\nu}$, we have

$$(x^\mu H^{\nu\alpha 0k}{}_{,\alpha})_{,k} - H^{\nu j 0\mu}{}_{,j} - H^{\nu 00\mu}{}_{,0} = H^{\nu\alpha 0\mu}{}_{,\alpha} + x^\mu H^{\nu\alpha 0k}{}_{,\alpha k} - H^{\nu j 0\mu}{}_{,j} - H^{\nu 00\mu}{}_{,0} = 16\pi x^\mu T^{\nu 0}. \quad (20.2.1)$$

So we have

$$\begin{aligned} J^{\mu\nu} &= \int (x^\mu T^{\nu 0} - x^\nu T^{\mu 0}) d^3x \\ &= \frac{1}{16\pi}\int \left[(x^\mu H^{\nu\alpha 0k}{}_{,\alpha})_{,k} - H^{\nu j 0\mu}{}_{,j} - H^{\nu 00\mu}{}_{,0} - (x^\nu H^{\mu\alpha 0k}{}_{,\alpha})_{,k} + H^{\mu j 0\nu}{}_{,j} + H^{\mu 00\nu}{}_{,0}\right] d^3x \\ &= \frac{1}{16\pi}\int \left[(x^\mu H^{\nu\alpha 0k}{}_{,\alpha})_{,k} - H^{\nu j 0\mu}{}_{,j} - (x^\nu H^{\mu\alpha 0k}{}_{,\alpha})_{,k} + H^{\mu j 0\nu}{}_{,j}\right] d^3x \\ &= \frac{1}{16\pi}\int_{\mathrm{S}} \left(x^\mu H^{\nu\alpha 0j}{}_{,\alpha} - H^{\nu j 0\mu} - x^\nu H^{\mu\alpha 0j}{}_{,\alpha} + H^{\mu j 0\nu}\right) d^2S_j \end{aligned} \quad (20.2.2)$$

20.3 FLUX INTEGRALS FOR AN ARBITRARY STATIONARY SOURCE

(a) Use the flux integrals (20.9) to calculate P^μ and $J^{\mu\nu}$ for an arbitrary stationary source. For the asymptotically flat metric around the source, use (19.13), with the gravitational radiation terms set to zero.

(b) Verify that the "auxiliary equations" (20.10) to (20.12) give the correct answer for this source's total mass-energy M and intrinsic angular momentum S^μ.

Solution:

(a)

$$\begin{aligned} H^{0i0j} &= h^{ij} - \delta^{ij} h^k{}_k \\ H^{0ijk} &= -\delta^{ik} h^{0j} + \delta^{ij} h^{0k} \\ H^{ijkl} &= -(\bar{h}^{ik}\delta^{jl} + \bar{h}^{jl}\delta^{ik} - \bar{h}^{jk}\delta^{il} - \bar{h}^{il}\delta^{jk}) \end{aligned} \quad (20.3.1)$$

The asymptotically flat metric around the source with the gravitational radiation terms set to zero is

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2M}{r} + \frac{2M^2}{r^2} + O\left(\frac{1}{r^3}\right)\right] dt^2 - \left[4\epsilon_{jkl} S^k \frac{x^l}{r^3} + O\left(\frac{1}{r^3}\right)\right] dt dx^j \\ &+ \left[\left(1 + \frac{2M}{r} + \frac{3M^2}{2r^2}\right)\delta_{jk} + O\left(\frac{1}{r^3}\right)\right] dx^j dx^k \end{aligned} \quad (20.3.2)$$

So we have

$$
\begin{aligned}
H^{0i0j} &= -\frac{4M}{r}\delta^{ij} + O\left(\frac{1}{r^2}\right) \\
H^{0ijk} &= 2\left(-\delta^{ik}\epsilon^{jmn} + \delta^{ij}\epsilon^{kmn}\right) S_m \frac{x_n}{r^3} + O\left(\frac{1}{r^3}\right) \\
H^{ijkl} &= O\left(\frac{1}{r^2}\right).
\end{aligned} \tag{20.3.3}
$$

Now we have

$$
\begin{aligned}
P^0 &= \frac{1}{16\pi}\int_{\mathcal{S}} H^{0i0j}{}_{,i} d^2S_j = \frac{M}{4\pi}\int_{\mathcal{S}} \frac{x^j}{r^3} d^2S_j = M; \\
P^i &= \frac{1}{16\pi}\int_{\mathcal{S}} H^{ik0j}{}_{,k} d^2S_j = \frac{1}{16\pi}\int_{\mathcal{S}} O\left(\frac{1}{r^3}\right) d^2S_j = 0; \\
J^{0i} &= \frac{1}{16\pi}\int_{\mathcal{S}} (x^0 H^{ik0j}{}_{,k} - x^i H^{0k0j}{}_{,k} + H^{0j0i} - H^{ij00}) d^2S_j \\
&= \frac{1}{16\pi}\int_{\mathcal{S}} \left[tO\left(\frac{1}{r^3}\right) - x^i x^j f(r) + g(r)\delta^{ij}\right] d^2S_j \\
&= 0; \\
J^{ij} &= \frac{1}{16\pi}\int_{\mathcal{S}} (x^i H^{jk0l}{}_{,k} - x^j H^{ik0l}{}_{,k} + H^{il0j} - H^{jl0i}) d^2S_l.
\end{aligned} \tag{20.3.4}
$$

As

$$
\begin{aligned}
\int_{\mathcal{S}} x^i H^{jk0l}{}_{,k} d^2S_l &= 2\int_{\mathcal{S}} \left(-\delta^{lk}\epsilon^{jmn} + \delta^{lj}\epsilon^{kmn}\right) S_m \left(\frac{\delta_{nk}}{r^3} - \frac{3x_n x_k}{r^5}\right) \frac{x^i x_l}{r} d^2S \\
&= -2\epsilon^{jmn} S_m \int_{\mathcal{S}} \left(\frac{\delta^l_n}{r^3} - \frac{3x_n x^l}{r^5}\right) \frac{x^i x_l}{r} d^2S \\
&= 4\epsilon^{jmn} S_m \int_{\mathcal{S}} \frac{x^i x_n}{r^4} d^2S \\
&= 4\delta^i_n \epsilon^{jmn} S_m \int_{\mathcal{S}} \frac{1}{3r^2} d^2S \\
&= \frac{16\pi}{3}\epsilon^{ijm} S_m
\end{aligned} \tag{20.3.5}
$$

and

$$
\begin{aligned}
\int_{\mathcal{S}} H^{il0j} d^2S_l &= 2\int_{\mathcal{S}} \left(-\delta^{jl}\epsilon^{imn} + \delta^{ji}\epsilon^{lmn}\right) S_m \frac{x_n}{r^3}\frac{x_l}{r} d^2S \\
&= -2\epsilon^{imn} S_m \int_{\mathcal{S}} \frac{x_n x^j}{r^4} d^2S \\
&= -2\delta^j_n \epsilon^{imn} S_m \int_{\mathcal{S}} \frac{1}{3r^2} d^2S \\
&= \frac{8\pi}{3}\epsilon^{ijm} S_m.
\end{aligned} \tag{20.3.6}
$$

We have

$$
J^{ij} = \frac{1}{3}\epsilon^{ijm} S_m - \frac{1}{3}\epsilon^{jim} S_m + \frac{1}{6}\epsilon^{ijm} S_m - \frac{1}{6}\epsilon^{jim} S_m = \epsilon^{ijm} S_m. \tag{20.3.7}
$$

(b) We have

$$
(-P^\mu P_\mu)^{1/2} = P^0 = M, \quad Y^\mu = -J^{\mu\nu} P_\nu / M^2 = J^{\mu 0}/M = 0, \tag{20.3.8}
$$

and so

$$
\frac{1}{2}\epsilon_{\mu\nu\sigma\rho}(J^{\mu\nu} - Y^\mu P^\nu + Y^\nu P^\mu) P^\sigma / M = \frac{1}{2}\epsilon_{ij0\rho} J^{ij} = (0, S_l). \tag{20.3.9}
$$

So the “auxiliary equations”give the correct answer for this source’s total mass-energy and intrinsic angular momentum.

20.4 FORM OF $H^{\mu\alpha\nu\beta}_{\rm L-L}$ FAR FROM SOURCE

Show that the entities $H^{\mu\alpha\nu\beta}_{\rm L-L}$ of equations (20.20) reduce to $H^{\mu\alpha\nu\beta}$ (20.3) in the weak-field region far outside the source.

Solution:

In the limit of weak field, keeping the first order of $h_{\mu\nu}$, we have

$$\begin{aligned} g &= -\epsilon^{\alpha\beta\gamma\delta} g_{0\alpha} g_{1\beta} g_{2\gamma} g_{3\delta} \\ &= -1 - \epsilon^{\alpha\beta\gamma\delta}(h_{0\alpha}\eta_{1\beta}\eta_{2\gamma}\eta_{3\delta} + \eta_{0\alpha}h_{1\beta}\eta_{2\gamma}\eta_{3\delta} + \eta_{0\alpha}\eta_{1\beta}h_{2\gamma}\eta_{3\delta} + \eta_{0\alpha}\eta_{1\beta}\eta_{2\gamma}h_{3\delta}) \\ &= -1 + h_{00} - h_{11} - h_{22} - h_{33} \\ &= -1 - h. \end{aligned} \tag{20.4.1}$$

So we obtain

$$\mathfrak{g}^{\alpha\beta} = (-g)^{1/2} g^{\alpha\beta} = (1 + \frac{h}{2})(\eta^{\alpha\beta} - h^{\alpha\beta}) = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta}. \tag{20.4.2}$$

And so

$$H^{\mu\alpha\nu\beta}_{\rm L-L} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta} = -(\eta^{\mu\nu}\bar{h}^{\alpha\beta} + \eta^{\alpha\beta}\bar{h}^{\mu\nu} - \eta^{\mu\beta}\bar{h}^{\alpha\nu} - \eta^{\alpha\nu}\bar{h}^{\mu\beta}) + \eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\alpha\nu}\eta^{\mu\beta}. \tag{20.4.3}$$

So in the limit of weak field, $H^{\mu\alpha\nu\beta}_{\rm L-L}$ differs from $H^{\mu\alpha\nu\beta}$ with only a constant term, which will vanish when taking derivatives.

20.5 TOTAL MASS-ENERGY IN NEWTONIAN LIMIT

(a) Calculate $t^{\alpha\beta}_{\rm L-L}$ for the nearly Newtonian metric

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{jk}dx^j dx^k. \tag{20.5.1}$$

Assume the source is slowly changing, so that time derivatives of Φ can be neglected compared to space derivatives.

(b) Let the source of the gravitational field be a perfect fluid with

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \quad p/\rho \sim v^2 \equiv (dx/dt)^2 \sim |\Phi|. \tag{20.5.2}$$

Let the Newtonian potential satisfy the source equation

$$\Phi_{,jj} = 4\pi\rho. \tag{20.5.3}$$

Show that the energy of the source is

$$P^0 = \int (T^{00} + t^{00})(-g)d^3x = \int \left[\rho/(1 - v^2)^{1/2}] + \frac{1}{2}\rho v^2 + \frac{1}{2}\rho\Phi\right](g_{xx}g_{yy}g_{zz})^{1/2}dxdydz \tag{20.5.4}$$

(c) Show that the "equations of motion" $T^{\mu\nu}_{\rm L-L\,eff\,,\nu} = 0$ reduce to the standard equations (16.3) of Newtonian hydrodynamics.

Solution:

(a) According to 20.4.2, we have

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta} + O(\Phi^2), \tag{20.5.5}$$

where

$$\bar{h}^{00} = -4\Phi, \quad \bar{h}^{\alpha\beta} = 0 \text{ if } \alpha \neq 0 \text{ or } \beta \neq 0. \tag{20.5.6}$$

So we have

$$\mathfrak{g}^{\alpha\beta}{}_{,\gamma} = 4\Phi_{,i} + O(\Phi^2) \text{ if } \alpha = 0, \beta = 0, \gamma = i, \tag{20.5.7}$$

and all other components of $\mathfrak{g}^{\alpha\beta}{}_{,\gamma}$ all vanish up to the first order of Φ.

According to (20.22) in the text book, we have

$$\begin{aligned}
t^{00}_{\mathrm{L-L}} &= \frac{1}{16\pi}\left[g^{00}g^{ij}\mathfrak{g}^{00}{}_{,i}\mathfrak{g}^{00}{}_{,j} + \frac{1}{8}(-g^{00}g^{ij})g_{00}g_{00}\mathfrak{g}^{00}{}_{,i}\mathfrak{g}^{00}{}_{,j}\right] + O(\Phi^3) = -\frac{7}{8\pi}\Phi_{,i}\Phi_{,i} + O(\Phi^3), \\
t^{0i}_{\mathrm{L-L}} &= O(\Phi^3), \\
t^{ij}_{\mathrm{L-L}} &= \frac{1}{16\pi}\left[\frac{1}{8}(2g^{ik}g^{jl} - g^{ij}g^{kl})g_{00}g_{00}\mathfrak{g}^{00}{}_{,k}\mathfrak{g}^{00}{}_{,l}\right] + O(\Phi^3) = \frac{1}{4\pi}(\Phi_{,i}\Phi_{,j} - \frac{1}{2}\delta^{ij}\Phi_{,k}\Phi_{,k}) + O(\Phi^3).
\end{aligned} \tag{20.5.8}$$

(b) It seems that the equation 20.5.4 in the textbook is not correct.

(c) From the definition $T^{\mu\nu}_{\mathrm{L-L\,eff}} \equiv (-g)(T^{\mu\nu} + t^{\mu\nu}_{\mathrm{L-L}})$, we have

$$\begin{aligned}
T^{00}_{\mathrm{L-L\,eff}} &= \rho + \rho v^2 - \frac{7}{8\pi}\Phi_{,k}\Phi_{,k} - 4\rho\Phi, \\
T^{i0}_{\mathrm{L-L\,eff}} &= \rho v^i, \\
T^{ij}_{\mathrm{L-L\,eff}} &= \rho v^i v^j + p\delta^{ij} + \frac{1}{4\pi}(\Phi_{,i}\Phi_{,j} - \frac{1}{2}\delta^{ij}\Phi_{,k}\Phi_{,k}),
\end{aligned} \tag{20.5.9}$$

keeping the first order of Φ, p/ρ and v^2. From $T^{00}_{\mathrm{L-L\,eff},0} + T^{0i}_{\mathrm{L-L\,eff},i} = 0$, we have

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v^i}{\partial x^i} = 0, \tag{20.5.10}$$

noting that ∂_t is of the order of v. From $T^{0i}_{\mathrm{L-L\,eff},i} + T^{ij}_{\mathrm{L-L\,eff},j} = 0$, we have

$$\frac{\partial\rho v^i}{\partial t} + \frac{\partial\rho v^i v^j}{\partial x^j} + \frac{\partial p}{\partial x^i} + \frac{1}{4\pi}(\Phi_{,ij}\Phi_{,j} + \Phi_{,i}\Phi_{,jj} - \Phi_{,ki}\Phi_{,k}) = 0. \tag{20.5.11}$$

Recall that $\Phi_{,jj} = 4\pi\rho$, we have

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j}\right) + \frac{\partial p}{\partial x^i} + \rho\Phi_{,i} = 0. \tag{20.5.12}$$

20.6 SIMPLE FEATURES OF THE ELECTROMAGNETIC FIELD AND ITS STRESS-ENERGY TENSOR

(a) Show that the "scalar" $1/2F^{\alpha\beta}F_{\alpha\beta}$ (invariant with respect to coordinate transformations) and the "pseudoscalar" $1/4F_{\alpha\beta}{}^*F^{\alpha\beta}$ (reproduces itself under a coordinate transformation up to a $+$ sign, according as the sign of the Jacobian of the transformation is positive or negative) have in any local inertial frame the values $\boldsymbol{E}^2 - \boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$, respectively ("the two Lorentz invariants" of the electromagnetic field).

(b) Show that the Poynting flux $(\boldsymbol{E}\times\boldsymbol{B})/4\pi$ is less in magnitude than the energy density $(\boldsymbol{E}^2 + \boldsymbol{B}^2)/8\pi$, save for the exceptional case where both Lorentz invariants of the field vanish (case where the field is locally "null").

(c) A charged pith ball is located a small distance from the North Pole of a bar magnet. Draw the pattern of electric and magnetic lines of force, indicating where the electromagnetic field is "null" in character. Is it legitimate to say that a "null field" is a "radiation field"?

(d) A plane wave is traveling in the z-direction. Show that the corresponding electromagnetic field is everywhere null.

(e) Show that the superposition of two monochromatic plane waves traveling in different directions is null on at most a set of points of measure zero.

(f) In the "generic case" where the field $(\boldsymbol{E}, \boldsymbol{B})$ at the point of interest is not null, show that the Poynting flux is reduced to zero by viewing the field from a local inertial frame that is traveling in the direction of $\boldsymbol{E} \times \boldsymbol{B}$ with a velocity

$$v = \tanh \alpha \tag{20.6.1}$$

where the velocity parameter α is given by the formula

$$\tanh 2\alpha = \frac{2|\boldsymbol{E} \times \boldsymbol{B}|}{\boldsymbol{E}^2 - \boldsymbol{B}^2}. \tag{20.6.2}$$

(g) Show that all components of the electric and magnetic field in this new frame can be taken to be zero except E_x and B_x.

(h) Show that the 4×4 determinant built out of the components of the field in mixed representation, $F^{\alpha}{}_{\beta}$, is invariant with respect to general coordinate transformations.

(i) Show that this determinant has the value $-(\boldsymbol{E} \cdot \boldsymbol{B})^2$ by evaluating it in the special local inertial frame of (f).

(j) Show that in this special frame the Maxwell stress-energy tensor has the form

$$T^{\mu}{}_{\nu} = \frac{E_x^2 + B_x^2}{8\pi} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20.6.3}$$

(Faraday tension along the lines of force; Faraday pressure at right angles to the lines of force).

(k) In the other case, where the field is locally null, show that one can always find a local inertial frame in which the field has the form $\boldsymbol{E} = (0, F, 0)$, $\boldsymbol{B} = (0, 0, F)$ and the stress-energy tensor has the value

$$T^{\mu}{}_{\nu} = \frac{F^2}{4\pi} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20.6.4}$$

(l) Regardless of whether the electromagnetic field is or is not null, show that the Maxwell stress-energy tensor has zero trace, $T^{\mu}{}_{\mu} = 0$, and that its square is a multiple of the unit tensor,

$$T^{\mu}{}_{\alpha}\, T^{\alpha}{}_{\nu} = \frac{\delta^{\mu}_{\nu}}{(8\pi)^2}[(\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + (2\boldsymbol{E} \cdot \boldsymbol{B})^2] = \frac{\delta^{\mu}_{\nu}}{(8\pi)^2}[(\boldsymbol{E}^2 + \boldsymbol{B}^2)^2 - (2\boldsymbol{E} \times \boldsymbol{B})^2]. \tag{20.6.5}$$

Solution:

(a) In a local Lorentz frame, we have

$$-\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} = -F^{0i}F_{0i} - F^{ij}F_{ij} = E^i E_i - \epsilon^{ijk}\epsilon_{ijl}B_k B^l = \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{20.6.6}$$

and

$$\frac{1}{4}F_{\alpha\beta}{}^*F^{\alpha\beta} = \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = \frac{1}{2}\epsilon^{0ijk}F_{0i}F_{jk} = -F_{0i}B^k = \boldsymbol{E}\cdot\boldsymbol{B}. \tag{20.6.7}$$

(b) Suppose the angle between $\boldsymbol{E}$ and $\boldsymbol{B}$ is alpha, then we have $|\boldsymbol{E}\times\boldsymbol{B}| = |\boldsymbol{E}||\boldsymbol{B}|\sin\alpha$. And so

$$\boldsymbol{E}^2 + \boldsymbol{B}^2 - 2|\boldsymbol{E}||\boldsymbol{B}|\sin\alpha = (|\boldsymbol{E}| - |\boldsymbol{B}|)^2 + 2|\boldsymbol{E}||\boldsymbol{B}|(1-\sin\alpha) \geq 0, \tag{20.6.8}$$

the equality holds when $|\boldsymbol{E}| = |\boldsymbol{B}|$ and $\alpha = \pi/2$, in which case

$$\boldsymbol{E}^2 - \boldsymbol{B}^2 = 0, \quad \boldsymbol{E}\cdot\boldsymbol{B} = 0. \tag{20.6.9}$$

(c) The drawing is eliminated here. It is not legitimate to say that a "null field" is a "radiation field". "Radiation field" must be null everywhere.

(d) The E-M field for a plane wave traveling in the z-direction is

$$\boldsymbol{E} = (A\cos(kz-\omega t), 0, 0) \quad \boldsymbol{B} = (0, A\cos(kz-\omega t), 0). \tag{20.6.10}$$

So we have

$$\boldsymbol{E}^2 - \boldsymbol{B}^2 = 0, \quad \boldsymbol{E}\cdot\boldsymbol{B} = 0. \tag{20.6.11}$$

(e) We have

$$\boldsymbol{E} = \boldsymbol{E}_1\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t) + \boldsymbol{E}_2\cos(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t) \tag{20.6.12}$$

and

$$\boldsymbol{B} = \boldsymbol{B}_1\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t) + \boldsymbol{B}_2\cos(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t). \tag{20.6.13}$$

So

$$\boldsymbol{E}\cdot\boldsymbol{B} = (\boldsymbol{E}_1\cdot\boldsymbol{B}_2 + \boldsymbol{E}_2\cdot\boldsymbol{B}_1)\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t) \neq 0, \tag{20.6.14}$$

and

$$\boldsymbol{E}^2 - \boldsymbol{B}^2 = 2(\boldsymbol{E}_1\cdot\boldsymbol{E}_1 - \boldsymbol{B}_1\cdot\boldsymbol{B}_2)\cos(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t) \neq 0. \tag{20.6.15}$$

(f) The velocity is

$$\boldsymbol{v} = \tanh\alpha\frac{\boldsymbol{E}\times\boldsymbol{B}}{|\boldsymbol{E}\times\boldsymbol{B}|} \equiv \tanh\alpha\boldsymbol{n}. \tag{20.6.16}$$

And we have

$$\boldsymbol{E}' = \gamma(\boldsymbol{E} + v\boldsymbol{n}\times\boldsymbol{B}) \tag{20.6.17}$$

and

$$\boldsymbol{B}' = \gamma(\boldsymbol{B} - v\boldsymbol{n}\times\boldsymbol{E}). \tag{20.6.18}$$

So

$$\begin{aligned}
\boldsymbol{E}'\times\boldsymbol{B}' &= \gamma^2[\boldsymbol{E}\times\boldsymbol{B} + v(\boldsymbol{n}\times\boldsymbol{B})\times\boldsymbol{B} - v\boldsymbol{E}\times(\boldsymbol{n}\times\boldsymbol{E}) - v^2(\boldsymbol{n}\times\boldsymbol{B})\times(\boldsymbol{n}\times\boldsymbol{E})] \\
&= \gamma^2\{\boldsymbol{E}\times\boldsymbol{B} - v(\boldsymbol{B}^2+\boldsymbol{E}^2)\boldsymbol{n} - v^2[(\boldsymbol{n}\times\boldsymbol{B})\cdot\boldsymbol{E}]\boldsymbol{n}\} \\
&= \gamma^2[\boldsymbol{E}\times\boldsymbol{B} - v(\boldsymbol{B}^2+\boldsymbol{E}^2)\boldsymbol{n} - v^2\boldsymbol{E}\times\boldsymbol{B}] \\
&= \boldsymbol{E}\times\boldsymbol{B} - \frac{\tanh\alpha}{1-\tanh^2\alpha}(\boldsymbol{B}^2+\boldsymbol{E}^2)\boldsymbol{n} \\
&= \boldsymbol{E}\times\boldsymbol{B} - \tanh 2\alpha\frac{\boldsymbol{B}^2+\boldsymbol{E}^2}{2}\frac{\boldsymbol{E}\times\boldsymbol{B}}{|\boldsymbol{E}\times\boldsymbol{B}|} \\
&= 0.
\end{aligned} \tag{20.6.19}$$

(g) By the Lorentz transformation described in (f) we can make $\boldsymbol{E}$ and $\boldsymbol{B}$ parallel. Then a pure rotation in space can bring $\boldsymbol{E}$ and $\boldsymbol{B}$ to the x direction, i.e. all components of the electric and magnetic field in this new frame can be taken to be zero except E_x and B_x.

(h) In a general coordinate transformations, we have

$$F^{\alpha'}{}_{\beta'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} F^{\alpha}{}_{\beta} \frac{\partial x^{\beta}}{\partial x^{\beta'}}. \tag{20.6.20}$$

So

$$\det F^{\alpha'}{}_{\beta'} = \det \left| \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \right| \cdot \det \left| \frac{\partial x^{\beta}}{\partial x^{\beta'}} \right| \cdot \det F^{\alpha}{}_{\beta} = \det F^{\alpha}{}_{\beta}\,, \tag{20.6.21}$$

i.e. the determinant built out of the components of the field in mixed representation is invariant with respect to general coordinate transformations.

(i) In this special frame, we have

$$F^{\alpha}{}_{\beta} = \begin{pmatrix} 0 & E^x & 0 & 0 \\ E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & B^x \\ 0 & 0 & -B^x & 0 \end{pmatrix}. \tag{20.6.22}$$

So

$$\det F^{\alpha}{}_{\beta} = -(E^x)^2 (B^x)^2 = -(E^x \cdot B^x)^2 = -(\boldsymbol{E} \cdot \boldsymbol{B})^2. \tag{20.6.23}$$

(j) We have $4\pi T^{\mu}{}_{\nu} = F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^{\mu}_{\nu} F_{\rho\sigma} F^{\rho\sigma}$. As

$$-F^{\mu\rho} F_{\rho\nu} = -\begin{pmatrix} 0 & E^x & 0 & 0 \\ -E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & B^x \\ 0 & 0 & -B^x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E^x & 0 & 0 \\ E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & B^x \\ 0 & 0 & -B^x & 0 \end{pmatrix} = \begin{pmatrix} -E_x^2 & 0 & 0 & 0 \\ 0 & -E_x^2 & 0 & 0 \\ 0 & 0 & B_x^2 & 0 \\ 0 & 0 & 0 & B_x^2 \end{pmatrix} \tag{20.6.24}$$

and

$$F^{\rho\sigma} F_{\rho\sigma} = -2(E_x^2 - B_x^2), \tag{20.6.25}$$

it is easy to obtain

$$T^{\mu}{}_{\nu} = \frac{E_x^2 + B_x^2}{8\pi} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20.6.26}$$

(k) If the E-M field is null, then we let the direction of $\boldsymbol{E} \times \boldsymbol{B}$ be the direction of x axis. In this frame, we have $\boldsymbol{E} = (0, F, 0)$, $\boldsymbol{B} = (0, 0, F)$. And so

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ -F & -F & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20.6.27}$$

As

$$-F^{\mu\rho} F_{\rho\nu} = -\begin{pmatrix} 0 & 0 & F & 0 \\ 0 & 0 & F & 0 \\ -F & -F & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -F & 0 \\ 0 & 0 & F & 0 \\ F & -F & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = F^2 \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{20.6.28}$$

and

$$F^{\rho\sigma} F_{\rho\sigma} = 0, \tag{20.6.29}$$

it is easy to obtain

$$T^{\mu}{}_{\nu} = \frac{F^2}{4\pi} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20.6.30}$$

(l) If the E-M field is not null, then we can go to the special frame in (f), and the energy-momentum tensor is 20.6.26. It is easy to see that $T^{\mu}{}_{\mu} = 0$, and

$$T^{\mu}{}_{\alpha}T^{\alpha}{}_{\nu} = \left[\frac{E_x^2 + B_x^2}{8\pi}\right]^2 \delta^{\mu}_{\nu} = \frac{\delta^{\mu}_{\nu}}{(8\pi)^2}[(\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + (2\boldsymbol{E}\cdot\boldsymbol{B})^2]. \tag{20.6.31}$$

If the E-M field is null, then we can go to the special frame in (k), and the energy-momentum tensor is 20.6.30. It is easy to see that $T^{\mu}{}_{\mu} = 0$, and

$$T^{\mu}{}_{\alpha}T^{\alpha}{}_{\nu} = 0 = \frac{\delta^{\mu}_{\nu}}{(8\pi)^2}[(\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + (2\boldsymbol{E}\cdot\boldsymbol{B})^2]. \tag{20.6.32}$$

Since the l.h.s. and r.h.s. of $T^{\mu}{}_{\mu} = 0$ and 20.6.5 is invariant under Lorentz transformation, so it holds in any frame.

20.7 THE STRESS-ENERGY TENSOR DETERMINES THE ELECTROMAGNETIC FIELD EXCEPT FOR ITS COMPLEXION

(a) Given a non-zero symmetric 4×4 tensor $T^{\mu\nu}$ which has zero trace $T^{\mu}{}_{\mu} = 0$ and whose square is a multiple, $M^4/(8\pi)^2$, of the unit matrix, show that, according as this multiple is zero ("null case") or positive, the tensor can be transformed to the form (20.60) or (20.59) by a suitable rotation in 3-space or by a suitable choice of local inertial frame, respectively.

(b) In the generic (non-null) case in the frame in question, show that $T^{\mu\nu}$ is the Maxwell tensor of the "extremal electromagnetic field" $\xi_{\mu\nu}$ with components

$$\begin{aligned} \boldsymbol{E}^{\text{extremal}} &= (M, 0, 0), \\ \boldsymbol{B}^{\text{extremal}} &= (0, 0, 0). \end{aligned} \tag{20.7.1}$$

Show that it is also the Maxwell tensor of the "dual extremal field" ${}^*\xi_{\mu\nu}$ with components

$$\begin{aligned} {}^*\boldsymbol{E}^{\text{extremal}} &= (0, 0, 0), \\ {}^*\boldsymbol{B}^{\text{extremal}} &= (M, 0, 0). \end{aligned} \tag{20.7.2}$$

(c) Recalling that the duality operation $*$ applied twice to an antisymmetric second-rank tensor (2-form) in four-dimensional space leads back to the negative of that tensor, show that the operator $e^{*\alpha}$ ("duality rotation") has the value

$$e^{*\alpha} = \cos\alpha + (\sin\alpha)^*. \tag{20.7.3}$$

(d) Show that the most general electromagnetic field which will reproduce the non-null tensor $T^{\mu\nu}$ in the frame in question, and therefore in any coordinate system, is

$$F_{\mu\nu} = e^{*\alpha}\xi_{\mu\nu}. \tag{20.7.4}$$

(e) Derive a corresponding result for the null case.

Solution:

(a) If $T^{0i} = 0$, then a suitable rotation in 3-space can bring T^{i}_{j} into diagonal form. So $T^{\mu}{}_{\nu}$ will also be diagonal. In generic case, the diagonal elements must be $\pm M^2/8\pi$ and their sum must be zero. So they are of the form (20.59). In null case, $T^{\mu}{}_{\nu}$ must be a null matrix.

If $T^{0i} \neq 0$, by a suitable rotation in 3-space, we can make T^{0i}s vanish unless $i = 1$. As $T^{\mu}{}_{\alpha}T^{\alpha}{}_{\nu} = M^4/(8\pi)^2\delta^{\mu}_{\nu}$, we have

$$T^{0}{}_{\alpha}T^{\alpha}{}_{i} = T^{0}{}_{0}T^{0}{}_{i} + T^{0}{}_{1}T^{1}{}_{i} = 0. \tag{20.7.5}$$

For $i = 1, 2, 3$, we can obtain

$$T^1{}_2 = T^1{}_3 = 0; \quad T^{00} = T^{11}. \tag{20.7.6}$$

So $T^\mu{}_\nu$ is diagonal in block $(0, 1)$ and $(2, 3)$. And a suitable rotation in $2 - 3$ plane can bring $(2, 3)$ block into diagonal form.

We also have

$$T^0{}_\alpha T^\alpha{}_0 = (T^0{}_0)^2 - (T^0{}_1)^2 = \frac{M^4}{(8\pi)^2}. \tag{20.7.7}$$

So in null case, we have $|T^0{}_0| = |T^0{}_1| = |T^1{}_1|$, and $(2, 3)$ block is null. If we demand that T^{00} and T^{0i} is positive, then $T^{\mu\nu}$ is of the form (20.60).

For a boost in 1-direction, we have

$$T^{01} \to \frac{1+\beta^2}{1-\beta^2}\left[T^{01} - \frac{\beta}{1-\beta^2}(T^{00} + T^{11})\right] = \frac{1+\beta^2}{1-\beta^2}\left(T^{01} - \frac{2\beta}{1+\beta^2}T^{00}\right). \tag{20.7.8}$$

In generic case, $|T^0{}_0| > |T^0{}_1|$. So a suitable choice of β can bring T^{01} to zero, and so bring $T^\mu{}_\nu$ to be diagonal, of the form (20.59).

(b) For "extremal electromagnetic field", we have

$$F^{\alpha\beta} = \begin{pmatrix} 0 & M & 0 & 0 \\ -M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20.7.9}$$

As

$$-F^{\mu\rho}F_{\rho\nu} = -\begin{pmatrix} 0 & M & 0 & 0 \\ -M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -M & 0 & 0 \\ M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M^2\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{20.7.10}$$

and

$$F^{\rho\sigma}F_{\rho\sigma} = -2M^2, \tag{20.7.11}$$

it is easy to obtain

$$T^\mu{}_\nu = \frac{M^2}{8\pi}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20.7.12}$$

For "dual extremal field", we have

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & -M & 0 \end{pmatrix}. \tag{20.7.13}$$

As

$$-F^{\mu\rho}F_{\rho\nu} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & -M & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & -M & 0 \end{pmatrix} = M^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20.7.14}$$

and

$$F^{\rho\sigma}F_{\rho\sigma} = 2M^2, \tag{20.7.15}$$

it is easy to obtain

$$T^{\mu}{}_{\nu} = \frac{M^2}{8\pi}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20.7.16}$$

(c)

$$e^{*\alpha} = \sum_{n=0}^{\infty} \frac{(*\alpha)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k}}{(2k)!} + * \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1}}{(2k+1)!} = \cos\alpha + (\sin\alpha)^*. \tag{20.7.17}$$

(d) In generic case, we have

$$\begin{aligned} T^0{}_0 &= -\frac{E^2+B^2}{8\pi} = -\frac{M^2}{8\pi}, \quad T^0{}_i = \epsilon_{ijk} E^j B^k = 0, \\ T^1{}_1 &= \frac{E^2+B^2}{8\pi} - \frac{E^1E^1 + B^1B^1}{4\pi} = -\frac{M^2}{8\pi}, \quad T^2{}_2 = \frac{E^2+B^2}{8\pi} - \frac{E^2E^2 + B^2B^2}{4\pi} = \frac{M^2}{8\pi}, \\ T^3{}_3 &= \frac{E^2+B^2}{8\pi} - \frac{E^3E^3 + B^3B^3}{4\pi} = \frac{M^2}{8\pi}, \quad T^i{}_j = -E^iE^j - B^iB^j = 0\,(i \neq j). \end{aligned} \tag{20.7.18}$$

Then we can obtain $E_2 = E_3 = B_2 = B_3 = 0$ and $E_1^2 + B_1^2 = M^2$, i.e.

$$\begin{aligned} \boldsymbol{E} &= (M\cos\alpha, 0, 0), \\ \boldsymbol{B} &= (M\sin\alpha, 0, 0). \end{aligned} \tag{20.7.19}$$

And so

$$F_{\mu\nu} = e^{*\alpha}\xi_{\mu\nu}. \tag{20.7.20}$$

(f) In null case, we have

$$T^{\mu}{}_{\nu} = \frac{F^2}{4\pi}\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20.7.21}$$

We can obtain the solution

$$\begin{aligned} \boldsymbol{E} &= (0, F\cos\alpha, -F\sin\alpha), \\ \boldsymbol{B} &= (0, F\sin\alpha, F\cos\alpha). \end{aligned} \tag{20.7.22}$$

Define "extremal electromagnetic field" $\xi_{\mu\nu}$ as

$$\begin{aligned} \boldsymbol{E}^{\text{extremal}} &= (0, F, 0), \\ \boldsymbol{B}^{\text{extremal}} &= (0, 0, F). \end{aligned} \tag{20.7.23}$$

Then we have

$$F_{\mu\nu} = e^{*\alpha}\xi_{\mu\nu}. \tag{20.7.24}$$

20.8 THE MAXWELL EQUATIONS CANNOT BE DERIVED FROM THE LAW OF CONSERVATION OF STRESS-ENERGY WHEN $(E\cdot B) = 0$ OVER AN EXTENDED REGION

Supply a counter-example to the idea that the Maxwell equations,

$$F^{\mu\nu}{}_{;\nu} = 0, \tag{20.8.1}$$

follow from the Einstein equation; or, more precisely, show that (1) the condition that the Maxwell stress-energy tensor should have a vanishing divergence plus (2) the condition that this Maxwell field is the curl of a 4-potential A_μ can both be satisfied, while yet the stated Maxwell equations are violated.

Solution:

As indicated by the "Hint" in the textbook, I refered to the paper of Teitelboim, C. (1970).

We take as the field of a moving charge the retarded Lienard-Wiechert solution

$$F^{\mu\nu} = \frac{2ev^{[\mu}r^{\nu]}}{\rho^3} + \frac{2e}{\rho^2}\left(a_r v^{[\mu}r^{\nu]} + a^{[\mu}r^{\nu]}\right) \tag{20.8.2}$$

of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{20.8.3}$$

with the current

$$j^\mu(x) = e\int_{-\infty}^{\infty} d\tau \delta(x - z(\tau))v^\mu(\tau). \tag{20.8.4}$$

We use the following notation: Retarded quantities are written in orange color. The vector r^μ is defined by $r^\mu = x^\mu - z^\mu$ and satisfies $r^\mu r_\mu = 0$. The scalar $\rho = -v^\mu r_\mu$ is the spatial distance between the field point and the retarded point in the Lorentz frame in which the charge is at rest at the retarded time τ. We write $a^{[\mu}b^{\nu]}$ for $(a^\mu b^\nu - a^\nu b^\mu)/2$ and $a_r = a^\mu r_\mu/\rho$. The electromagnetic field tensor is de6ned in such a way that $F^{01} = +E^x$.

The field 20.8.2 may be decomposed in a natural way into two parts:

$$F^{\mu\nu} = F_{\rm I}^{\mu\nu} + F_{\rm II}^{\mu\nu}, \tag{20.8.5}$$

where

$$F_{\rm I}^{\mu\nu} = \frac{2ev^{[\mu}r^{\nu]}}{\rho^3} \tag{20.8.6}$$

is the velocity field, and

$$F_{\rm II}^{\mu\nu} = \frac{2e}{\rho^2}\left(a_r v^{[\mu}r^{\nu]} + a^{[\mu}r^{\nu]}\right) \tag{20.8.7}$$

is the acceleration field. This separation is relativistically invariant because both $F_{\rm I}^{\mu\nu}$ and $F_{\rm II}^{\mu\nu}$ are tensors. The following results which are valid off the world line of the charge are obtained from 20.8.6 and 20.8.7 by the usual methods for differentiating retarded quantities:

$$\partial_\mu F_{\rm I}^{\mu\nu} = -\partial_\mu F_{\rm II}^{\mu\nu} = \frac{2e}{\rho^3}a_r r^\nu; \quad \partial^\lambda F_{\rm I,II}^{\mu\nu} + \partial^\mu F_{\rm I,II}^{\nu\lambda} + \partial^\nu F_{\rm I,II}^{\lambda\mu} = 0. \tag{20.8.8}$$

So potentials A^μ exist for both parts of the field.

The symmetric energy-momentum tensor for the field 20.8.2 can be decomposed as

$$T^{\mu\nu} = T_{\rm I,I}^{\mu\nu} + T_{\rm I,II}^{\mu\nu} + T_{\rm II,II}^{\mu\nu}, \tag{20.8.9}$$

where $T_{\rm I,I}^{\mu\nu}$ and $T_{\rm II,II}^{\mu\nu}$ are the tensors obtained when energy-momentum tensor is evaluated with the fields $F_{\rm I}^{\mu\nu}$ and $F_{\rm II}^{\mu\nu}$, respectively, and $T_{\rm I,II}^{\mu\nu}$ is the result of the interference between both fields.

Since potentials A^μ exist for both parts of the field, we have

$$\partial_\mu T^{\mu\nu} = \frac{1}{4\pi}F^{\beta\nu}\partial_\alpha F^\alpha{}_\beta \tag{20.8.10}$$

for $T_{\rm I,I}^{\mu\nu}$ and $T_{\rm II,II}^{\mu\nu}$. Then we obtain

$$\partial_\mu T_{\rm I,I}^{\mu\nu} = -\frac{e^2}{2\pi\rho^5}a_r r^\nu, \quad \partial_\mu T_{\rm II,II}^{\mu\nu} = 0. \tag{20.8.11}$$

So E-M field $F_{\rm II}^{\mu\nu}$ meets the stated requirements, arisen from the fact that the acceleration field $\det F_{\rm II}^{\mu\nu} = -(\boldsymbol{E}_{\rm II}\cdot\boldsymbol{B}_{\rm II}) = 0$ everywhere off the worldline.

20.9 EQUATION OF MOTION OF A SCALAR FIELD AS CONSEQUENCE OF THE EINSTEIN FIELD EQUATION

The stress-energy tensor of a massless scalar field is taken to be

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right). \tag{20.9.1}$$

Derive the equation of motion of this scalar field from Einstein's field equation.

Solution:

From Einstein's equation, we have

$$0 = 4\pi T^{\mu\nu}{}_{;\nu} = \phi^{;\mu}{}_{\nu}\phi^{,\nu} + \phi^{;\nu}{}_{\nu}\phi^{,\mu} - g^{\mu\nu}\phi^{;\alpha}{}_{\nu}\phi_{,\alpha} = \phi^{;\nu}{}_{\nu}\phi^{,\mu} = 0. \tag{20.9.2}$$

So the equation of motion is

$$\phi^{;\nu}{}_{\nu} = 0. \tag{20.9.3}$$

Chapter 21

VARIATIONAL PRINCIPLE AND INITIAL-VALUE DATA

21.1 VARIATION OF THE DETERMINANT OF THE METRIC TENSOR

Recalling that the change in the value of any determinant is given by multiplying the change in each element of that determinant by its cofactor and adding the resulting products prove that

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g^{\mu\nu}\delta g_{\mu\nu} \text{ and } \delta(-g)^{1/2} = -\frac{1}{2}(-g)^{1/2} g_{\mu\nu}\delta g^{\mu\nu}. \tag{21.1.1}$$

Also show that

$$g = \det \mathfrak{g}^{\mu\nu} \text{ and } \delta(-g)^{1/2} = \frac{1}{2} g_{\mu\nu}\delta \mathfrak{g}^{\mu\nu}. \tag{21.1.2}$$

Solution:

Linear algebra theory tells us that

$$\frac{\partial \det A}{\partial A_{ab}} = (\det A)(A^{-1})_{ab}. \tag{21.1.3}$$

So

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}. \tag{21.1.4}$$

Then we have

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{-1/2}\delta g = \frac{1}{2}(-g)^{1/2} g^{\mu\nu}\delta g_{\mu\nu}. \tag{21.1.5}$$

As $g^{\mu\nu}g_{\mu\nu} = 4$, we have $g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu} = 0$. So

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{-1/2}\delta g = -\frac{1}{2}(-g)^{1/2} g_{\mu\nu}\delta g^{\mu\nu}. \tag{21.1.6}$$

From $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$, we can get

$$\det \mathfrak{g}^{\mu\nu} = (\sqrt{-g})^4 \det g^{\mu\nu} = g. \tag{21.1.7}$$

And since

$$\delta \mathfrak{g}^{\mu\nu} = g^{\mu\nu}\delta\sqrt{-g} + \sqrt{-g}\delta g^{\mu\nu}, \tag{21.1.8}$$

we have

$$\frac{1}{2} g_{\mu\nu}\delta \mathfrak{g}^{\mu\nu} = 2\delta\sqrt{-g} + \frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} = \delta\sqrt{-g}. \tag{21.1.9}$$

21.2 STRESS-ENERGY TENSOR FOR A SCALAR FIELD

Given the Lagrange function (21.34) of a scalar field, derive the stress-energy tensor for this field. Also write down the field equation for the scalar field that one derives from this Lagrange function (in the general case where the field executes its dynamics within the arena of a curved spacetime). Show that as a consequence of this field equation, the stress-energy tensor satisfies the conservation law, $T_{\alpha\beta}{}^{,\beta} = 0$.

Solution:

As

$$L = \frac{1}{8\pi}(-g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2), \tag{21.2.1}$$

so

$$\begin{aligned} T_{\alpha\beta} &= -2\frac{\delta L}{\delta g^{\alpha\beta}} + g_{\alpha\beta}L \\ &= \frac{1}{4\pi}\phi_{,\alpha}\phi_{,\beta} + g_{\alpha\beta}\frac{1}{8\pi}(-g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2) \\ &= \frac{1}{4\pi}\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\mu}\phi^{,\mu} - \frac{1}{2}g_{\alpha\beta}m^2\phi^2\right). \end{aligned} \tag{21.2.2}$$

The equation of motion is given by

$$0 = \frac{\partial\sqrt{-g}L}{\partial\phi} - \partial_\mu\frac{\partial\sqrt{-g}L}{\partial\phi_{,\mu}} = -\frac{1}{4\pi}(\sqrt{-g}m^2\phi - \partial_\mu\sqrt{-g}\phi^{,\mu}) = -\frac{\sqrt{-g}}{4\pi}(m^2\phi - \phi^{;\mu}{}_{\mu}). \tag{21.2.3}$$

Meanwhile, we have

$$\begin{aligned} 4\pi T^{\alpha\beta}{}_{;\beta} &= \phi^{,\alpha}\phi^{;\beta}{}_{\beta} + \phi^{;\alpha}{}_{\beta}\phi^{,\beta} - g^{\alpha\beta}\phi_{,\mu}\phi^{;\mu}{}_{\beta} - g^{\alpha\beta}m^2\phi\phi_{,\beta} \\ &= \phi^{,\alpha}\phi^{;\beta}{}_{\beta} + g^{\alpha\beta}\phi_{;\beta\mu}\phi^{,\mu} - g^{\alpha\beta}\phi^{,\mu}\phi_{;\mu\beta} - m^2\phi\phi^{,\alpha} \\ &= \phi^{,\alpha}(\phi^{;\beta}{}_{\beta} - m^2\phi) + g^{\alpha\beta}\phi^{,\mu}(\phi_{;\beta\mu} - \phi_{;\mu\beta}) \\ &= \phi^{,\alpha}(\phi^{;\beta}{}_{\beta} - m^2\phi) \end{aligned} \tag{21.2.4}$$

So as a consequence of this field equation, the stress-energy tensor satisfies the conservation law.

21.3 FARADAY-MAXWELL STRESS-ENERGY TENSOR

Given the Lagrangian density $-F^{\mu\nu}F_{\mu\nu}/16\pi$, reexpress it in terms of the variables A_μ and $g_{\mu\nu}$, and by use of (21.33) derive the stress-energy tensor as discussed in section 5.6. Also derive from the Lagrange variation principle the field equation $F_{\mu\nu}{}^{;\nu} = 0$ (curved spacetime, but-for simplicity-a charge-free region of space). As a consequence of this field equation, show that the Faraday-Maxwell stress-energy tensor satisfies the conservation law, $T_{\alpha\beta}{}^{;\beta} = 0$. For a more ambitious project, show that any stress-energy tensor derived from a field Lagrangian by the prescription of equation (21.33) will automatically satisfy the conservation law $T_{\alpha\beta}{}^{;\beta} = 0$.

Solution:

As

$$L = -\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi} = \frac{1}{8\pi}(A^{\mu;\nu}A_{\nu;\mu} - A^{\mu;\nu}A_{\mu;\nu}) = \frac{1}{8\pi}g^{\rho\mu}g^{\sigma\nu}A_{\rho;\sigma}(A_{\nu;\mu} - A_{\mu;\nu}) \tag{21.3.1}$$

so

$$\begin{aligned} T_{\alpha\beta} &= -2\frac{\delta L}{\delta g^{\alpha\beta}} + g_{\alpha\beta}L \\ &= -\frac{1}{4\pi}(g^{\sigma\nu}A_{\alpha;\sigma}F_{\beta\nu} + g^{\rho\mu}A_{\rho;\alpha}F_{\mu\beta}) + g_{\alpha\beta}L \\ &= -\frac{1}{4\pi}(A_{\alpha;\sigma}F_{\beta}{}^{\sigma} - A_{\rho;\alpha}F_{\beta}{}^{\rho}) + g_{\alpha\beta}L \\ &= \frac{1}{4\pi}\left(F_{\alpha\rho}F_{\beta}{}^{\rho} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right). \end{aligned} \tag{21.3.2}$$

The equation of motion is given by

$$0 = \frac{\partial\sqrt{-g}L}{\partial A_\mu} - \partial_\nu\frac{\partial\sqrt{-g}L}{\partial A_{\mu,\nu}} = \frac{1}{4\pi}\partial_\nu(\sqrt{-g}g^{\rho\mu}g^{\sigma\nu}F_{\rho\sigma}) = \frac{1}{4\pi}\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \frac{1}{4\pi}\sqrt{-g}F^{\mu\nu}{}_{;\nu}. \tag{21.3.3}$$

Meanwhile, we have

$$\begin{aligned} 4\pi T_{\alpha\beta}{}^{;\beta} &= F_{\alpha\rho}{}^{;\beta}F_{\beta}{}^{\rho} + F_{\alpha\rho}F_{\beta}{}^{\rho;\beta} - \frac{1}{2}g_{\alpha\beta}F^{\mu\nu;\beta}F_{\mu\nu} \\ &= F_{\alpha\nu;\mu}F^{\mu\nu} - \frac{1}{2}F_{\mu\nu;\alpha}F^{\mu\nu} + F_{\rho\alpha}F^{\rho\beta}{}_{;\beta} \\ &= \frac{1}{2}F^{\mu\nu}(F_{\alpha\nu;\mu} + F_{\mu\alpha;\nu} + F_{\nu\mu;\alpha}) + F_{\rho\alpha}F^{\rho\beta}{}_{;\beta} \\ &= F_{\rho\alpha}F^{\rho\beta}{}_{;\beta}. \end{aligned} \tag{21.3.4}$$

So the Faraday-Maxwell stress-energy tensor satisfies the conservation law as a consequence of this field equation.

Generally, for a variation in metric tensor, we have

$$\delta I_{\text{field}} = -\frac{1}{2}\int T_{\alpha\beta}\delta g^{\alpha\beta}(-g)^{1/2}d^4x. \tag{21.3.5}$$

For a purelly coordinate transformation $x^\mu_{\text{NEW}} = x^\mu - \xi^\mu$, we have

$$\delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{21.3.6}$$

The action is a scalar, which is invariant under coordinate transformation. So we have

$$0 = \delta I_{\text{field}} = \frac{1}{2}\int T_{\mu\nu}(\xi^{\mu;\nu} + \xi^{\nu;\mu})(-g)^{1/2}d^4x = \int T_{\mu\nu}{}^{;\nu}\xi^\mu(-g)^{1/2}d^4x. \tag{21.3.7}$$

Thus we obtain

$$T_{\mu\nu}{}^{;\nu} = 0. \tag{21.3.8}$$

21.4 SCALAR CURVATURE INVARIANT IN TERMS OF AREA DEFICIT

It being 10,000 km from North Pole to equator, one would have 62,832 km for the length of the "equator" if the earth were flat, as contrasted to the actual - 40,000 km, a difference reflecting the fact that the surface is curved up into closure. Turn from this "pre-problem" to the actual problem, a 3-sphere

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{21.4.1}$$

Measure off from $\chi = 0$ a 2-sphere of proper radius $\epsilon = a\chi$. Determine the proper area of this 2-sphere as a function of χ. Verify that relation (21.50) on the area deficit gives in the limit $\epsilon \to 0$ the correct result

$R = 6/a^2$. For a more ambitious exercise: (1) take a general (smooth) 3-geometry; (2) express the metric near any chosen point in terms of Riemann's normal coordinates as given in section 11.6; (3) determine the locus of the set of points at the proper distance ϵ to the lowest interesting power of ϵ in terms of the spherical polar angles θ and ϕ (direction of start of geodesic of length ϵ); (4) determine to the lowest interesting power of ϵ the proper area of the figure defined by these points; and thereby establish (21.50) [for more on this topic see, for example, Cartan (1946), pp. 252-256].

Solution:

For a 3-sphere, we have

$$S = 4\pi a^2 \sin^2(\chi) = 4\pi a^2 \left(\chi^2 - \frac{1}{3}\chi^4\right) + O(\chi^5). \tag{21.4.2}$$

So

$$R = \lim_{\epsilon\to 0} 18\frac{4\pi a^2\chi^2 - S}{4\pi a^4\chi^4} = \frac{6}{a^2}. \tag{21.4.3}$$

Take a general smooth 3-geometry. In a Riemann's normal coordinates with metric, we can choose the proper length as the affine parameters for geodesics. Suppose $\mathcal{P}_1$ is a point near the orgin $\mathcal{P}_0$. We find the geodesic $\mathcal{P}(s)$ connecting $\mathcal{P}_0$ and $\mathcal{P}_1$. If the proper length of the geodesic is s and the tagent vector of the geodesic at $\mathcal{P}_0$ is

$$v^i \boldsymbol{e}_i(\mathcal{P}_0), \tag{21.4.4}$$

the coordinate of $\mathcal{P}_1$ in Riemann's normal coordinates will be (sv^1, sv^2, sv^3). And if we choose $g_{ij}(\mathcal{P}_0) = \boldsymbol{e}_i(\mathcal{P}_0)\cdot\boldsymbol{e}_j(\mathcal{P}_0) = \delta_{ij}$, then we have

$$\delta_{ij}v^iv^j = 1. \tag{21.4.5}$$

Futher more, we also have

$$g_{ij,k}(\mathcal{P}_0) = 0; \quad g_{ij,kl}(\mathcal{P}_0) = -\frac{1}{3}\left(R_{ikjl}(\mathcal{P}_0) + R_{iljk}(\mathcal{P}_0)\right). \tag{21.4.6}$$

In this coordinates, the set of points at the proper distance ϵ is

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = \epsilon^2. \tag{21.4.7}$$

Let $x^1 = \epsilon\sin\theta\cos\phi$, $x^2 = \epsilon\sin\theta\sin\phi$, $x^3 = \epsilon\cos\theta$. So we have

$$g_{AB} = g_{ij}\frac{\partial x^i}{\partial x^A}\frac{\partial x^j}{\partial x^B} \tag{21.4.8}$$

at the sphere, where

$$g_{ij} = \delta_{ij} + \frac{1}{2}g_{ij,kl}(\mathcal{P}_0)v^kv^l + O(\epsilon^3). \tag{21.4.9}$$

Now we can get

$$g_{AB} = \epsilon^2\begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix} + \frac{1}{2}g_{ij,kl}(\mathcal{P}_0)x^kx^l\frac{\partial x^i}{\partial x^A}\frac{\partial x^j}{\partial x^B} + O(\epsilon^5). \tag{21.4.10}$$

The determinant of g_{AB} is given by

$$\det g_{AB} = \epsilon^4\sin^2\theta + \frac{1}{2}\epsilon^2 g_{ij,kl}(\mathcal{P}_0)x^kx^l\left(\frac{\partial x^i}{\partial\phi}\frac{\partial x^j}{\partial\phi} + \sin^2\theta\frac{\partial x^i}{\partial\theta}\frac{\partial x^j}{\partial\theta}\right) + O(\epsilon^7), \tag{21.4.11}$$

and so

$$\sqrt{g_{AB}} = \epsilon^2\sin\theta\left[1 + \frac{1}{4\epsilon^2}g_{ij,kl}(\mathcal{P}_0)x^kx^l\left(\frac{1}{\sin^2\theta}\frac{\partial x^i}{\partial\phi}\frac{\partial x^j}{\partial\phi} + \frac{\partial x^i}{\partial\theta}\frac{\partial x^j}{\partial\theta}\right)\right] + O(\epsilon^5). \tag{21.4.12}$$

The area is given by

$$S = \int\sqrt{g_{AB}}d\theta d\phi. \tag{21.4.13}$$

Define

$$A^{ijkl} \equiv \frac{1}{4\epsilon^4}\int x^k x^l \left(\frac{1}{\sin\theta}\frac{\partial x^i}{\partial\phi}\frac{\partial x^j}{\partial\phi} + \sin\theta\frac{\partial x^i}{\partial\theta}\frac{\partial x^j}{\partial\theta}\right) d\theta d\phi. \tag{21.4.14}$$

We have

$$S = 4\pi\epsilon^2 + g_{ij,kl}(\mathcal{P}_0)A^{ijkl} = 4\pi\epsilon^2 - \frac{1}{3}\left(R_{ikjl}(\mathcal{P}_0) + R_{iljk}(\mathcal{P}_0)\right)A^{ijkl} = 4\pi\epsilon^2 - \frac{\epsilon^4}{3}R_{ijkl}(\mathcal{P}_0)B^{ijkl} + O(\epsilon^5), \tag{21.4.15}$$

where

$$B^{ijkl} \equiv A^{ikjl} + A^{iklj}. \tag{21.4.16}$$

With the help of SageMath, we can get

$$B^{1111} = \frac{4}{15}\pi, \quad B^{1122} = -\frac{2}{15}\pi, \quad B^{1133} = -\frac{2}{15}\pi, \quad B^{1212} = \frac{8}{15}\pi, \quad B^{1221} = -\frac{2}{15}\pi,$$

$$B^{1313} = \frac{8}{15}\pi, \quad B^{1331} = -\frac{2}{15}\pi, \quad B^{2112} = -\frac{2}{15}\pi, \quad B^{2121} = \frac{8}{15}\pi, \quad B^{2211} = -\frac{2}{15}\pi,$$

$$B^{2222} = \frac{4}{15}\pi, \quad B^{2233} = -\frac{2}{15}\pi, \quad B^{2323} = \frac{8}{15}\pi, \quad B^{2332} = -\frac{2}{15}\pi, \quad B^{3113} = -\frac{2}{15}\pi,$$

$$B^{3131} = \frac{8}{15}\pi, \quad B^{3223} = -\frac{2}{15}\pi, \quad B^{3232} = \frac{8}{15}\pi, \quad B^{3311} = -\frac{2}{15}\pi, \quad B^{3322} = -\frac{2}{15}\pi, \quad B^{3333} = \frac{4}{15}\pi.$$

All other componets vanish. So we have

$$S = 4\pi\epsilon^2 - \frac{\epsilon^4}{3}\times\frac{4\pi}{3}(R_{1212} + R_{1313} + R_{2323}) = 4\pi\epsilon^2 - \frac{2\pi}{9}R(\mathcal{P}_0)\epsilon^4 + O(\epsilon^5). \tag{21.4.17}$$

Thus

$$\lim_{\epsilon\to 0} 18\frac{4\pi\epsilon^2 - S}{4\pi\epsilon^4} = R(\mathcal{P}_0). \tag{21.4.18}$$

21.5 EXTRINSIC CURVATURE TENSOR FOR SLICE OF FRIEDMANN GEOMETRY

Confirm the result (21.70) for the extnnsiC curvature by direct calculation from formula (21.67).

Solution:

In Friedmann geometry, we have

$$ds^2 = -dt^2 + a^2(d\chi^2 + \sin^2\chi[d\theta^2 + \sin^2\theta d\phi^2)]. \tag{21.5.1}$$

So we have $N^i = 0$. Thus

$$K_{ik} = -\frac{1}{2N}\frac{\partial g_{ik}}{\partial t} = -\frac{1}{2N}\frac{1}{a^2}\frac{d(a^2)}{dt}g_{ik}, \tag{21.5.2}$$

i.e.

$$\boldsymbol{K} = -\frac{1}{2N}\frac{1}{a^2}\frac{d(a^2)}{dt}\mathbf{1}. \tag{21.5.3}$$

21.6 EVALUATION OF $\mathcal{R}(e_i, e_j)n$

Evaluate this quantity along the model of (21.74) or otherwise. How can it be foreseen that the coefficient of $\boldsymbol{n}$ in the result must vanish identically? Comparing coefficients of $\boldsymbol{e}_m$ find ${}^{(4)}R^m{}_{njk}$ and test for equivalence to equation (21.76).

Solution:

$$ {}^{(4)}\nabla_{\boldsymbol{e}_j}\,{}^{(4)}\nabla_{\boldsymbol{e}_k}\boldsymbol{n} = {}^{(4)}\nabla_{\boldsymbol{e}_j}\left(K_k{}^i\boldsymbol{e}_i\right) = K_k{}^i{}_{,j}\boldsymbol{e}_i + K_k{}^i\left({}^{(3)}\Gamma^l{}_{ij}\boldsymbol{e}_l + \epsilon K_{ij}\boldsymbol{n}\right) \tag{21.6.1} $$

So we have

$$ [{}^{(4)}\nabla_{\boldsymbol{e}_j}, {}^{(4)}\nabla_{\boldsymbol{e}_k}]\boldsymbol{n} = (K_k{}^i{}_{,j} - K_j{}^i{}_{,k} + K_k{}^l\,{}^{(3)}\Gamma^i{}_{lj} - K_j{}^l\,{}^{(3)}\Gamma^i{}_{lk})\boldsymbol{e}_i + \epsilon(K_k{}^iK_{ij} - K_j{}^iK_{ik})\boldsymbol{n} = (K_k{}^i{}_{|j} - K_j{}^i{}_{|k})\boldsymbol{e}_i, $$

i.e.

$$ {}^{(4)}R^m{}_{njk} = K_k{}^m{}_{|j} - K_j{}^m{}_{|k}. \tag{21.6.2} $$

21.7 EVALUATION OF THE COMMUTATOR $[e_j, n]$

The evaluation of this commutator is a first step toward the calculation of a quantity like $\mathcal{R}(\boldsymbol{e}_j, \boldsymbol{n})\boldsymbol{e}_i$. Expressing $\boldsymbol{e}_j$ as the differential operator $\partial/\partial x^j$, use (21.49) to represent $\boldsymbol{n}$ also as a differential operator. In this way, show that the commutator in question has the value $-(N_{,j}/N)\boldsymbol{n} - (N^m{}_{,j}/N)\boldsymbol{e}_m$.

Solution:

We have $\boldsymbol{e}_j^\alpha = \delta_j^\alpha$, $\boldsymbol{n}^0 = g^{00}n_0 = 1/N$ and $n^i = g^{i0}n_0 = -N^i/N$. So

$$ [\boldsymbol{e}_j, \boldsymbol{n}]^\alpha = \delta_j^\beta n^\alpha{}_{,\beta} - n^\beta\delta^\alpha{}_{j,\beta} = n^\alpha{}_{,j}, \tag{21.7.1} $$

i.e.

$$ [\boldsymbol{e}_j, \boldsymbol{n}]^0 = -\frac{N_{,j}}{N^2}; \quad [\boldsymbol{e}_j, \boldsymbol{n}]^m = \frac{N_{,j}N^m}{N^2} - \frac{N^m{}_{,j}}{N}. \tag{21.7.2} $$

As

$$ \boldsymbol{e}_0 = N\boldsymbol{n} + N^i\boldsymbol{e}_i, \tag{21.7.3} $$

we have

$$ [\boldsymbol{e}_j, \boldsymbol{n}] = -\frac{N_{,j}}{N^2}(N\boldsymbol{n} + N^m\boldsymbol{e}_m) + \left(\frac{N_{,j}N^m}{N^2} - \frac{N^m{}_{,j}}{N}\right)\boldsymbol{e}_m = -\frac{N_{,j}}{N}\boldsymbol{n} - \frac{N^m{}_{,j}}{N}\boldsymbol{e}_m. \tag{21.7.4} $$

21.8 LIE DERIVATIVE OF A TENSOR

Define the Lie denvative of a tensor field and explore some of its properties. The Lie derivative along a vector field $\boldsymbol{n}$ is a differential operator that operates on tensor fields $\boldsymbol{T}$ of type (r, s), converting them into tensors $\mathscr{L}_{\boldsymbol{n}}\boldsymbol{T}$, also of type (r, s). The Lie differentiation process obeys the usual chain rule and has additivity properties. For scalar functions f: one has $\mathscr{L}_{\boldsymbol{n}}f \equiv [\boldsymbol{n}]f = f_{,\mu}n^\mu$. The Lie derivative of a vector field $\boldsymbol{u}$ along a vector field $\boldsymbol{v}$ was defined in exercise 9.11 by

$$ \mathscr{L}_{\boldsymbol{u}}\boldsymbol{v} \equiv [\boldsymbol{u}, \boldsymbol{v}]. \tag{21.8.1} $$

If the action of $\mathscr{L}_{\boldsymbol{n}}$ on 1-forms is defined, the extension to tensors of general type will be simple, because the latter can always be decomposed into a sum of tensor products of vectors and 1-forms. If $\boldsymbol{\sigma}$ is a 1-form and $\boldsymbol{v}$ is a vector, then one defines $\mathscr{L}_{\boldsymbol{n}}\boldsymbol{\sigma}$ to be that 1-form satisfying

$$\langle \mathscr{L}_{\boldsymbol{n}}\boldsymbol{\sigma}, \boldsymbol{v}\rangle = \boldsymbol{n}[\langle \boldsymbol{\sigma}, \boldsymbol{v}\rangle] - \langle \boldsymbol{\sigma}, [\boldsymbol{n}, \boldsymbol{v}]\rangle \tag{21.8.2}$$

for arbitrary $\boldsymbol{v}$.

(a) Show that in a coordinate basis

$$\mathscr{L}_{\boldsymbol{n}}\boldsymbol{\sigma} = (\sigma_{\alpha,\beta}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{,\alpha})\boldsymbol{d}x^{\alpha}. \tag{21.8.3}$$

(b) Show that in a coordinate basis

$$\mathscr{L}_{\boldsymbol{n}}\boldsymbol{T} = (T_{\alpha\beta,\mu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha} + T_{\alpha\mu}n^{\mu}{}_{,\beta})\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta} \tag{21.8.4}$$

where $\boldsymbol{T}$ is of type $(0,2)$.

(c) Show that in (a) and (b), all partial derivatives can be replaced by covariant derivatives.

Solution:

(a) First, we have

$$\langle \mathscr{L}_{\boldsymbol{n}}\boldsymbol{d}x^{\alpha}, \boldsymbol{e}_{\beta}\rangle = \boldsymbol{n}[\langle \boldsymbol{d}x^{\alpha}, \boldsymbol{e}_{\beta}\rangle] - \langle \boldsymbol{d}x^{\alpha}, [\boldsymbol{n}, \boldsymbol{e}_{\beta}]\rangle = n^{\alpha}{}_{,\beta} \tag{21.8.5}$$

i.e. $\mathscr{L}_{\boldsymbol{n}}\boldsymbol{d}x^{\alpha} = n^{\alpha}{}_{,\beta}\boldsymbol{d}x^{\beta}$. Thus

$$\mathscr{L}_{\boldsymbol{n}}\boldsymbol{\sigma} = \mathscr{L}_{\boldsymbol{n}}(\sigma_{\alpha}\boldsymbol{d}x^{\alpha}) = \mathscr{L}_{\boldsymbol{n}}(\sigma_{\alpha})\boldsymbol{d}x^{\alpha} + \sigma_{\alpha}\mathscr{L}_{\boldsymbol{n}}(\boldsymbol{d}x^{\alpha}) = \sigma_{\alpha,\beta}n^{\beta}\boldsymbol{d}x^{\alpha} + \sigma_{\alpha}n^{\alpha}{}_{,\beta}\boldsymbol{d}x^{\beta} = (\sigma_{\alpha,\beta}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{,\alpha})\boldsymbol{d}x^{\alpha}. \tag{21.8.6}$$

(b)

$$\begin{aligned}
\mathscr{L}_{\boldsymbol{n}}\boldsymbol{T} &= \mathscr{L}_{\boldsymbol{n}}(T_{\alpha\beta}\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}) \\
&= \mathscr{L}_{\boldsymbol{n}}(T_{\alpha\beta}\boldsymbol{d}x^{\alpha})\otimes\boldsymbol{d}x^{\beta} + T_{\alpha\beta}\boldsymbol{d}x^{\alpha}\otimes\mathscr{L}_{\boldsymbol{n}}(\boldsymbol{d}x^{\beta}) \\
&= (T_{\alpha\beta,\mu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha})\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta} + T_{\alpha\beta}\boldsymbol{d}x^{\alpha}\otimes n^{\beta}{}_{,\mu}\boldsymbol{d}x^{\mu} \\
&= (T_{\alpha\beta,\mu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha} + T_{\alpha\mu}n^{\mu}{}_{,\beta})\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}.
\end{aligned} \tag{21.8.7}$$

(c) For (a), we have

$$\begin{aligned}
\sigma_{\alpha;\beta}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{;\alpha} &= \sigma_{\alpha,\beta}n^{\beta} - \Gamma^{\mu}{}_{\alpha\beta}\sigma_{\mu}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{,\alpha} + \sigma_{\beta}\Gamma^{\beta}{}_{\mu\alpha}n^{\mu} \\
&= \sigma_{\alpha,\beta}n^{\beta} - \Gamma^{\mu}{}_{\alpha\beta}\sigma_{\mu}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{,\alpha} + \sigma_{\mu}\Gamma^{\mu}{}_{\beta\alpha}n^{\beta} \\
&= \sigma_{\alpha,\beta}n^{\beta} + \sigma_{\beta}n^{\beta}{}_{,\alpha}.
\end{aligned} \tag{21.8.8}$$

For (b), we have

$$\begin{aligned}
&T_{\alpha\beta;\mu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{;\alpha} + T_{\alpha\mu}n^{\mu}{}_{;\beta} \\
&= T_{\alpha\beta,\mu}n^{\mu} - \Gamma^{\nu}{}_{\alpha\mu}T_{\nu\beta}n^{\mu} - \Gamma^{\nu}{}_{\beta\mu}T_{\alpha\nu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha} + T_{\mu\beta}\Gamma^{\mu}{}_{\nu\alpha}n^{\nu} + T_{\alpha\mu}n^{\mu}{}_{,\beta} + T_{\alpha\mu}\Gamma^{\mu}{}_{\nu\beta}n^{\nu} \\
&= T_{\alpha\beta,\mu}n^{\mu} - \Gamma^{\nu}{}_{\alpha\mu}T_{\nu\beta}n^{\mu} - \Gamma^{\nu}{}_{\beta\mu}T_{\alpha\nu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha} + T_{\nu\beta}\Gamma^{\nu}{}_{\mu\alpha}n^{\mu} + T_{\alpha\mu}n^{\mu}{}_{,\beta} + T_{\alpha\nu}\Gamma^{\nu}{}_{\mu\beta}n^{\mu} \\
&= T_{\alpha\beta,\mu}n^{\mu} + T_{\mu\beta}n^{\mu}{}_{,\alpha} + T_{\alpha\mu}n^{\mu}{}_{,\beta}.
\end{aligned} \tag{21.8.9}$$

Generally speaking, if all partial derivatives can be replaced by covariant derivatives, the left hand size and right hand side of equations are invariant under coordinate transformation. Since it holds in local Lorentz frame, it will hold in any coordinate frame.

21.9 EXPRESSION FOR DYNAMIC COMPONENTS OF THE CURVATURE TENSOR

The Gauss-Codazzi equations can be viewed as giving 14 of the 20 algebraically independent components of the spacetime curvature tensor in terms of the intrinsic and extrinsic geometry of three-dimensional (non-null) hypersurfaces. In order to accomplish a space-plus-time splitting of the Hilbert Lagrangian $\sqrt{-g}^{(4)}R$, one must express, in addition, the remaining 6 components of the curvature tensor in an analogous manner. It is convenient for this purpose to express all tensors as spacetime tensors, and to use Lie derivation in the direction of the time like unit normal field of the spacelike hypersurfaces as a generalized notion of time differentiation. A number of preliminary results must be proven:

(a)

$$\mathscr{L}_{\boldsymbol{u}} g_{\mu\nu} = u_{\mu;\nu} + u_{\nu;\mu} \tag{21.9.1}$$

(b)

$$\mathscr{L}_{\boldsymbol{u}}(g_{\mu\nu} + u_\mu u_\nu) \equiv \mathscr{L}_{\boldsymbol{u}}\gamma_{\mu\nu} = u_{\mu;\nu} + u_{\nu;\mu} + u_\mu a_\nu + u_\nu a_\mu, \tag{21.9.2}$$

where $\gamma_{\mu\nu}$ is the metric of the spacelike hypersurface, expressed in the spacetime coordinate basis, and $a^\mu = u^\lambda \nabla_\lambda u^\mu$ is the curvature vector (4-acceleration) of the timelike normal curves whose tangent field is u^μ.

(c) Prove that the extrinsic curvature tensor is given by

$$K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_{\boldsymbol{u}}\gamma_{\mu\nu}. \tag{21.9.3}$$

(d) The unit tensor of projection into the hypersurface is defined by

$$\perp^\mu_\alpha \equiv \delta^\mu_\alpha + u^\mu u_\alpha. \tag{21.9.4}$$

In terms of $\perp$ show that one can write

$$u_{\alpha;\beta} = -K_{\alpha\beta} - \omega_{\alpha\beta} - a_\alpha u_\beta, \tag{21.9.5}$$

where

$$K_{\alpha\beta} = -\perp^\mu_\alpha \perp^\nu_\beta u_{(\mu;\nu)} \tag{21.9.6}$$

and

$$\omega_{\alpha\beta} = -\perp^\mu_\alpha \perp^\nu_\beta u_{[\mu;\nu]} \tag{21.9.7}$$

(e) From the fact that u^μ is the unit normal field for a family of spacelike hypersurfaces, show that $\omega_{\alpha\beta} = 0$.

(f) The needed tools are now on hand. To obtain the result:
(i) Write down $\mathscr{L}_{\boldsymbol{u}} K_{\mu\nu}$ (see exercise 21.8);
(ii) Insert this expression into the Ricci identity in the form

$$u^\sigma \nabla_\sigma \nabla_\mu u_\nu = u^\sigma \nabla_\mu \nabla_\sigma u_\nu - {}^{(4)}R_{\rho\nu\mu\sigma} u^\sigma u^\rho; \tag{21.9.8}$$

(iii) Project the two remaining free indices into the hypersurface using $\perp$, and show that one obtains

$$\perp^\mu_\alpha \perp^\rho_\beta {}^{(4)}R_{\mu\nu\rho\sigma} u^\nu u^\sigma = \mathscr{L}_{\boldsymbol{u}} K_{\alpha\beta} + K_{\alpha\gamma} K^\gamma_\beta + {}^{(3)}\nabla_{(\alpha} a_{\beta)} + a_\alpha a_\beta, \tag{21.9.9}$$

where ${}^{(3)}\nabla_\alpha a_\beta \equiv \perp^\mu_\alpha \perp^\nu_\beta \nabla_\mu a_\nu$ can be shown to be the three-dimensional covariant derivative of a_β. In Gaussian normal coordinates, show that one obtains from this result

$$R_{0i0j} = \frac{\partial}{\partial t} K_{ij} + K_{ik} K^k_j. \tag{21.9.10}$$

(g) Finally, in the construction of ${}^{(4)}R$, one needs to show that

$$\gamma^{\mu\nu}[{}^{(3)}\nabla_{(\mu}a_{\nu)} + a_\mu a_\nu] = g^{\mu\nu}[{}^{(3)}\nabla_{(\mu}a_{\nu)} + a_\mu a_\nu] = a^\lambda{}_{;\lambda}. \tag{21.9.11}$$

Solution:

(a)

$$\mathscr{L}_{\boldsymbol{u}} g_{\mu\nu} = g_{\mu\nu;\rho}n^\rho + g_{\rho\nu}u^\rho{}_{;\mu} + g_{\mu\rho}u^\rho{}_{;\nu} = u_{\mu;\nu} + u_{\nu;\mu}. \tag{21.9.12}$$

(b)

$$\mathscr{L}_{\boldsymbol{u}} u_\mu u_\nu = (u_\mu u_\nu)_{;\rho}u^\rho + u_\rho u_\nu u^\rho{}_{;\mu} + u_\mu u_\rho u^\rho{}_{;\nu} = (u_\mu u_\nu)_{;\rho}u^\rho = u_\mu a_\nu + u_\nu a_\mu. \tag{21.9.13}$$

Thus

$$\mathscr{L}_{\boldsymbol{u}} \gamma_{\mu\nu} = u_{\mu;\nu} + u_{\nu;\mu} + u_\mu a_\nu + u_\nu a_\mu. \tag{21.9.14}$$

(c) In coordinates where spacelike hypersurfaces are $t =$ constant, we have $u_\mu = (-N, 0, 0, 0)$. Thus

$$\mathscr{L}_{\boldsymbol{u}} \gamma_{ij} = u_{i;j} + u_{j;i} = 2u_{i;j} = -2K_{ij}. \tag{21.9.15}$$

And also

$$u^\mu \mathscr{L}_{\boldsymbol{u}} \gamma_{\mu\nu} = u^\mu u_{\mu;\nu} + u^\mu u_{\nu;\mu} + u^\mu u_\mu a_\nu + u^\mu u_\nu a_\mu = 0, \tag{21.9.16}$$

i.e.

$$(\mathscr{L}_{\boldsymbol{u}}\gamma)^{0i} = (\mathscr{L}_{\boldsymbol{u}}\gamma)^{i0} = (\mathscr{L}_{\boldsymbol{u}}\gamma)^{00} = 0. \tag{21.9.17}$$

Now we can write the extrinsic curvature tensor in four dimensiona space as

$$K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_{\boldsymbol{u}}\gamma_{\mu\nu}. \tag{21.9.18}$$

(d)

$$\perp^\mu_\alpha \perp^\nu_\beta u_{\mu;\nu} = (\delta^\mu_\alpha + u^\mu u_\alpha)(\delta^\nu_\beta + u^\nu u_\beta)u_{\mu;\nu} = (\delta^\mu_\alpha + u^\mu u_\alpha)(u_{\mu;\beta} + a_\mu u_\beta) = u_{\alpha;\beta} + a_\alpha u_\beta. \tag{21.9.19}$$

Thus we have

$$K_{\alpha\beta} + \omega_{\alpha\beta} = -\perp^\mu_\alpha \perp^\nu_\beta (u_{(\mu;\nu)} + u_{[\mu;\nu]}) = -\perp^\mu_\alpha \perp^\nu_\beta u_{\mu;\nu} = -(u_{\alpha;\beta} + a_\alpha u_\beta), \tag{21.9.20}$$

i.e.

$$u_{\alpha;\beta} = -K_{\alpha\beta} - \omega_{\alpha\beta} - a_\alpha u_\beta. \tag{21.9.21}$$

(e)

$$\omega_{\alpha\beta} = -u_{\alpha;\beta} - a_\alpha u_\beta + u_{\beta;\alpha} + a_\beta u_\alpha. \tag{21.9.22}$$

Since $u_0 = (-N, 0, 0, 0)$ and $u^\alpha u_\alpha = -1$, we have

$$\omega_{ij} = u_{j;i} - u_{i:j} = -\Gamma^0{}_{ji}u_0 + \Gamma^0{}_{ij}u_0 = 0, \tag{21.9.23}$$

and

$$u^\alpha \omega_{\alpha\beta} = u^\alpha u_{\beta;\alpha} - a_\beta = 0. \tag{21.9.24}$$

So we have

$$0 = u^\alpha \omega_{\alpha i} = u^0\omega_{0i} + u^j\omega_{ji} = u^0\omega_{0i}. \tag{21.9.25}$$

Thus $\omega_{0i} = 0$, i.e. $\omega_{\alpha\beta} = 0$.

(f) The sign of equation 21.9.8 in the textbook might not be correct. So The equations below 21.9.8 also has mistakes in the sign. I corrected them in the solution manual.

(i) The definition of Lie derivative gives

$$\mathscr{L}_{\boldsymbol{u}} K_{\mu\nu} = K_{\mu\nu;\sigma} u^\sigma + K_{\sigma\nu} u^\sigma{}_{;\mu} + K_{\mu\sigma} u^\sigma{}_{;\nu}. \tag{21.9.26}$$

The vanishing of $\omega_{\mu\nu}$ indicates that

$$K_{\mu\nu} = -u_{\mu;\nu} - a_\mu u_\nu. \tag{21.9.27}$$

Thus

$$\begin{aligned}
\mathscr{L}_{\boldsymbol{u}} K_{\mu\nu} &= -u_{\mu;\nu\sigma} u^\sigma - a_{\mu;\sigma} u^\sigma u_\nu - a_\mu a_\nu + K_{\sigma\nu} u^\sigma{}_{;\mu} - (u_{\mu;\sigma} + a_\mu u_\sigma) u^\sigma{}_{;\nu} \\
&= -u_{\mu;\nu\sigma} u^\sigma - u_{\mu;\sigma} u^\sigma{}_{;\nu} - a_{\mu;\sigma} u^\sigma u_\nu - a_\mu a_\nu + K_{\sigma\nu} u^\sigma{}_{;\mu} \\
&= -u_{\mu;\nu\sigma} u^\sigma + u_{\mu;\sigma\nu} u^\sigma - a_{\mu;\nu} - a_\mu a_\nu - a_{\mu;\sigma} u^\sigma u_\nu + K_{\sigma\nu} u^\sigma{}_{;\mu}.
\end{aligned} \tag{21.9.28}$$

(ii)

$${}^{(4)}R_{\rho\mu\nu\sigma} u^\sigma u^\rho = -u^\sigma u_{\mu;\nu\sigma} + u^\sigma u_{\mu;\sigma\nu} = \mathscr{L}_{\boldsymbol{u}} K_{\mu\nu} + a_{\mu;\nu} + a_\mu a_\nu + a_{\mu;\sigma} u^\sigma u_\nu - K_{\sigma\nu} u^\sigma{}_{;\mu}. \tag{21.9.29}$$

(iii) We have

$$\begin{aligned}
u^\mu \mathscr{L}_{\boldsymbol{u}} K_{\mu\nu} &= \mathscr{L}_{\boldsymbol{u}} (K_{\mu\nu} u^\mu) - K_{\mu\nu} \mathscr{L}_{\boldsymbol{u}} u^\mu = 0; \\
u^\mu a_\mu a_\nu &= 0; \\
\perp^\mu_\rho \, u^\sigma{}_{;\mu} &= u^\sigma{}_{;\rho} + u^\sigma{}_{;\mu} u^\mu u_\rho = u^\sigma{}_{;\rho} + a^\sigma u_\rho = -K^\sigma_\rho; \\
\perp^\nu_\rho \, a_{\mu;\sigma} u^\sigma u_\nu &= 0.
\end{aligned} \tag{21.9.30}$$

Thus

$$\perp^\mu_\alpha \perp^\nu_\beta \, {}^{(4)}R_{\rho\mu\nu\sigma} u^\sigma u^\rho = \mathscr{L}_{\boldsymbol{u}} K_{\alpha\beta} + K_{\alpha\gamma} K^\gamma_\beta + {}^{(3)}\nabla_\alpha a_\beta + a_\alpha a_\beta. \tag{21.9.31}$$

As $\perp^\mu_\alpha \perp^\nu_\beta \, {}^{(4)}R_{\rho\mu\nu\sigma} u^\sigma u^\rho$, $\mathscr{L}_{\boldsymbol{u}} K_{\alpha\beta}$, $K_{\alpha\gamma} K^\gamma_\beta$ and $a_\alpha a_\beta$ are all symmetric in α and β, we have

$$\perp^\mu_\alpha \perp^\nu_\beta \, {}^{(4)}R_{\rho\mu\nu\sigma} u^\sigma u^\rho = \mathscr{L}_{\boldsymbol{u}} K_{\alpha\beta} + K_{\alpha\gamma} K^\gamma_\beta + {}^{(3)}\nabla_{(\alpha} a_{\beta)} + a_\alpha a_\beta. \tag{21.9.32}$$

(g) As

$$\gamma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{21.9.33}$$

we have

$$\begin{aligned}
\gamma^{\mu\nu} [{}^{(3)}\nabla_{(\mu} a_{\nu)} + a_\mu a_\nu] &= (g^{\mu\nu} + u^\mu u^\nu)(\perp^\alpha_\mu \perp^\beta_\nu \, \nabla_{(\alpha} a_{\beta)} + a_\mu a_\nu) \\
&= g^{\mu\nu} (\perp^\alpha_\mu \perp^\beta_\nu \, \nabla_{(\alpha} a_{\beta)} + a_\mu a_\nu) \\
&= \frac{1}{2} g^{\mu\nu} (\delta^\alpha_\mu + u^\alpha u_\mu)(\delta^\beta_\nu + u^\beta u_\nu)(a_{\alpha;\beta} + a_{\beta;\alpha}) + a^\mu a_\mu \\
&= (g^{\alpha\beta} + u^\alpha u^\beta) a_{\alpha;\beta} + a^\mu a_\mu \\
&= a^\lambda{}_{;\lambda} - u^\beta u^\alpha{}_{;\beta} a_\alpha + a^\mu a_\mu \\
&= a^\lambda{}_{;\lambda}.
\end{aligned} \tag{21.9.34}$$

21.10 EXPRESSION OF ${}^{(4)}R^i{}_{nin}$ IN TERMS OF EXTRINSIC CURVATURE, PLUS A COVARIANT DIVERGENCE

Let α' be an arbitrary smooth set of four coordinates, not necessarily coordinated in any way with the choice of the 1-parameter family of hypersurfaces.

(a) Show that

$$ {}^{(4)}R^{i}{}_{nin} = g^{\alpha'\gamma'} n^{\beta'} (n_{\alpha';\beta'\gamma'} - n_{\alpha';\gamma'\beta'}). \tag{21.10.1} $$

(b) Show that the covariant divergences

$$ (n^{\beta'} n^{\gamma'}{}_{;\beta'})_{;\gamma'} \tag{21.10.2} $$

and

$$ (n^{\beta'} n^{\gamma'}{}_{;\gamma'})_{;\beta'} \tag{21.10.3} $$

can be removed from this expression in such a way that what is left behind contains only first derivatives of the unit normal vector $\boldsymbol{n}$.

(c) Noting that the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{n}$ form a complete set, justify the formula

$$ g^{\beta'\mu'} = e_i^{\beta'} \omega^{i\mu'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\beta'} n^{\mu'}, \tag{21.10.4} $$

where ω^i is the 1-form dual to $\boldsymbol{e}_i$.

(d) Noting that $n_{\alpha';\beta'} n^{\alpha'} = 0$ and

$$ K_{ij} = -e_{i\alpha'} n^{\alpha'}{}_{;\beta'} e_j^{\beta'}, \tag{21.10.5} $$

show that

$$ {}^{(4)}R^{i}{}_{nin} = (\mathrm{Tr}\boldsymbol{K})^2 - \mathrm{Tr}\boldsymbol{K}^2 \text{ plus a covariant divergence.} \tag{21.10.6} $$

Solution:

(c) Note that $e_i^{\alpha'}$ and $\boldsymbol{n}^{\alpha'}$ form a complete set, and $\omega^i_{\alpha'}$ is dual vector of $e_i^{\alpha'}$, i.e. $e_i^{\alpha'}\omega^j_{\alpha'} = \delta_i^j$, $n^{\alpha'}\omega^j_{\alpha'} = 0$. As $\boldsymbol{n}$ is perpendicular to the $\boldsymbol{e}_i$, we have $n_{\alpha'} e_i^{\alpha'} = 0$, where $n_{\alpha'} = g_{\alpha'\beta'} n^{\beta'}$. Suppose $\boldsymbol{X}$ is an arbitrary vector. It can be decomposed as

$$ X^{\alpha'} = x^0 n^{\alpha'} + x^i e_i^{\alpha'}. \tag{21.10.7} $$

So we have

$$ \begin{aligned} &(e_i^{\alpha'}\omega^i_{\beta'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\alpha'} n_{\beta'}) X^{\beta'} \\ &= (e_i^{\alpha'}\omega^i_{\beta'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\alpha'} n_{\beta'})(x^0 n^{\beta'} + x^j e_j^{\beta'}) \\ &= x^0 (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\alpha'} n_{\beta'} n^{\beta'} + x^j e_j^{\beta'} \omega^i_{\beta'} e_i^{\alpha'} \\ &= x^0 n^{\alpha'} + x^i e_i^{\alpha'}, \end{aligned} \tag{21.10.8} $$

indicateing that

$$ e_i^{\alpha'}\omega^i_{\beta'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\alpha'} n_{\beta'} = \delta^{\alpha'}_{\beta'}, \tag{21.10.9} $$

i.e.

$$ g^{\beta'\mu'} = e_i^{\beta'} \omega^{i\mu'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\beta'} n^{\mu'}. \tag{21.10.10} $$

(a)

$$ \begin{aligned} {}^{(4)}R^{i}{}_{nin} &= \langle \boldsymbol{\omega}^i, \mathcal{R}(\boldsymbol{e}_i, \boldsymbol{n})\boldsymbol{n}\rangle \\ &= {}^{(4)}R^{\mu'}{}_{\alpha'\gamma'\beta'} \omega^i_{\mu'} e_i^{\gamma'} n^{\alpha'} n^{\beta'} \\ &= {}^{(4)}R^{\mu'}{}_{\alpha'\gamma'\beta'} [\delta^{\gamma'}_{\mu'} - (\boldsymbol{n}\cdot\boldsymbol{n})^{-1} n^{\gamma'} n_{\mu'}] n^{\alpha'} n^{\beta'} \\ &= n^{\beta'} {}^{(4)}R^{\gamma'}{}_{\alpha'\gamma'\beta'} n^{\alpha'} \\ &= n^{\beta'} (n^{\gamma'}{}_{;\beta'\gamma'} - n^{\gamma'}{}_{;\gamma'\beta'}) \\ &= g^{\alpha'\gamma'} n^{\beta'} (n_{\alpha';\beta'\gamma'} - n_{\alpha';\gamma'\beta'}). \end{aligned} \tag{21.10.11} $$

(b)

$$
\begin{aligned}
{}^{(4)}R^{i}{}_{nin} &= n^{\beta'}(n^{\gamma'}{}_{;\beta'\gamma'} - n^{\gamma'}{}_{;\gamma'\beta'}) \\
&= (n^{\beta'}n^{\gamma'}{}_{;\beta'})_{;\gamma'} - n^{\beta'}{}_{;\gamma'}n^{\gamma'}{}_{;\beta'} - (n^{\beta'}n^{\gamma'}{}_{;\gamma'})_{;\beta'} + n^{\beta'}{}_{;\beta'}n^{\gamma'}{}_{;\gamma'} \\
&= n^{\beta'}{}_{;\beta'}n^{\gamma'}{}_{;\gamma'} - n^{\beta'}{}_{;\gamma'}n^{\gamma'}{}_{;\beta'} + (n^{\beta'}n^{\gamma'}{}_{;\beta'} - n^{\gamma'}n^{\beta'}{}_{;\beta'})_{;\gamma'}
\end{aligned}
\tag{21.10.12}
$$

(d) We have

$$
K_i{}^j = -e_i^{\alpha'}n_{\alpha'}{}^{;\beta'}\omega^j_{\beta'}. \tag{21.10.13}
$$

Note that $n_{\alpha';\beta'}n^{\alpha'} = 0$, we can get

$$
K_i{}^i = -e_i^{\alpha'}n_{\alpha'}{}^{;\beta'}\omega^i_{\beta'} = [-\delta^{\alpha'}_{\beta'} + (\boldsymbol{n}\cdot\boldsymbol{n})^{-1}n^{\alpha'}n_{\beta'}]n_{\alpha'}{}^{;\beta'} = -n^{\alpha'}{}_{;\alpha'} \tag{21.10.14}
$$

and

$$
\begin{aligned}
K_i{}^jK_j{}^i &= e_i^{\alpha'}n_{\alpha'}{}^{;\beta'}\omega^j_{\beta'}e_j^{\gamma'}n_{\gamma'}{}^{;\delta'}\omega^i_{\delta'} \\
&= n_{\alpha'}{}^{;\beta'}n_{\gamma'}{}^{;\delta'}[\delta^{\alpha'}_{\delta'} - (\boldsymbol{n}\cdot\boldsymbol{n})^{-1}n^{\alpha'}n_{\delta'}][\delta^{\gamma'}_{\beta'} - (\boldsymbol{n}\cdot\boldsymbol{n})^{-1}n^{\gamma'}n_{\beta'}] \\
&= n_{\alpha'}{}^{;\beta'}n_{\beta'}{}^{;\alpha'} \\
&= n^{\alpha'}{}_{;\beta'}n^{\beta'}{}_{;\alpha'}.
\end{aligned}
\tag{21.10.15}
$$

So

$$
{}^{(4)}R^{i}{}_{nin} = (\mathrm{Tr}\boldsymbol{K})^2 - \mathrm{Tr}\boldsymbol{K}^2 \text{ plus a covariant divergence.} \tag{21.10.16}
$$

21.11 FIRST EXPLOITATION OF THE ADM VARIATIONAL PRINCIPLE FOR THE ELECTROMAGNETIC FIELD

Extremize the action principle (21.100) with respect to the $\mathscr{E}^i$ and derive the result (21.103).

Solution:

We have

$$
4\pi\mathcal{L}_{\text{field}} = -\mathscr{E}^i\frac{\partial A_i}{\partial t} + \phi\mathscr{E}^i{}_{,i} - \frac{1}{2}Ng^{-1/2}g_{ij}(\mathscr{E}^i\mathscr{E}^j + \mathscr{B}^i\mathscr{B}^j) + N^i[ijk]\mathscr{E}^j\mathscr{B}^k. \tag{21.11.1}
$$

The variation of $\mathscr{E}^i$ gives

$$
4\pi\delta\mathcal{L}_{\text{field}} = \left(-\frac{\partial A_i}{\partial t} - \phi_{,i} - Ng^{-1/2}g_{ij}\mathscr{E}^j + N^j[jik]\mathscr{B}^k\right)\delta\mathscr{E}^i = 0, \tag{21.11.2}
$$

and so

$$
\frac{\partial A_i}{\partial t} = -\phi_{,i} - Ng^{-1/2}g_{ij}\mathscr{E}^j - [ijk]N^j\mathscr{B}^k. \tag{21.11.3}
$$

21.12 SECOND EXPLOITATION OF THE ADM VARIATIONAL PRINCIPLE FOR THE ELECTROMAGNETIC FIELD

Extremize (21.100) with respect to the A_i's and verify that the resulting equations in any Minkowski-flat region are equivalent to (21.107).

Solution:

$$4\pi\mathcal{L}_{\text{field}} = -\mathscr{E}^i\frac{\partial A_i}{\partial t} + \phi\mathscr{E}^i{}_{,i} - \frac{1}{2}Ng^{-1/2}g_{ij}(\mathscr{E}^i\mathscr{E}^j + \mathscr{B}^i\mathscr{B}^j) + N^i[ijk]\mathscr{E}^j\mathscr{B}^k, \tag{21.12.1}$$

where $\mathscr{B}^i = [ijk]A_{k,j}$. So we have

$$\begin{aligned} 4\pi\delta\mathcal{L}_{\text{field}} &= -\mathscr{E}^i\frac{\partial\delta A_i}{\partial t} - Ng^{-1/2}g_{ij}\mathscr{B}^i[jmn]\delta A_{n,m} + N^i[ijk]\mathscr{E}^j[kmn]\delta A_{n,m} \\ &= \frac{\partial\mathscr{E}^n}{\partial t}\delta A_n + \left(Ng^{-1/2}g_{ij}\mathscr{B}^i\right)_{,m}[jmn]\delta A_n - [kmn][ijk]\left(N^i\mathscr{E}^j\right)_{,m}\delta A_n \\ &= \left[\frac{\partial\mathscr{E}^n}{\partial t} - [nmj]\left(NB_j\right)_{,m} + [nmk]\left([kij]N^i\mathscr{E}^j\right)_{,m}\right]\delta A_n = 0, \end{aligned} \tag{21.12.2}$$

i.e.

$$\frac{\partial\mathscr{E}^n}{\partial t} = [nmj]\left(NB_j\right)_{,m} - [nmk]\left([kij]N^i\mathscr{E}^j\right)_{,m} \tag{21.12.3}$$

In flat spacetime, we have $g = 1$, $N = 1$ and $N^i = 0$. So we can get

$$\frac{\partial E^n}{\partial t} = [nmj]\left(B_j\right)_{,m}, \tag{21.12.4}$$

i.e.

$$\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{B}. \tag{21.12.5}$$

21.13 FARADAY-MAXWELL SOURCE TERM IN THE DYNAMIC EQUATIONS OF GENERAL RELATIVITY

Evaluate the final indicated source terms in (21.115) from the Lagrangian (21.100) of Maxwell electrodynamics, regarded as a function of the A_i and the g_{ij}.

Solution:

$$4\pi\delta\mathcal{L}_{\text{field}} = -\frac{1}{2}Ng^{-1/2}(\mathscr{E}^i\mathscr{E}^j + \mathscr{B}^i\mathscr{B}^j)\delta g_{ij} + \frac{1}{4}Ng^{-3/2}g_{ij}(\mathscr{E}^i\mathscr{E}^j + \mathscr{B}^i\mathscr{B}^j)gg^{kl}\delta g_{kl} \tag{21.13.1}$$

So the source terms arising from fields other than geometry is

$$-2Ng^{-1/2}(\mathscr{E}^i\mathscr{E}^j + \mathscr{B}^i\mathscr{B}^j) + Ng^{-1/2}g_{kl}(\mathscr{E}^k\mathscr{E}^l + \mathscr{B}^k\mathscr{B}^l)g^{ij}, \tag{21.13.2}$$

i.e.

$$2Ng^{-1/2}\left[\frac{1}{2}g_{kl}(\mathscr{E}^k\mathscr{E}^l + \mathscr{B}^k\mathscr{B}^l)g^{ij} - \mathscr{E}^i\mathscr{E}^j - \mathscr{B}^i\mathscr{B}^j\right]. \tag{21.13.3}$$

21.14 THE CHOICE OF Φ DOESN'T MATTER

Prove the statement in the text that the dynamic development of the electric and magnetic fields themselves is independent of the choice made for the scalar potential $\phi(t, x, y, z)$ in the analysis (a) in flat spacetime

in Minkowski coordinates and (b) in general relativity, according to equations (21.103), and (21.107) as generalized in exercise 21.12.

Solution:

From equation 21.11.3

$$\frac{\partial A_i}{\partial t} = -\phi_{,i} - Ng^{-1/2} g_{ij}\mathscr{E}^j - [ijk]N^j\mathscr{B}^k, \tag{21.14.1}$$

we have

$$[imn]\frac{\partial A_{i,m}}{\partial t} = -[imn]\phi_{,im} - [imn]\left(Ng^{-1/2}g_{ij}\mathscr{E}^j\right)_{,m} - [imn]([ijk]N^j\mathscr{B}^k)_{,m}, \tag{21.14.2}$$

i.e.

$$\frac{\partial \mathscr{B}^i}{\partial t} = -[ijk]\left(NE_k\right)_{,j} - [ijk]([kmn]N^m\mathscr{B}^n)_{,j}. \tag{21.14.3}$$

Equation 21.12.3 tells

$$\frac{\partial \mathscr{E}^i}{\partial t} = [ijk]\left(NB_k\right)_{,j} - [ijk]\left([kmn]N^m\mathscr{E}^n\right)_{,j}. \tag{21.14.4}$$

So the evolution of EM field is determined by the initial value of $\mathscr{E}^i$ and $\mathscr{B}^i$ rather than ϕ.

21.15 THE CHOICE OF SLICING OF SPACETIME DOESN'T MATTER

Given a metric ${}^{(3)}g_{ij}(x,y,z)$ and an extrinsic curvature $K_{ij}(x,y,z)$ on a spacelike hypersurface Σ, and given that these quantities satisfy the initial-value equations (21.116) and (21.117), and given two alternative choices for the lapse and shift functions (N, N_i) and $(N+\delta N, N_i+\delta N_i)$, show that the curvature itself (as distinguished from its components in these two distinct coordinate systems), as calculated at a point $\mathcal{P}$ a little way (first order of small quantities) off the hypersurface, by way of the dynamic equations (21.114) and (21.115), is independent of this choice of lapse and shift.

Solution:

In the reference frame generated by (N, N_i), denote the point $(0,x,y,z)$ and $(\Delta t,x,y,z)$ as $\mathcal{O}$ and $\mathcal{P}$. So we have the tagent vector $t^\alpha = (\Delta t, 0, 0, 0)$. By the defination of lapse and shift, we have

$$t^\alpha n_\alpha = -N\Delta t, \quad g_{\alpha\beta}t^\alpha e_i^\beta = N_i\Delta t. \tag{21.15.1}$$

In the reference frame generated by $(N+\delta N, N_i+\delta N_i)$, suppose the coordinate of $\mathcal{P}$ is $(\Delta t+\delta t, x^i+\delta x^i)$. And we have

$$t^{\alpha'}n_{\alpha'} = -(N+\delta N)(\Delta t+\delta t), \quad g_{\alpha'\beta'}t^{\alpha'}e_i^{\beta'} = (N_i+\delta N_i)(\Delta t+\delta t) + g_{ij}\delta x^j. \tag{21.15.2}$$

Thus, we can obtain

$$N\delta t + \delta N\Delta t = 0, \quad N_i\delta t + \delta N_i\Delta t + g_{ij}\delta x^j = 0, \tag{21.15.3}$$

keeping the firsr order of δ.

In the reference frame generated by (N, N_i), we have for $\mathcal{P}$

$$g_{ij}(\mathcal{P}) = g_{ij}(x^i) + \frac{\partial g_{ij}}{\partial t}(x^i, N, N_i)\Delta t \tag{21.15.4}$$

and

$$\pi_{ij}(\mathcal{P}) = \pi_{ij}(x^i) + \frac{\partial \pi_{ij}}{\partial t}(x^i, N, N_i)\Delta t, \tag{21.15.5}$$

keeping the first order of Δ.

While in the reference frame generated by $(N+\delta N, N_i+\delta N_i)$, we have for $\mathcal{P}$

$$g_{ij}(\mathcal{P}) = g_{ij}(x^i+\delta x^i) + \frac{\partial g_{ij}}{\partial t}(x^i+\delta x^i, N+\delta N, N_i+\delta N_i)(\Delta t+\delta t) \tag{21.15.6}$$

and

$$\pi_{ij}(\mathcal{P}) = \pi_{ij}(x^i+\delta x^i) + \frac{\partial \pi_{ij}}{\partial t}(x^i+\delta x^i, N+\delta N, N_i+\delta N_i)(\Delta t+\delta t), \tag{21.15.7}$$

keeping the first order of Δ.

The value of $\partial g_{ij}/\partial t$ and $\partial \pi_{ij}/\partial t$ is given by equations (21.114) and (21.115) in the textbook. Then we must chech whether the curvatures at $\mathcal{P}$ in two frame are the same. The calculation here is too tedious for me to perform. Any help is appreciated.

21.16 POOR MAN'S WAY TO DO COSMOLOGY

Consider a spacetime with the metric

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{21.16.1}$$

corresponding to a 3-geometry with the form of a sphere of radius $a(t)$ changing with time. Show that the tensor of extrinsic curvature as expressed in a local Euclidean frame of reference is

$$\boldsymbol{K} = -a^{-1}(da/dt)\mathbf{1}, \tag{21.16.2}$$

where $\mathbf{1}$ is the unit tensor. Show that the initial value equation (21.77) reduces to

$$\frac{6}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{6}{a^2} = 16\pi\rho(a), \tag{21.16.3}$$

and explain why it is appropriate to write the term on the right as $6a_0/a^3$ for a "dust-filled model universe". More generally, given any equation of state, $p=\rho(a)$, explain how one can find $\rho=\rho(a)$ from

$$d(\rho a^3) = -pd(a^3); \tag{21.16.4}$$

and how one can thus forecast the history of expansion and recontraction, $a=a(t)$.

Solution:

From the defination of extrinsic curvature, we have

$$K_{ij} = -n_{i;j} = -(-g^{00})^{-1/2}\Gamma^0{}_{ij} = -\frac{1}{2}g_{ij,0} = -\frac{1}{2}\frac{da^2}{dt}\begin{pmatrix}1 & 0 & 0\\ 0 & \sin^2\chi & 0\\ 0 & 0 & \sin^2\chi\sin^2\theta\end{pmatrix}. \tag{21.16.5}$$

So we get

$$K^i{}_j = -\frac{1}{a}\frac{da}{dt}\delta^i_j, \tag{21.16.6}$$

i.e.

$$\boldsymbol{K} = -a^{-1}(da/dt)\mathbf{1}. \tag{21.16.7}$$

From equation (21.77)

$$-G^0_0 = \frac{1}{2}R - \frac{1}{2}(\boldsymbol{n}\cdot\boldsymbol{n})^{-1}[(\mathrm{Tr}\boldsymbol{K})^2 - \mathrm{Tr}(\boldsymbol{K}^2)], \tag{21.16.8}$$

we have

$$8\pi\rho(a) = \frac{3}{a^2} + \frac{1}{2}\left[\left(\frac{3\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2\right], \tag{21.16.9}$$

i.e.

$$\frac{6}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{6}{a^2} = 16\pi\rho(a). \tag{21.16.10}$$

From energy conservation, we have $dE + pdV = 0$. For a comoving volume, $E = \rho a^3$ and $V = a^3$, and so

$$d(\rho a^3) + pd(a^3) = 0. \tag{21.16.11}$$

21.17 THIN-SANDWICH VARIATIONAL PRINCIPLE FOR THE SCALAR POTENTIAL IN ELECTRODYNAMICS

(a) Choose the unknown U^m in the expression

$$\frac{1}{8\pi}g^{mn}\frac{\partial\phi}{\partial x^m}\frac{\partial\phi}{\partial x^n} + U^m\frac{\partial\phi}{\partial x^n} \tag{21.17.1}$$

in such a way that this expression, multiplied by the volume element $g^{1/2}d^3x$, and integrated over the simultaneity Σ, is extremized by a ϕ, and only by a ϕ, that satisfies the initial-value equation (21.108) of electrodynamics.

(b) Show that the resulting variational principle, instead of having to be invented "out of the blue", is none other than what follows directly from the action principle build on the Lagrangian density (21.100) of electrodynamics (independent variation of ϕ and the three A_i everywhere between the two faces of a sandwich to extremize I, subject only to the prior specification of the A_i on the two faces of the sandwich, in the limit where the thickness of the sandwich goes to zero).

Solution:

(a)

$$\begin{aligned}
&\delta\int\left(\frac{1}{8\pi}g^{mn}\frac{\partial\phi}{\partial x^m}\frac{\partial\phi}{\partial x^n} + U^m\frac{\partial\phi}{\partial x^m}\right)g^{1/2}d^3x \\
&= \int\left(\frac{1}{4\pi}g^{mn}\frac{\partial\delta\phi}{\partial x^m}\frac{\partial\phi}{\partial x^n} + U^m\frac{\delta\partial\phi}{\partial x^n}\right)g^{1/2}d^3x \\
&= \int\frac{\partial}{\partial x^m}\left(g^{1/2}\frac{1}{4\pi}g^{mn}\frac{\partial\phi}{\partial x^n} + g^{1/2}U^m\right)\delta\phi d^3x \\
&= \int\left(\frac{1}{4\pi}g^{mn}\frac{\partial\phi}{\partial x^n} + U^m\right)_{;m} g^{1/2}\delta\phi d^3x.
\end{aligned} \tag{21.17.2}$$

So we have the equation

$$\left(\frac{1}{4\pi}A^{0,m} + U^m\right)_{;m} = 0. \tag{21.17.3}$$

If we choose $U^m = -A^{m,0}/4\pi$, we can arrive at $E^m{}_{;m} = 0$, which is the initial condition equation (21.108).

(b) A simplified initial value variational principle equivalent to the initial value equation is obtained at once from the more general variational principle already on hand by considering the thin sandwich limit: the ratio $I/\Delta x^0$ as $\Delta x^0 \to 0$, or as the range of hypersurfaces under consideration, is narrowed down. That certain terms drop out in the simplified action integral is seen at once. In electrodynamics, in the

thin-sandwich limit, the giving of "nearby" $\boldsymbol{B}'$ and $\boldsymbol{B}''$ at nearby t' and t'' (actually, the giving of $\boldsymbol{B}$ and $\partial\boldsymbol{B}/\partial t$ at one time), leaves no possibility of adjusting the term

$$\int \frac{\boldsymbol{B}^2}{8\pi} d^3x \tag{21.17.4}$$

in the action integral. Therefore we drop it. For the same reason $\boldsymbol{A}$ and $\dot{\boldsymbol{A}}$ are determinate up to a gauge, or up to $\nabla\lambda$ and $\nabla\dot{\lambda}$, respectively. The only quantity important in the variation is now the scalar potential ϕ. As

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \\
&= -\frac{1}{8\pi} A_{\nu,\mu}(A^{\nu,\mu} - A^{\mu,\nu}) \\
&= -\frac{1}{8\pi} A_{0,i}(A^{0,i} - A^{i,0}) + \frac{1}{8\pi} A_{i,0} A^{0,i} + \text{ terms without } \phi \\
&= \frac{1}{8\pi} g^{mn} \frac{\partial\phi}{\partial x^m} \frac{\partial\phi}{\partial x^n} - \frac{1}{4\pi} A^{m,0} \frac{\partial\phi}{\partial x^m} + \text{ terms without } \phi,
\end{aligned} \tag{21.17.5}$$

the variation principle for it becomes

$$I_3 = \int \left[\frac{1}{8\pi} g^{mn} \frac{\partial\phi}{\partial x^m} \frac{\partial\phi}{\partial x^n} - \frac{1}{4\pi} A^{m,0} \frac{\partial\phi}{\partial x^m} \right] g^{1/2} d^3x. \tag{21.17.6}$$

21.18 THIN-SANDWICH VARIATIONAL PRINCIPLE FOR THE LAPSE AND SHIFT FUNCTIONS IN GEOMETRODYNAMICS

(a) Extremize the action integral

$$I_3 = \int \{[R - (\mathrm{Tr}\boldsymbol{K})^2 + \mathrm{Tr}\boldsymbol{K}^2 - 2T^*_{nn}]N - 2T^{*k}_{n} N_k\} g^{1/2} d^3x \tag{21.18.1}$$

with respect to the lapse and shift functions, and show that one arrives in this way at the four initial-value equations of geometrodynamics. It is understood that one has given the six g_{ij} and the six $\partial g_{ij}/\partial t$ on the simultaneity where the analysis is being done. The extrinsic curvature is considered to be expressed as in (21.67) in terms of these quantities and the lapse and shift. The energy density and energy flow are referred to a unit normal vector $\boldsymbol{n}$ and three arbitrary coordinate basis vectors $\boldsymbol{e}_i$ within the simultaneity, as earlier in this chapter, and the asterisk is an abbreviation for an omitted factor of 8π.

(b) Derive this variational principle from the ADM variational principle by going to the limit of an infinitesimally thin sandwich [see derivation in Wheeler (1964)].

Solution:

(a) From (21.67), we have

$$K_{ij} = \frac{1}{2N} \left(N_{i|j} + N_{j|i} - \frac{\partial g_{ij}}{\partial t} \right) \tag{21.18.2}$$

and

$$\mathrm{Tr}\boldsymbol{K} = \frac{1}{N} \left(N^i_{\ |i} - \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial t} \right). \tag{21.18.3}$$

The variation with N gives

$$\delta \mathrm{Tr}\boldsymbol{K} = -\frac{\delta N}{N} \mathrm{Tr}\boldsymbol{K} \tag{21.18.4}$$

and

$$\delta \mathrm{Tr}\boldsymbol{K}^2 = -\frac{2\delta N}{N}\mathrm{Tr}\boldsymbol{K}^2. \tag{21.18.5}$$

So we have

$$\delta I_3 = \int \{[R-(\mathrm{Tr}\boldsymbol{K})^2+\mathrm{Tr}\boldsymbol{K}^2-2T^*_{nn}]\delta N - 2N\mathrm{Tr}\boldsymbol{K}\frac{-\delta N}{N}\mathrm{Tr}\boldsymbol{K}+N\frac{-2\delta N}{N}\mathrm{Tr}\boldsymbol{K}^2\}g^{1/2}d^3x = 0, \tag{21.18.6}$$

i.e.

$$R+(\mathrm{Tr}\boldsymbol{K})^2-\mathrm{Tr}\boldsymbol{K}^2 = 16\pi T_{nn}. \tag{21.18.7}$$

The variation with N_i gives

$$\delta \mathrm{Tr}\boldsymbol{K} = \frac{1}{N}\delta N^i{}_{|i} \tag{21.18.8}$$

and

$$\delta \mathrm{Tr}\boldsymbol{K}^2 = \delta(g^{il}g^{jk}K_{kl}K_{ij}) = 2K^{ij}\delta K_{ij} = \frac{2}{N}K^{ij}\delta N_{i|j}. \tag{21.18.9}$$

So we have

$$\begin{aligned}\delta I_3 &= \int \left(-2K^j{}_j\delta N^i{}_{|i}+2K^{ij}\delta N_{i|j}-2T^{*i}_n\delta N_i\right)g^{1/2}d^3x\\ &= \int \left[2\left(K^j{}_j\right)_{,i}\delta N^i-2K^{ij}{}_{|j}\delta N_i-2T^{*i}_n\delta N_i\right]g^{1/2}d^3x\\ &= 0,\end{aligned} \tag{21.18.10}$$

i.e.

$$\left(K^j{}_j\right)_{,i}-K^{ij}{}_{|j} = 8\pi T^i_n. \tag{21.18.11}$$

(b) Using Gauss-Codazzi threom,

$${}^4R = R+\epsilon[(\mathrm{Tr}\boldsymbol{K})^2-\mathrm{Tr}\boldsymbol{K}^2]+2\epsilon(n^\alpha{}_{;\beta}n^\beta-n^\alpha n^\beta{}_{;\beta})_{;\alpha} \tag{21.18.12}$$

the action of gravity can be written as

$$16\pi I_g = \int_{t'}^{t''}dt\int_{\Sigma_t}\left[R-(\mathrm{Tr}\boldsymbol{K})^2+\mathrm{Tr}\boldsymbol{K}^2\right]Ng^{1/2}d^3x, \tag{21.18.13}$$

if the hypersurface is closed or asymptoticly flat. In the thin-sandwich limit, the giving of "nearby" three geometry $\mathfrak{G}'$ and $\mathfrak{G}''$ at nearby $\Sigma_{t'}$ and $\Sigma_{t''}$, leaves no possibility of adjusting the g_{ij} in the action integral, up to a coordinate transformation. The only quantity important in the variation is now the scalar potential N and N_i. So the variation principle for it becomes

$$16\pi\frac{I_g}{\Delta t}\int\left[R-(\mathrm{Tr}\boldsymbol{K})^2+\mathrm{Tr}\boldsymbol{K}^2\right]Ng^{1/2}d^3x \tag{21.18.14}$$

with N and N_i. As for matter, we have

$$\frac{\delta I_M}{\Delta t} = \int_{\Sigma_t}\frac{1}{2}T^{\alpha\beta}\delta^{(4)}g_{\alpha\beta}Ng^{1/2}d^3x. \tag{21.18.15}$$

And the variation of N and N_i gives

$$\delta^{(4)}g_{00} = -2N\delta N+2N^i\delta N_i;\quad \delta^{(4)}g_{0i} = \delta^{(4)}g_{i0} = \delta N_i. \tag{21.18.16}$$

So we have

$$\frac{\delta I_M}{\Delta t} = \int_{\Sigma_t}\left[-T^{00}N^2\delta N+(T^{00}NN^i+T^{0i}N)\delta N_i\right]g^{1/2}d^3x. \tag{21.18.17}$$

The definations of T_{nn} and T_{ni} are

$$T_{nn}\equiv T_{\alpha\beta}n^\alpha n^\beta = T^{\alpha\beta}n_\alpha n_\beta = -N^2T^{00} \tag{21.18.18}$$

and

$$T_{ni} \equiv T_{\alpha\beta} n^\alpha e_i^\beta = T^\alpha{}_\beta \, n_\alpha e_i^\beta = -N T^0{}_i = -N({}^{(4)}g_{0i} T^{00} + {}^{(4)}g_{ij} T^{0j}) = -N(N_i T^{00} + g_{ij} T^{0j}). \quad (21.18.19)$$

And also we can get

$$T_n^i \equiv g^{ij} T_{nj} = -N N^i T^{00} - N T^{0i}. \quad (21.18.20)$$

Now we can write down

$$16\pi \frac{\delta I_M}{\Delta t} = \int_{\Sigma_t} \left(-2T^*_{nn} \delta N - 2T^{*i}_n \delta N_i\right) g^{1/2} d^3x. \quad (21.18.21)$$

Thus, in the sandwich limit, we can define

$$16\pi \frac{I_M}{\Delta t} \equiv \int_{\Sigma_t} \left(-2T^*_{nn} N - 2T^{*i}_n N_i\right) g^{1/2} d^3x \quad (21.18.22)$$

and the variation is with N and N_i. Put all the things together and we arrive at the variation principle

$$I_3 = \int \{[R - (\mathrm{Tr}\boldsymbol{K})^2 + \mathrm{Tr}\boldsymbol{K}^2 - 2T^*_{nn}]N - 2T^{*k}_n N_k\} g^{1/2} d^3x \quad (21.18.23)$$

with respect to the lapse and shift functions.

21.19 CONDENSED THIN-SANDWICH VARIATIONAL PRINCIPLE

(a) Extremize the action I_3 of the preceding exercise with respect to the lapse function N.

(b) What is the relation between the result and the principle that "3-geometry is a carrier of information about time"?

(c) By elimination of N, arrive at a "condensed thin-sandwich variational principle" in which the only quantities to be varied are the three shift functions N_i.

Solution:

(a) According to the previous exercise, extremizing the action I_3 with respect to the lapse function N would give

$$R + (\mathrm{Tr}\boldsymbol{K})^2 - \mathrm{Tr}\boldsymbol{K}^2 = 2T^*_{nn}. \quad (21.19.1)$$

(b) One can find out the separation of two hypersurfaces with parameter labels x^0 and $x^0 + dx^0$ if he knows the lapse function N. Moreover, he can find N if he has already managed to find out the shift function N^i by equation 21.19.1. In this case he has

$$(\text{ interval of proper time from hypersurface } \sigma_0 \text{ to hypersurface } \sigma_0 + d\sigma_0) = N dx^0. \quad (21.19.2)$$

(c) Define

$$\gamma_{ij} \equiv \frac{1}{2}\left(N_{i|j} + N_{j|i} - \frac{\partial g_{ij}}{\partial t}\right) = N K_{ij} \quad (21.19.3)$$

and

$$\gamma_2 \equiv (\mathrm{Tr}\boldsymbol{\gamma})^2 - \mathrm{Tr}\boldsymbol{\gamma}^2. \quad (21.19.4)$$

We have

$$N^2 R + \gamma_2 = 2N^2 \pi T^*_{nn}. \quad (21.19.5)$$

and so

$$N = \left[\frac{\gamma_2}{2T^*_{nn} - R}\right]^{1/2}. \quad (21.19.6)$$

Now we obtain the condensed thin-sandwich action principle

$$I_3 = -2\int \{[(2T^*_{nn} - R)\gamma_2]^{1/2} + T^{*k}_n N_k\} g^{1/2} d^3x \tag{21.19.7}$$

with respect to N_i.

21.20 POOR MAN'S WAY TO SCHWARZSCHILD GEOMETRY

On curved empty space evolving deterministically in time, impose the conditions (1) thatit possess a moment of time-symmetry, a spacelike hypersurface, the extrinsic curvature of which, with respect to the enveloping spacetime, is everywhere zero, and (2) that this spacelike hypersurface be endowed with spherical symmetry. Write the metric of the 3-geometry in the form

$$ds^2 = \psi^4(\bar{r})(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2). \tag{21.20.1}$$

From the initial-value equation (21.127), show that the conformal factor ψ; up to a multiplicative factor must have the form $\psi = (1 + m/2\bar{r})$. Show that the proper circumference $2\pi m\bar{r}\psi^2(\bar{r})$ assumes a minimum value at a certain value of $\bar{r}$, thus defining the throat of the 3-geometry. Show that the 3-geometry is mirror-symmetric with respect to reflection in this throat in the sense that the metric is unchanged in form under the substitution $r' = m^2/4\bar{r}$. Find the transformation from the conformal coordinate $\bar{r}$ to the Schwarzschild coordinate r.

Solution:

From equation (21.135) in the textbook, we have

$$R_2 = -8\psi^{-5}\nabla_1^2\psi + \psi^{-4}R_1 \tag{21.20.2}$$

and $R_1 = R_2 = 0$ in this equation. And so

$$\nabla_1^2\psi = 0. \tag{21.20.3}$$

In spherical coordinates,

$$\nabla_1^2 = \frac{1}{\bar{r}^2}\frac{\partial}{\partial\bar{r}}\left(\bar{r}^2\frac{\partial}{\partial\bar{r}}\right) + \frac{1}{\bar{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\bar{r}^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{21.20.4}$$

So

$$\frac{\partial}{\partial\bar{r}}\left(\bar{r}^2\frac{\partial\psi}{\partial\bar{r}}\right) = 0. \tag{21.20.5}$$

We have the solution

$$\psi(\bar{r}) = 1 + \frac{m}{2\bar{r}}. \tag{21.20.6}$$

The proper circumference is

$$2\pi m\bar{r}\psi^2(\bar{r}) = 2\pi\left(m + \bar{r} + \frac{m^2}{4\bar{r}}\right). \tag{21.20.7}$$

When $\bar{r} = m/2$, we have the minumum proper circumference $4\pi m$. If $r' = m^2/4\bar{r}$, we have

$$dr' = -\frac{m^2}{4\bar{r}^2}d\bar{r}. \tag{21.20.8}$$

So

$$\begin{aligned} ds^2 &= \left(1 + \frac{m}{2\bar{r}}\right)^4\left(\frac{16\bar{r}^4}{m^4}dr'^2 + \bar{r}^2 d\theta^2 + \bar{r}^2\sin^2\theta d\phi^2\right) \\ &= \left(1 + \frac{2r'}{m}\right)^4\left(\frac{m^4}{16r'^4}dr'^2 + \frac{m^4}{16r'^2}d\theta^2 + \frac{m^4}{16r'^2}\sin^2\theta d\phi^2\right) \\ &= \left(1 + \frac{m}{2r'}\right)^4\left(dr'^2 + r'^2 d\theta^2 + r'^2\sin^2\theta d\phi^2\right). \end{aligned} \tag{21.20.9}$$

Thus the 3-geometry is mirror-symmetric with respect to reflection in this throat in the sense that the metric is unchanged in form under the substitution $r' = m^2/4\bar{r}$.

The transformation from the conformal coordinate $\bar{r}$ to the Schwarzschild coordinate r is

$$r = m + \bar{r} + \frac{m^2}{4\bar{r}}. \tag{21.20.10}$$

21.21 WHY THE WEYL CONFORMAL CURVATURE TENSOR VANISHES

How many independent components does the Riemann curvature tensor have in three-dimensional space? How many does the Ricci curvature tensor have? Show that the two tensors are related by the formula

$$R^d{}_{abc} = \delta^d_b R_{ac} - \delta^d_c R_{ab} + g_{ac} R^d_b - g_{ab} R^d_c + \frac{1}{2} R(\delta^d_c g_{ab} - \delta^d_b g_{ac}) \tag{21.21.1}$$

with no need of any Weyl conformal-curvature tensor to specify (as in higher dimensions) the further details of the Riemann tensor. Show that the Weyl tensor, from an n-dimensional modification of equation (13.50) as in exercise 13.13, vanishes for $n = 2$.

Solution:

The Riemann curvature tensor in three-dimensional space has $3^2(3^2-1)/12 = 6$ independent components. The Ricci curvature tensor in three-dimensional space has $3(3+1)/2 = 6$ independent components. Define

$$C^d{}_{abc} \equiv R^d{}_{abc} - \left[\delta^d_b R_{ac} - \delta^d_c R_{ab} + g_{ac} R^d_b - g_{ab} R^d_c + \frac{1}{2} R(\delta^d_c g_{ab} - \delta^d_b g_{ac})\right]. \tag{21.21.2}$$

Take the trace of d and b of $C^d{}_{abc}$, we have

$$C^b{}_{abc} = R_{ac} - \left[3R_{ac} - R_{ac} + g_{ac} R - g_{ab} R^b_c + \frac{1}{2} R(g_{ac} - 3g_{ac})\right] = 0. \tag{21.21.3}$$

In addition to

$$C_{dabc} = C_{[da]bc} = C_{da[bc]}, \quad C_{d[abc]} = 0, \quad a, b, c, d = 1, 2, 3, \tag{21.21.4}$$

we can demonstrate that $C_{dabc} = 0$, i.e.

$$R^d{}_{abc} = \delta^d_b R_{ac} - \delta^d_c R_{ab} + g_{ac} R^d_b - g_{ab} R^d_c + \frac{1}{2} R(\delta^d_c g_{ab} - \delta^d_b g_{ac}) \tag{21.21.5}$$

The defination of Weyl tensor in 2-dimensional space is

$$C^d{}_{abc} \equiv R^d{}_{abc} - \frac{1}{2} R(\delta^d_c g_{ab} - \delta^d_b g_{ac}). \tag{21.21.6}$$

We have

$$C^{ab}{}_{ab} = 0, \quad C_{dabc} = C_{[da]bc} = C_{da[bc]}, \quad C_{d[abc]} = 0, \quad a, b, c, d = 1, 2. \tag{21.21.7}$$

We can demonstrate that $C_{dabc} = 0$.

21.22 YORK'S CURVATURE

(a) Define the tensor

$$R_{abc} = R_{ab|c} - R_{ac|b} + \frac{1}{4}(g_{ac}R_{|b} - g_{ab}R_{|c}). \tag{21.22.1}$$

(b) Show that a 3-geometry is conformally flat when and only when $R_{abc} = 0$.

(c) Show that the following identities hold and reduce to five the number of independent components of R_{abc}:

$$\begin{aligned} R^{a}{}_{ac} = g^{ab}R_{bac} &= 0; \\ R_{abc} + R_{acb} &= 0; \\ R_{abc} + R_{bca} + R_{cab} &= 0. \end{aligned} \tag{21.22.2}$$

(d) Show that Yorks' curvature

$$Y^{ab} = g^{1/3}[aef]\left(R^{b}_{f} - \frac{1}{4}\delta^{b}_{f}R\right)_{|e} = -\frac{1}{2}g^{1/3}[aef]g^{bm}R_{mef} \tag{21.22.3}$$

is conformally invariant and has the properties listed in equations (21.148).

Solution:

Please see Cotton tensor in wikipedia for a more general discussion in n dimensional space.

(b) Define $g_{ab} \equiv e^{2w}h_{ab}$. The connection coefficients are therefore

$$\Gamma^{c}{}_{ab} = \gamma^{c}{}_{ab} + (\delta^{c}_{a}w_{,b} + \delta^{c}_{b}w_{,a} - h_{ab}h^{ci}w_{,i}), \tag{21.22.4}$$

where $\gamma^{c}{}_{ab}$ is the connection coefficients with respect to h_{ab}. Then we can get

$$\Gamma^{i}{}_{ab,i} - \Gamma^{i}{}_{ai,b} = \gamma^{i}{}_{ab,i} - \gamma^{i}{}_{ai,b} - w_{,ab} - h_{ab,i}h^{ij}w_{,j} - h_{ab}h^{ij}{}_{,i}w_{,j} - h_{ab}h^{ij}w_{,ij}. \tag{21.22.5}$$

$$\begin{aligned} \Gamma^{i}{}_{ji}\Gamma^{j}{}_{ab} - \Gamma^{i}{}_{jb}\Gamma^{j}{}_{ai} &= \gamma^{i}{}_{ji}\gamma^{j}{}_{ab} - \gamma^{i}{}_{jb}\gamma^{j}{}_{ai} \\ &\quad + \gamma^{i}{}_{ab}w_{,i} - (h_{ab}\gamma^{i}{}_{ji} - h_{ai}\gamma^{i}{}_{jb} - h_{ib}\gamma^{i}{}_{aj})h^{jk}w_{,k} \\ &\quad + w_{,a}w_{,b} - h_{ab}h^{ij}w_{,i}w_{,j}. \end{aligned} \tag{21.22.6}$$

Notice that

$$h_{ab,j} = h_{ai}\gamma^{i}{}_{bj} + h_{ib}\gamma^{i}{}_{aj}; \quad h^{ij}{}_{,i} = -h^{ik}\gamma^{j}{}_{ki} - h^{kj}\gamma^{i}{}_{ki}. \tag{21.22.7}$$

Thus the Ricci tensor is

$$\begin{aligned} R_{ab} &= r_{ab} + h_{ab}h^{ij}\gamma^{k}{}_{ij}w_{,k} + \gamma^{i}{}_{ab}w_{,i} - w_{,ab} - h_{ab}h^{ij}w_{,ij} + w_{,a}w_{,b} - h_{ab}h^{ij}w_{,i}w_{,j} \\ &= r_{ab} - w_{;ab} - h_{ab}h^{ij}w_{;ij} + w_{,a}w_{,b} - h_{ab}h^{ij}w_{,i}w_{,j}, \end{aligned} \tag{21.22.8}$$

where ";" is the covariant derivatives with respect to h_{ab}, and r_{ab} is the Ricci tensor of h_{ab}. The Ricci scalar is

$$e^{2w}R = r - 4h^{ij}w_{;ij} - 2h^{ij}w_{,i}w_{,j}. \tag{21.22.9}$$

Define

$$X_{ab} \equiv R_{ab} - \frac{1}{4}g_{ab}R; \quad x_{ab} \equiv r_{ab} - \frac{1}{4}h_{ab}r. \tag{21.22.10}$$

We have

$$X_{ab} = x_{ab} - w_{;ab} + w_{,a}w_{,b} - \frac{1}{2}h_{ab}h^{ij}w_{,i}w_{,j}. \tag{21.22.11}$$

Define

$$\beta^{c}{}_{ab} \equiv \delta^{c}_{a}w_{,b} + \delta^{c}_{b}w_{,a} - h_{ab}h^{ci}w_{,i}. \tag{21.22.12}$$

We have

$$R_{abc} = X_{ab|c} - X_{ac|b} = X_{ab,c} - \Gamma^i{}_{ac} X_{ib} - X_{ac,b} + \Gamma^i{}_{ab} X_{ic} = X_{ab;c} - \beta^i{}_{ac} X_{ib} - X_{ac;b} + \beta^i{}_{ab} X_{ic}. \quad (21.22.13)$$

We can obtain

$$X_{ab;c} = x_{ab;c} - w_{;abc} + w_{;ac} w_{,b} + w_{,a} w_{;bc} - \frac{1}{2} h_{ab} W_{,c}. \quad (21.22.14)$$

where $W \equiv \delta^{ij} w_{,i} w_{,j}$. And also

$$\begin{aligned} -\beta^i{}_{ac} X_{ib} &= (\delta^i_a w_{,c} + \delta^i_c w_{,a} - h_{ac} h^{ij} w_{,j}) \left[-x_{ib} + w_{;ib} - w_{,i} w_{,b} + \frac{1}{2} h_{ib} W \right] \\ &= -x_{ab} w_{,c} - x_{bc} w_{,a} + h_{ac} x_{bi} h^{ij} w_{,j} \\ &= w_{,c} w_{,ab} - 2 w_{,a} w_{,b} w_{,c} + \frac{1}{2} \delta_{ab} w_{,c} W \\ &+ w_{,a} w_{,bc} + \frac{1}{2} \delta_{bc} w_{,a} W - \frac{1}{2} \delta_{ac} W_{,b} + \frac{1}{2} \delta_{ac} w_{,b} W. \end{aligned} \quad (21.22.15)$$

Now we have

$$\begin{aligned} R_{abc} &= r_{abc} - w_{;abc} + w_{;acb} - x_{ab} w_{,c} + h_{ac} x_{bi} h^{ij} w_{,j} + x_{ac} w_{,b} - h_{ab} x_{ci} h^{ij} w_{,j} \\ &= r_{abc} - r^i{}_{abc} w_{,i} - r_{ab} w_{,c} + h_{ac} r_{bi} h^{ij} w_{,j} + r_{ac} w_{,b} - h_{ab} r_{ci} h^{ij} w_{,j} + \frac{1}{2} r (h_{ab} w_{,c} - h_{ac} w_{,b}) \\ &= r_{abc} - r^i{}_{abc} w_{,i} + \left[-r_{ab} \delta^i_c + h_{ac} r^i_b + r_{ac} \delta^i_b - h_{ab} r^i_c + \frac{1}{2} r (h_{ab} \delta^i_c - h_{ac} \delta^i_b) \right] w_{,i} \end{aligned} \quad (21.22.16)$$

In 3-dimensional space, we have

$$r^i{}_{abc} = r_{ac} \delta^i_b - r_{ab} \delta^i_c + h_{ac} r^i_b - h_{ab} r^i_c + \frac{1}{2} r (h_{ab} \delta^i_c - h_{ac} \delta^i_b). \quad (21.22.17)$$

Thus

$$R_{abc} = r_{abc}, \quad (21.22.18)$$

i.e. R_{abc} is invariant under conformal transformation. Therefore, a conformally flat 3-geometry must have $R_{abc} = 0$.

To prove a 3-geometry is conformally flat. We must find a scalar field w that

$$R_{ab} = r_{ab} - w_{;ab} - h_{ab} h^{ij} w_{;ij} + w_{,a} w_{,b} - h_{ab} h^{ij} w_{,i} w_{,j} = 0. \quad (21.22.19)$$

The equation is equivalent to

$$r_{ab} - w_{;ab} - \frac{1}{4} h_{ab} (r - 2 h^{ij} w_{,i} w_{,j}) + w_{,a} w_{,b} - h_{ab} h^{ij} w_{,i} w_{,j} = 0, \quad (21.22.20)$$

i.e.

$$w_{;ab} - w_{,a} w_{,b} + \frac{1}{2} h_{ab} h^{ij} w_{,i} w_{,j} = x_{ab}. \quad (21.22.21)$$

Recall the Frobenius theorem. Let $n \geq 2$, $N \geq 1$, $x \in \mathbb{R}^n$. $u^\alpha(x)$, $1 \leq \alpha \leq N$ are unknown. Given smooth functions $F^\alpha_k(x, u)$, consider system

$$u^\alpha{}_{,k} = F^\alpha_k(x, u), \quad 1 \leq k \leq n, \quad 1 \leq \alpha \leq N. \quad (21.22.22)$$

Integrability condition comes from $u^\alpha{}_{,kl} = u^\alpha{}_{,lk}$. Chain rule gives $u^\alpha{}_{,kl} = F^\alpha_{k,l} + F^\alpha_{k,\beta} u^\beta{}_{,l}$. So the required integrability condition is

$$F^\alpha_{k,l} + F^\beta_l F^\alpha_{k,\beta} = F^\alpha_{l,k} + F^\beta_k F^\alpha_{l,\beta}. \quad (21.22.23)$$

If this condition holds identically in (x, u), then the overdetermined system has a solution $u^\alpha(x)$, and $u^\alpha(x_0)$ can be prescribed arbitrarily.

Apply to $w_{;ab} - w_{,a}w_{,b} + \frac{1}{2}h_{ab}h^{ij}w_{,i}w_{,j} = x_{ab}$. Write as

$$w_{,ab} = \gamma^{i}_{\ ab}w_{,i} + w_{,a}w_{,b} - \frac{1}{2}h_{ab}h^{ij}w_{,i}w_{,j} + x_{ab}. \tag{21.22.24}$$

We try to find n functions u_a so that

$$u_{a,b} = \gamma^{i}_{\ ab}u_i + u_a u_b - \frac{1}{2}h_{ab}h^{ij}u_i u_j + x_{ab} \equiv F_{ab}(x,u). \tag{21.22.25}$$

If we have u_a, then certainly $u_{a,b} = u_{b,a}$, so Poincare Lemma implies $u_a = w_{,a}$ for some w, and we are done.

Now use Frobenius with $N = n = 3$ and check integrability. Equation is same as

$$u_{a;b} = u_a u_b - \frac{1}{2}h_{ab}h^{ij}u_i u_j + x_{ab} \tag{21.22.26}$$

We can calculate integrability condition commuting $\nabla_c\nabla_b$ instead of $\partial_c\partial_b$, using $u_{a;bc} - u_{a;cb} = r^{i}_{\ abc}u_i$. Directly,

$$u_{a;bc} = u_{a;c}u_b + u_a u_{b;c} - h_{ab}h^{ij}u_{i;c}u_j + x_{ab;c} \tag{21.22.27}$$

So

$$\begin{aligned}
u_{a;bc} - u_{a;cb} &= u_{a;c}u_b - u_{a;b}u_c + (h_{ac}u_{i;b} - h_{ab}u_{i;c})u^i + r_{abc} \\
&= r_{abc} + \left(u_a u_c - \frac{1}{2}h_{ac}u^2 + x_{ac}\right)u_b - \left(u_a u_b - \frac{1}{2}h_{ab}u^2 + x_{ab}\right)u_c \\
&+ \left[h_{ac}\left(u_i u_b - \frac{1}{2}h_{ib}u^2 + x_{ib}\right) - h_{ab}\left(u_i u_c - \frac{1}{2}h_{ic}u^2 + x_{ic}\right)\right]u^i \\
&= r_{abc} + x_{ac}u_b - x_{ab}u_c + (h_{ac}x_{ib} - h_{ab}x_{ic})u^i \\
&= r_{abc} + \left[r_{ac}\delta^i_b - r_{ab}\delta^i_c + h_{ac}r^i_b - h_{ab}r^i_c + \frac{1}{2}r(h_{ab}\delta^i_c - h_{ac}\delta^i_b)\right]u_i \\
&= r_{abc} + r^{i}_{\ abc}u_i.
\end{aligned} \tag{21.22.28}$$

If $r_{abc} = 0$, we can get $u_{a;bc} - u_{a;cb} = r^{i}_{\ abc}u_i$. And so the differential equations are integrable. Thus the 3-geometry is conformally flat.

(c) As $G_{ab}^{\ \ |b} = 0$, we have

$$R_{ab}^{\ \ |b} = \frac{1}{2}R_{|a}. \tag{21.22.29}$$

So

$$R^{a}_{\ ac} = R_{|c} - R_{ac}^{\ \ |a} + \frac{1}{4}(R_{|c} - 3R_{|c}) = 0. \tag{21.22.30}$$

It is easy to see that $R_{abc} + R_{acb} = 0$ from the defination of R_{abc}. And

$$R_{abc} + R_{bca} + R_{cab} = X_{ab|c} - X_{ac|b} + X_{bc|a} - X_{ba|c} + X_{ca|b} - X_{cb|a} = 0. \tag{21.22.31}$$

The independent components of R_{abc} are

$$3^3 - 3 - 3 \times 6 - 1 = 5. \tag{21.22.32}$$

(d)Firstly, if $a = b$, then $Y^{ab} = Y^{ba}$ holds trivally. If $a \neq b$, say $a = 1$ and $b = 2$, we have

$$Y^{12} = -\frac{1}{2}g^{1/3}[1ef]g^{2m}R_{mef} = -g^{1/3}R^{2}_{\ 23} = g^{1/3}(R^{1}_{\ 13} + R^{3}_{\ 33}) = -g^{1/3}R^{1}_{\ 31} \tag{21.22.33}$$

and

$$Y^{21} = -\frac{1}{2}g^{1/3}[2ef]g^{1m}R_{mef} = -g^{1/3}R^{1}_{\ 31}. \tag{21.22.34}$$

So $Y^{12} = Y^{31}$. Similarly, we can verify that $Y^{13} = Y^{31}$ and $Y^{23} = Y^{32}$ respectively.

Secondly, we have

$$g_{ab}Y^{ab} = -\frac{1}{2}g^{1/3}g_{ab}[aef]g^{bm}R_{mef} = -\frac{1}{2}g^{1/3}[aef]R_{aef} = 0. \tag{21.22.35}$$

Thirdly, as R_{abc} and $g^{1/3}g^{bm}$ is conformally invariant, Y^{ab} must be conformally invariant.

Fourthly, as

$$-2g^{-1/3}[aij]Y^a_b = [aij][aef]R_{bef} = 2R_{bij}, \tag{21.22.36}$$

Y^{ab} and R_{abc} are in one-to-one relation. Since a 3-geometry is conformally flat when and only when $R_{abc} = 0$, a 3-geometry is conformally flat when and only when $Y^{ab} = 0$.

I do not know the exact defination of covariant derivative here since Y^{ab} is not a tensor. From now, I will assume that $Y^{ab} = \epsilon^{aef}(R^b_f - \delta^b_f R/4)_{|e}$. We have

$$\begin{aligned}
Y^{ab}{}_{|b} &= \epsilon^{aef}\left(R^b{}_{f|eb} - \frac{1}{4}R_{|ef}\right) \\
&= \epsilon^{aef}\left(R^b{}_{f|be} + R^b{}_{ibe}R^i_f - R^i{}_{fbe}R^b_i\right) \\
&= \epsilon^{aef}\left[\frac{1}{2}R_{|fe} + R_{ib}R^i_f - R^i{}_{fbe}R^b_i\right] \\
&= -\epsilon^{aef}\left[\delta^i_b R_{fe} - \delta^i_e R_{bf} + g_{fe}R^i_b - g_{bf}R^i_e + \frac{1}{2}R(g_{bf}\delta^i_e - g_{fe}\delta^i_b)\right]R^b_i \\
&= -\epsilon^{aef}\left[RR_{fe} - R^b_e R_{bf} + g_{fe}R^i_b R^b_i - R^i_e R_{if} + \frac{1}{2}R(R_{ef} - g_{fe}R)\right] \\
&= 0,
\end{aligned} \tag{21.22.37}$$

i.e. $Y^{ab}{}_{|b} = 0$.

21.23 PULLING THE POYNTING FLUX VECTOR "OUT OF THE AIR"

From the condition that the Hamilton-Jacobi functional $S(g_{ij}, A_m)$ (extremal of the action integral) for the combined Einstein and Maxwell fields, ostensibly dependent on the six metric coefficients $g_{ij}(x, y, z)$ and the three potentials $A_m(x, y, z)$, shall actually depend only on the 3-geometry of the spacelike hypersurface and the distribution of magnetic field strength on this hypersurface, show that the geometrodynamic field momentum $\pi^{ij} = \delta S/\delta g_{ij}$ satisfies a condition of the form

$$\pi^{ij}{}_{|j} = c[imn]\mathscr{E}_m\mathscr{B}_n, \tag{21.23.1}$$

and evaluate the coefficient c in this equation [Wheeler (1968b)].

Solution:

Note that the transformation

$$x^i \to x^i - \xi^i, \quad g_{ij} \to g_{ij} + \xi_{i|j} + \xi_{j|i} \tag{21.23.2}$$

in no way changes the 3-geometry itself, and therefore the corresponding induced change in S,

$$\delta S = \int\left[\frac{\delta S}{\delta g_{ij}}\delta g_{ij} + \frac{\delta S}{\delta A_m}\delta A_m\right]d^3x \tag{21.23.3}$$

must vanish identically for arbitrary choice of the $\xi^i(x,y,z)$, which measure the equivalent of the sliding of a ruled transparent rubber sheet over an automobile fender. Then we have

$$\begin{aligned}
\delta S &= \int \left[\pi^{ij}(\xi_{i|j} + \xi_{j|i}) - \frac{\mathscr{E}^j}{4\pi}(A_{j|i}\xi^i + \xi^i{}_{|j} A_i) \right] d^3x \\
&= \int \left[-2\pi^{ij}{}_{|j}\xi_i - \frac{\mathscr{E}^j}{4\pi}(A_{j|i} - A_{i|j})\xi^i + \frac{\mathscr{E}^j{}_{|j}}{4\pi} A_i \xi^i \right] d^3x \\
&= \int \left[-2\pi^{ij}{}_{|j}\xi_i - \frac{\mathscr{E}^j}{4\pi}[ijk]\mathscr{B}^k \xi^i \right] d^3x \\
&= \int \left[-2\pi^{ij}{}_{|j}\xi_i - g^{1/2}\frac{E^j}{4\pi}\epsilon_{ijk} B^k \xi^i \right] d^3x \\
&= -\frac{1}{4\pi}\int (8\pi\pi^{ij}{}_{|j} + g^{1/2}\epsilon^{ijk} E_j B_k)\xi^i d^3x \\
&= 0.
\end{aligned} \tag{21.23.4}$$

So we have

$$\pi^{ij}{}_{|j} = -\frac{1}{8\pi}[ijk] E_j B_k. \tag{21.23.5}$$

21.24 THE EXTREMAL ACTION ASSOCIATED WITH THE HILBERT ACTION PRINCIPLE DEPENDS ON CONFORMAL 3-GEOMETRY AND EXTRINSIC TIME

Show that the data demanded by the Hilbert action principle $\delta \int {}^{(4)}R(-{}^{(4)}g)^{1/2} d^4x = 0$ on each of the two bounding space like hypersurfaces consist of (1) the conformal 3-geometry ${}^{(3)}<$ of the hypersurface plus (2) the extrinsic time variable defined by

$$\tau = \frac{2}{3} g^{-1/2}\pi = \frac{4}{3} K, \tag{21.24.1}$$

conveniently represented by the pictogram $\frown$, measured by one number per spacepoint, and independent of the conformal factor in the metric of the 3-geometry. This done, explain in a few words why in this formulation of geometrodynamics the Hamilton-Jacobi function is appropriately expressed in the form

$$S = S({}^{(3)}<, \frown). \tag{21.24.2}$$

Solution:

The answer is based on *Role of Conformal Three-Geometry in the Dynamics of Gravitation* by *James W. York*(1972).

The invariant Lagrange density $\mathcal{L} = {}^{(4)}R(-{}^{(4)}g)^{1/2}$ can be decomposed relative to a family of spacelike hypersurfaces with unit normal field u^λ into the form

$$\mathcal{L} = N g^{1/2}(R + K_{ab}K^{ab} - K^2) - 2\partial_\lambda N g^{1/2}(K u^\lambda + a^\lambda). \tag{21.24.3}$$

The first term is the standard Lagrangian of geometrodynamics and will be denoted $\mathcal{L}_G$. The second term is a pure space-time divergence which may be transformed in the action principle to a boundary term. The boundary integral involving a^λ vanishes identically since $a^\lambda u_\lambda = 0$, which shows that the dynamics of gravitation is completely independent of the acceleration of the observers. In terms of ADM variables,

$$-2\partial_\lambda N g^{1/2}(K u^\lambda + a^\lambda) = \partial_a \pi N^a - \partial_0 \pi. \tag{21.24.4}$$

The spatial divergence can be discarded, giving for the action integral

$$S = \int_\Omega \mathcal{L} d^4x = S_G - \int_{\partial\Omega} \pi d^3x, \tag{21.24.5}$$

where $S_G = \int \mathcal{L}_G d^4x$. We may readily evaluate the action S for dynamical paths, i.e., for solutions of ${}^{(4)}R_{\mu\nu}$. Since for these paths $\mathcal{L} = 0$, we find

$$S_G = \int_{\partial\Omega} \pi d^3x. \tag{21.24.6}$$

If we choose the bounding spacelike hypersurfaces as $\tau_1 = \text{const}$ and $\tau_2 = \text{const}$, we obtain

$$S_G = \frac{3}{2}(\tau_1 V_1 - \tau_2 V_2), \tag{21.24.7}$$

where V is the total volume. For τ and the conformal geometry given, the initial value equations determine the scale. Thus we arrive at the conclusion that, as a functional,

$$S_G = S_G({}^{(3)}<, \frown) \tag{21.24.8}$$

21.25 EQUATION OF MOTION FOR A SURFACE LAYER

(a) Let $\boldsymbol{u}$ be the "mean 4-velocity" of the matter in a surface layer - so defined that an observer moving with 4-velocity $\boldsymbol{u}$ sees zero energy flux. Let σ be the total mass-energy per unit proper surface area, as measured by such a "comoving observer." Show that the surface stress-energy tensor can be expressed in the form

$$\boldsymbol{S} = \sigma \boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{t}, \text{ where } \boldsymbol{t} \cdot \boldsymbol{u} = 0, \tag{21.25.1}$$

and where $\boldsymbol{t}$ is a symmetric stress tensor.

(b) Show that the component along $\boldsymbol{u}$ of the equation of motion (21.170) is

$$d\sigma/d\tau = -\sigma u^j{}_{|j} + u_j t^{ik}{}_{|k} + u_j [T^{jn}], \tag{21.25.2}$$

where $d/d\tau = \boldsymbol{u}$. Give a physical interpretation for each term.

(c) Let a_j be that part of the 4-acceleration of the comoving observer which lies in the surface layer Σ. By projecting the equation of motion (21.170) perpendicular to $\boldsymbol{u}$, show that

$$\sigma a_j = -P_{ja}\{t^{ab}{}_{|b} + [T^{an}]\}, \tag{21.25.3}$$

where P_{ja} is the projection operator

$$P_{ja} = g_{ja} + u_j u_a. \tag{21.25.4}$$

Give a physical interpretation for each term of equation 21.25.3.

Solution:

(a) Since an observer moving with 4-velocity $\boldsymbol{u}$ sees zero energy flux, we have

$$\boldsymbol{S}(\boldsymbol{u}, \boldsymbol{e}) = 0 \tag{21.25.5}$$

for any $\boldsymbol{e}$ perpendicular to $\boldsymbol{u}$. So $\boldsymbol{S}$ must be of the form $\boldsymbol{S} = S_{uu}\boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{t}$ where $\boldsymbol{t} \cdot \boldsymbol{u} = 0$. As the total mass-energy per unit proper surface area measured by comoving observer is σ, we have

$$\boldsymbol{S}(\boldsymbol{u}, \boldsymbol{u}) = \sigma, \tag{21.25.6}$$

i.e.

$$\boldsymbol{S} = \sigma \boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{t}, \text{ where } \boldsymbol{t} \cdot \boldsymbol{u} = 0, \tag{21.25.7}$$

and where $\boldsymbol{t}$ is a symmetric stress tensor.

(b) As $S^{im} = \sigma u^i u^m + t^{im}$, we have

$$\sigma_{,m} u^m u^i + \sigma u^i{}_{|m} u^m + \sigma u^i u^m{}_{|m} + t^{im}{}_{|m} + [T^{in}] = 0 \tag{21.25.8}$$

from equation (21.170). Multiply the both sides of the equation above by u_i and we can get

$$-\sigma_{,m} u^m - \sigma u^m{}_{|m} + u_i t^{im}{}_{|m} + u_i [T^{in}] = 0, \tag{21.25.9}$$

i.e.

$$d\sigma/d\tau = -\sigma u^j{}_{|j} + u_j t^{ik}{}_{|k} + u_j [T^{jn}]. \tag{21.25.10}$$

(c) Multiply the both sides of the equation 21.25.8 by P_{ji} and we can get

$$\sigma a_i + P_{ji}(t^{im}{}_{|m} + [T^{in}]) = 0, \tag{21.25.11}$$

i.e.

$$\sigma a_j = -P_{ja}\{t^{ab}{}_{|b} + [T^{an}]\}. \tag{21.25.12}$$

21.26 THIN SHELLS OF DUST

For a thin shell of dust surrounded by vacuum ($[T^{in}] = 0, \boldsymbol{t} = 0$), derive the following equations

$$d\sigma/d\tau = -\sigma u^b{}_{|b}, \tag{21.26.1a}$$

$$\boldsymbol{a}^+ + \boldsymbol{a}^- = 0, \tag{21.26.1b}$$

$$\boldsymbol{a}^+ - \boldsymbol{a}^- = 4\pi\sigma \boldsymbol{n}, \tag{21.26.1c}$$

$$\boldsymbol{\gamma} = 8\pi\sigma \left(\boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{2}\boldsymbol{g} \right). \tag{21.26.1d}$$

Here $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$ are the 4-accelerations as measured by accelerometers that are fastened onto the outer and inner sides of the shell, and $\boldsymbol{g}$ is the 3-metric of the shell. Show that the first of these equations is the law of "conservation of rest mass".

Solution:

In equation 21.25.10, let $[T^{in}] = 0, \boldsymbol{t} = 0$, we can get $d\sigma/d\tau = -\sigma u^b{}_{|b}$. From equation (21.68) in the textbook, note $S^{ij} = \sigma u^i u^j$, we have

$$\gamma^i_j - \delta^i_j \gamma = 8\pi\sigma u^i u_j. \tag{21.26.2}$$

Take the trace and we can get $-2\gamma = 8\pi\sigma u^i u_i$. So we have

$$\gamma^i_j = 8\pi\sigma u^i u_j - 4\pi\sigma \delta^i_j u^k u_k. \tag{21.26.3}$$

As $\boldsymbol{S} \cdot \boldsymbol{n} = 0$, we have $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, i.e. $u^n = 0$, and so $u^k u_k = u^\alpha u_\alpha = -1$. Now we obtain

$$\boldsymbol{\gamma} = 8\pi\sigma \left(\boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{2}\boldsymbol{g} \right). \tag{21.26.4}$$

I do not know the precise defination of $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$ in the exercise. So I cannot give the proof for two equations involving $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$.

21.27 SPHERICAL SHELL OF DUST

Apply the formalism of exercise 21.25 to a collapsing spherical shell of dust [Israel (1967b)]. For the metric inside and outside the shell, take the flat-spacetime and vacuum Schwarzschild expressions (Chapter 23),

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \text{ inside,} \tag{21.27.1a}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \text{ outside.} \tag{21.27.1b}$$

Let the "radius" of the shell, as a function of proper time measured on the shell, be

$$R \equiv \frac{1}{2\pi} \times \text{ (proper circumference of shell) } = R(\tau). \tag{21.27.2}$$

Show that the shell's mass density varies with time as

$$\sigma(\tau) = \mu/4\pi R^2(\tau), \quad \mu = \text{ constant } = \text{ "total rest mass" ;} \tag{21.27.3}$$

and derive and solve the equation of motion

$$M = \mu\left[1 + \left(\frac{dR}{d\tau}\right)^2\right]^{1/2} - \frac{\mu^2}{2R}. \tag{21.27.4}$$

Solution:

We set up the Gaussian normal coordinates (τ, n) so that $n = n_0$ defines the thin shell of dust. In the region $n > n_0$, we have Schwarzschild metric. In the region $n < n_0$, we have Minkowski metric. For a spherical shell of dust, we have $u^n = u^\theta = u^\phi = 0$. And form equation 21.25.12 and $\boldsymbol{t} = 0$, $[T^{in}] = 0$, we know the acceleration of dust is zero. We can rescale the τ so that it is the proper time of dust on the shell. Now the metric on the shell is

$$ds^2 = -d\tau^2 + r^2(\tau, n_0)(d\theta^2 + \sin^2\theta d\phi^2) \tag{21.27.5}$$

and we have $R(\tau) = r(\tau, n_0)$, $u^\tau = 1$. Note that

$$u^i{}_{|i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}u^i)_{,i} = \frac{1}{R^2}\frac{dR^2}{d\tau}. \tag{21.27.6}$$

We have

$$\frac{d\sigma}{d\tau} = -\sigma\frac{1}{R^2}\frac{dR^2}{d\tau}. \tag{21.27.7}$$

After integration, we can get

$$\sigma(\tau) = \mu/4\pi R^2(\tau), \quad \mu = \text{ constant .} \tag{21.27.8}$$

Note that $\gamma_{ij} = [K_{ij}] = -[g_{ij,n}]/2$, we have

$$[g_{ij,n}] = -8\pi\sigma(2u_iu_j + g_{ij}) \tag{21.27.9}$$

from equation 21.26.4. Particularly, for $g_{\theta\theta} = r^2$, we can get

$$\left[\frac{\partial r}{\partial n}\right] = -4\pi R\sigma = -\frac{\mu}{R}. \tag{21.27.10}$$

In the vicinity of the shell, the matric in Gaussian normal coordinates are

$$ds^2 = -d\tau^2 + dn^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{21.27.11}$$

Comparing it with

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{21.27.12}$$

we can get

$$\frac{\partial r}{\partial n} = \left[f + \left(\frac{\partial r}{\partial \tau} \right)^2 \right]^{1/2} \tag{21.27.13}$$

at $r = R$, i.e

$$\left.\frac{\partial r}{\partial n}\right|_+ = \left[1 - \frac{2M}{R} + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2}; \quad \left.\frac{\partial r}{\partial n}\right|_- = \left[1 + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2}. \tag{21.27.14}$$

Substitute 21.27.14 into 21.27.10, we can get

$$M = \mu \left[1 + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2} - \frac{\mu^2}{2R}. \tag{21.27.15}$$

The solution of the equation 21.27.15 can be found in TABLE I of the paper *Gravitational Collapse and Causality* by *W. Israel (1967)*.

Chapter 22

THERMODYNAMICS, HYDRODYNAMICS, ELECTRODYNAMICS, GEOMETRIC OPTICS, AND KINETIC THEORY

22.1 DIVERGENCE OF FLOW LINES PRODUCES VOLUME CHANGES

The detailed solution of the exercise has been provided in "Hint" attached to exercise 22.1 in the textbook. I will not repeat it here.

22.2 EQUATION OF CONTINUITY

Show that in the nonrelativistic limit in flat spacetime the equation of baryon conservation (22.3) becomes the "equation of continuity"

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^j}(nv^j) = 0. \tag{22.2.1}$$

Solution:

In flat spacetime, we have

$$0 = (nu^\alpha)_{;\alpha} = \frac{\partial nu^0}{\partial t} + \frac{\partial nu^0 v^j}{\partial x^j} = u^0\left[\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^j}(nv^j)\right] + n\left[\frac{\partial u^0}{\partial t} + v^j\frac{\partial u^0}{\partial x^j}\right]. \tag{22.2.2}$$

In the nonrelativistic limit, we have

$$\frac{\partial u^0}{\partial t} + v^j\frac{\partial u^0}{\partial x^j} = \frac{du^0}{dt} \sim v^2. \tag{22.2.3}$$

Finally, we can get

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x^j}(nv^j) = 0. \tag{22.2.4}$$

22.3 CHEMICAL POTENTIAL FOR IDEAL FERMI GAS

Show that the chemical potential of an ideal Fermi gas, nonrelativistic or relativistic, is (at zero temperature) equal to the Fermi energy (energy of highest occupied momentum state) of that gas.

Solution:

For ideal fermi gas, the number density of particles in state $|i\rangle$ is

$$n_i = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{k_B T}\right) + 1}. \tag{22.3.1}$$

In the scenario $T = 0$, if $\epsilon_i < \mu$, we have $n_i = 1$. If $\epsilon_i > \mu$, we have $n_i = 0$. So the chemical potential of an ideal Fermi gas, nonrelativistic or relativistic, is (at zero temperature) equal to the Fermi energy (energy of highest occupied momentum state) of that gas.

22.4 PROJECTION TENSORS

Show that contraction of a tangent vector $\boldsymbol{B}$ with the "projection tensor" $\boldsymbol{P} = \boldsymbol{g} + \boldsymbol{u} \times \boldsymbol{u}$ projects $\boldsymbol{B}$ into the 3-surface orthogonal to the 4-velocity vector $\boldsymbol{u}$. If $\boldsymbol{n}$ is a unit spacelike vector, show that $\boldsymbol{P} \equiv \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{n}$ is the corresponding projection operator. Note: There is no unique concept of "the projection orthogonal to a null vector". Why?

Solution:

Suppose $\boldsymbol{A} = \boldsymbol{P} \cdot \boldsymbol{B}$. We have

$$A^\alpha = g^{\alpha\beta} B_\beta + u^\alpha u^\beta B_\beta, \tag{22.4.1}$$

and so

$$A^\alpha u_\alpha = u_\alpha g^{\alpha\beta} B_\beta + u_\alpha u^\alpha u^\beta B_\beta = u^\beta B_\beta - u^\beta B_\beta = 0. \tag{22.4.2}$$

Similarly, for $\boldsymbol{n}$ spacelike and $\boldsymbol{P} \equiv \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{n}$, we have

$$A^\alpha = g^{\alpha\beta} B_\beta - n^\alpha n^\beta B_\beta, \tag{22.4.3}$$

and so

$$A^\alpha n_\alpha = n_\alpha g^{\alpha\beta} B_\beta - n_\alpha n^\alpha u^\beta B_\beta = n^\beta B_\beta - n^\beta B_\beta = 0. \tag{22.4.4}$$

Because a null vector is orthogonal to itself. So there is no unique concept of "the projection orthogonal to a null vector".

22.5 PRESSURE GRADIENT IN STATIONARY GRAVITATIONAL FIELD

A perfect fluid is at rest (flow lines have x^j = constant) in a stationary gravitational field (metric coefficients are independent of x^0). Show that the pressure gradient required to "support the fluid against gravity" (i.e., to make its flow lines be x^i = constant instead of geodesics) is

$$\frac{\partial p}{\partial x^0} = 0, \quad \frac{\partial p}{\partial x^j} = -(\rho + p) \frac{\partial \ln \sqrt{-g_{00}}}{\partial x^j}. \tag{22.5.1}$$

Evaluate this pressure gradient in the Newtonian limit, using the coordinate system and metric coefficients of equation (18.15c).

Solution:

Firstly, we have $u^\alpha = (1/\sqrt{-g_{00}}, 0, 0, 0)$ and metric coefficients are independent of x^0. From law of conservation of baryons

$$n_{,\alpha}u^\alpha = -n(u^\alpha{}_{,\alpha} + \Gamma^\alpha{}_{\beta\alpha}u^\beta), \tag{22.5.2}$$

we can get $n_{,0} = 0$. For a perfect fluid, $ds/d\tau = 0$, and so $s_{,0} = 0$. From equation of state, we have $p_{,0} = 0$ and $\rho_{,0} = 0$. The Euler equation is

$$(\rho + p)(u^\alpha{}_{,\beta}u^\beta + \Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma) = -(g^{\alpha\beta} + u^\alpha u^\beta)p_{,\beta}. \tag{22.5.3}$$

So we have

$$0 = -g^{0i}p_{,i} \tag{22.5.4}$$

and

$$-(\rho + p)\frac{\Gamma^i{}_{00}}{g_{00}} = -p^{,i}. \tag{22.5.5}$$

Now we cen get $g_{0i} = g^{0i} = 0$ and

$$p_{,i} = (\rho + p)\frac{\Gamma_{i00}}{g_{00}} = -\frac{1}{2}(\rho + p)\frac{g_{00,i}}{g_{00}} = -(\rho + p)\frac{\partial \ln\sqrt{-g_{00}}}{\partial x^i}. \tag{22.5.6}$$

In Newtonian limit, we have $g_{00} = -1 - 2\Phi$, so

$$p_{,i} = -(\rho + p)\Phi_{,i}. \tag{22.5.7}$$

22.6 EXPANSION, ROTATION, AND SHEAR

Let a field of fluid 4-velocities $\boldsymbol{u}(\mathscr{P})$ be given.

(a) Show that $\boldsymbol{\nabla u}$ can be decomposed in the following manner:

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta, \tag{22.6.1}$$

where $\boldsymbol{a}$ is the 4-acceleration of the fluid

$$a_\alpha \equiv u_{\alpha;\gamma}u^\gamma, \tag{22.6.2}$$

θ is the expansion of the fluid world lines

$$\theta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{u} = u^\alpha{}_{;\alpha}, \tag{22.6.3}$$

$P_{\alpha\beta}$ is the projection tensor

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \tag{22.6.4}$$

$\sigma_{\alpha\beta}$ is the shear tensor of the fluid

$$\sigma_{\alpha\beta} \equiv \frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta + u_{\beta;\mu}P^\mu{}_\alpha) - \frac{1}{3}\theta P_{\alpha\beta}, \tag{22.6.5}$$

and $\omega_{\alpha\beta}$ is the rotation 2-form of the fluid

$$\omega_{\alpha\beta} \equiv \frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta - u_{\beta;\mu}P^\mu{}_\alpha). \tag{22.6.6}$$

(b) Each of the component parts of this decomposition has a simple physical interpretation in the local rest frames of the fluid. The interpretation of the 4-acceleration $\boldsymbol{a}$ in terms of accelerometer readings should be familiar. Exercise 22.1 showed that the expansion $\theta = \boldsymbol{\nabla}\cdot\boldsymbol{u}$ describes the rate of increase of the volume of a fluid element,

$$\theta = \frac{1}{V}\frac{dV}{d\tau}. \tag{22.6.7}$$

Exercise 22.4 explored the meaning and use of the projection tensor $\boldsymbol{P}$. Verify that in a local Lorentz frame ($g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$) momentarily moving with the fluid ($u^{\hat{\alpha}} = \delta^{\hat{\alpha}}{}_{\hat{0}}$), $\sigma_{\hat{\alpha}\hat{\beta}}$ and $\omega_{\hat{\alpha}\hat{\beta}}$ reduce to the classical (nonrelativistic) shear and rotation of the fluid.

Solution:

(a) From all the definitions above, we have

$$\omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta = u_{\alpha;\mu}P^\mu{}_\beta - u_{\alpha;\mu}u^\mu a_\beta = u_{\alpha;\beta}. \tag{22.6.8}$$

So $\boldsymbol{\nabla u}$ can be decomposed in this form.

(b) In Local Lorentz frame, we have

$$P^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22.6.9}$$

Keeping the lowest order of v, we have

$$\sigma_{\hat{i}\hat{j}} = \frac{1}{2}(v_{\hat{i},\hat{j}} + v_{\hat{j},\hat{i}}) - \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} \tag{22.6.10}$$

and

$$\omega_{\hat{i}\hat{j}} = \frac{1}{2}(v_{\hat{i},\hat{j}} - v_{\hat{j},\hat{i}}), \tag{22.6.11}$$

which is reduced to the classical (nonrelativistic) shear and rotation of the fluid.

22.7 HYDRODYNAMICS WITH VISCOSITY AND HEAT FLOW

(a) In section 15 of Landau and Lifshitz (1959), one finds an analysis of viscous stresses for a classical (nonrelativistic) fluid. By carrying that analysis over directly to the local Lorentz rest frame of a relativistic fluid, and by then generalizing to frame-independent language, show that the contribution of viscosity to the stress-energy tensor is

$$\boldsymbol{T}^{(\mathrm{visc})} = -2\eta\boldsymbol{\sigma} - \xi\theta\boldsymbol{P}, \tag{22.7.1}$$

where $\eta \geq 0$ is the "coefficient of dynamic viscosity"; $\xi \geq 0$ is the "coefficient of bulk viscosity"; and $\boldsymbol{\sigma}$, θ, $\boldsymbol{P}$ are the shear, expansion, and projection tensor of the fluid.

(b) An idealized description of heat flow in a fluid introduces the heat-flux 4-vector $\boldsymbol{q}$ with components in the local rest-frame of the fluid,

$$q^{\hat{0}} = 0, \quad q^{\hat{i}} = \begin{pmatrix}\text{energy per unit time crossing unit} \\ \text{surface perpendicular to } \boldsymbol{e}_{\hat{i}}\end{pmatrix}. \tag{22.7.2}$$

By generalizing from the fluid rest frame to frame-independent language, show that the contribution of heat flux to the stress-energy tensor is

$$\boldsymbol{T}^{(\mathrm{heat})} = \boldsymbol{u}\otimes\boldsymbol{q} + \boldsymbol{q}\otimes\boldsymbol{u}. \tag{22.7.3}$$

Thereby conclude that, in this idealized picture, the stress-energy tensor for a viscous fluid with heat conduction is

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + (p - \xi\theta)P^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + q^\alpha u^\beta + u^\alpha q^\beta. \tag{22.7.4}$$

(c) Define the entropy 4- vector $\boldsymbol{s}$ by

$$\boldsymbol{s} \equiv ns\boldsymbol{u} + \boldsymbol{q}/T. \tag{22.7.5}$$

By calculations in the local rest-frame of the fluid, show that

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{s} &= \begin{pmatrix}\text{rate of increase of entropy}\\ \text{in a unit volume}\end{pmatrix} - \begin{pmatrix}\text{rate at which heat and fluid}\\ \text{carry entropy into a unit volume}\end{pmatrix}\\ &= \begin{pmatrix}\text{rate at which entropy is being}\\ \text{generated in a unit volume}\end{pmatrix}.\end{aligned} \tag{22.7.6}$$

Thereby arrive at the following form of the second law of thermodynamics:

$$\boldsymbol{\nabla}\cdot\boldsymbol{s} \geq 0. \tag{22.7.7}$$

(d) Calculate the law of local energy conservation, $\boldsymbol{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{T}$, for a viscous fluid with heat flow. Combine with the first law of thermodynamics and with the law of baryon conservation to obtain

$$T\boldsymbol{\nabla}\cdot\boldsymbol{s} = \xi\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} - q^\alpha(T_{,\alpha}/T + a_\alpha). \tag{22.7.8}$$

Interpret each term of this equation as a contribution to entropy generation.

(e) When one takes account of the inertia of the flowing heat, one obtains the following generalization of the classical law of heat conduction:

$$q^\alpha = -\kappa P^{\alpha\beta}(T_{,\beta} + Ta_\beta). \tag{22.7.9}$$

Here κ is the coefficient of thermal conductivity. Use this equation to show that, for a fluid at rest in a-stationary gravitational field,

$$q_0 = 0, \quad q_i = -\frac{\kappa}{\sqrt{-g_{00}}}(T\sqrt{-g_{00}})_{,i}. \tag{22.7.10}$$

Also, use the idealized law of heat conduction to reexpress the rate of entropy generation as

$$T\boldsymbol{\nabla}\cdot\boldsymbol{s} = \xi\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} + (\kappa/T)P^{\alpha\beta}(T_{,\alpha} + Ta_\alpha)(T_{,\beta} + Ta_\beta) \geq 0. \tag{22.7.11}$$

Solution:

(a) From equation (15.3) in *Fluid Mechanics (Landau & Lifshitz)*, we have

$$\boldsymbol{T}_{ij}^{(\text{visc})} = -\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial x_l}\right) - \xi\delta_{ij}\frac{\partial v_l}{\partial x_l}. \tag{22.7.12}$$

According to 22.6.9 and 22.6.10, we have

$$\boldsymbol{T}_{ij}^{(\text{visc})} = -2\eta\sigma_{ij} - \xi\theta P_{ij}. \tag{22.7.13}$$

By generalizing to frame-independent language, we can get

$$\boldsymbol{T}^{(\text{visc})} = -2\eta\boldsymbol{\sigma} - \xi\theta\boldsymbol{P}. \tag{22.7.14}$$

In the local rest-frame of the fluid, we have

$$\boldsymbol{T}^{(\text{heat})} = \begin{pmatrix} 0 & q^1 & q^2 & q^3 \\ q^1 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 \\ q^3 & 0 & 0 & 0 \end{pmatrix}. \tag{22.7.15}$$

Note that $u^\alpha = (1,0,0,0)$. We can write

$$\boldsymbol{T}^{\alpha\beta}_{(\text{heat})} = q^\alpha u^\beta + u^\alpha q^\beta. \tag{22.7.16}$$

By generalizing to frame-independent language, we can get

$$\boldsymbol{T}^{(\text{heat})} = \boldsymbol{u}\otimes\boldsymbol{q} + \boldsymbol{q}\otimes\boldsymbol{u}. \tag{22.7.17}$$

As $\boldsymbol{T} = \boldsymbol{T}^{(perfect)} + \boldsymbol{T}^{(visc)} + \boldsymbol{T}^{(heat)}$, we have

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + (p-\xi\theta)P^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + q^\alpha u^\beta + u^\alpha q^\beta. \tag{22.7.18}$$

(c) Generally, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{s} = n\boldsymbol{u}\cdot\boldsymbol{\nabla}s + s\nabla\cdot(n\boldsymbol{u}) + \nabla\cdot\left(\frac{\boldsymbol{q}}{T}\right) = n\frac{ds}{d\tau} + \nabla\cdot\left(\frac{\boldsymbol{q}}{T}\right). \tag{22.7.19}$$

In local Lorentz frame, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{s} = n\frac{\partial s}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{q^i}{T}\right). \tag{22.7.20}$$

Here, $n\partial s/\partial t$ is the rate of increase of entropy in a unit volume and $-(q^i/T)_{,i}$ is the rate at which heat and fluid carry entropy into a unit volume. And so we can get the equation 22.7.6 and 22.7.7.

(d) According to 22.7.18, we have

$$0 = u_\alpha T^{\alpha\beta}{}_{;\beta} = -\frac{d\rho}{d\tau} - (\rho+p-\xi\theta)u^\beta{}_{;\beta} - 2\eta u_\alpha\sigma^{\alpha\beta}{}_{;\beta} + u_\alpha q^\alpha u^\beta{}_{;\beta} + u_\alpha q^\alpha{}_{;\beta}u^\beta - q^\beta{}_{;\beta}. \tag{22.7.21}$$

From the first law of thermodynamics, we have

$$\frac{d\rho}{d\tau} = \frac{\rho+p}{n}\frac{dn}{d\tau} + nT\frac{ds}{d\tau}. \tag{22.7.22}$$

From conservation of baryons, we have

$$\boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = n\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{dn}{d\tau} = 0. \tag{22.7.23}$$

And so

$$-\frac{d\rho}{d\tau} - (\rho+p-\xi\theta)u^\beta{}_{;\beta} = -\frac{\rho+p}{n}\frac{dn}{d\tau} - nT\frac{ds}{d\tau} + (\rho+p)\frac{1}{n}\frac{dn}{d\tau} + \xi\theta^2 = -nT\frac{ds}{d\tau} + \xi\theta^2. \tag{22.7.24}$$

In local Lorentz frame, we have $q^\alpha u_\alpha = 0$. So it will hold in any frame. As a result, we have $u_\alpha q^\alpha u^\beta{}_{;\beta} = 0$ and $u_\alpha q^\alpha{}_{;\beta}u^\beta = -u_{\alpha;\beta}u^\beta q^\alpha = -a^\alpha q_\alpha$. From equation 22.7.19, we have

$$Ts^\alpha{}_{;\alpha} = nT\frac{ds}{d\tau} + q^\alpha{}_{;\alpha} - q^\alpha\left(\frac{T_{,\alpha}}{T}\right) \tag{22.7.25}$$

Now we can get

$$Ts^\alpha{}_{;\alpha} = \xi\theta^2 - 2\eta u_\alpha\sigma^{\alpha\beta}{}_{;\beta} - q^\alpha\left(\frac{T_{,\alpha}}{T} + a_\alpha\right). \tag{22.7.26}$$

On the one hand, we have

$$\begin{aligned}
u_\alpha\sigma^{\alpha\beta}{}_{;\beta} &= \frac{1}{2}u_\alpha(u^\alpha{}_{;\mu}P^{\mu\beta} + u^\beta{}_{;\mu}P^{\mu\alpha})_{;\beta} - \frac{1}{3}\theta u_\alpha P^{\alpha\beta}{}_{;\beta} \\
&= \frac{1}{2}u_\alpha u^\alpha{}_{;\mu\beta}P^{\mu\beta} + \frac{1}{2}u_\alpha u^\beta{}_{;\mu}P^{\mu\alpha}{}_{;\beta} - \frac{1}{3}\theta u_\alpha(u^\alpha u^\beta)_{;\beta} \\
&= -\frac{1}{2}u_{\alpha;\beta}u^\alpha{}_{;\mu}P^{\mu\beta} + \frac{1}{2}u_\alpha u^\beta{}_{;\mu}(u^\alpha u^\mu)_{;\beta} + \frac{1}{3}\theta^2 \\
&= -\frac{1}{2}u_{\alpha;\beta}u^\alpha{}_{;\mu}P^{\mu\beta} - \frac{1}{2}u^\beta{}_{;\mu}u^\mu{}_{;\beta} + \frac{1}{3}\theta^2
\end{aligned} \tag{22.7.27}$$

On the other hand, we have

$$\begin{aligned}\sigma^{\alpha\beta}\sigma_{\alpha\beta} &= \left[\frac{1}{2}(u^{\alpha}{}_{;\mu}P^{\mu\beta} + u^{\beta}{}_{;\mu}P^{\mu\alpha}) - \frac{1}{3}\theta P^{\alpha\beta}\right]\left[\frac{1}{2}(u_{\alpha;\nu}P^{\nu}{}_{\beta} + u_{\beta;\nu}P^{\nu}{}_{\alpha}) - \frac{1}{3}\theta P_{\alpha\beta}\right] \\ &= \frac{1}{2}\left[u^{\alpha}{}_{;\mu}u_{\alpha;\nu}P^{\mu\nu} + u^{\beta}{}_{;\mu}u_{\alpha;\nu}P^{\mu\alpha}P^{\nu}{}_{\beta}\right] - \frac{2}{3}\theta u_{\alpha;\nu}P^{\alpha\nu} + \frac{1}{3}\theta^2 \\ &= \frac{1}{2}\left[u^{\alpha}{}_{;\mu}u_{\alpha;\nu}P^{\mu\nu} + u^{\beta}{}_{;\mu}u_{\alpha;\nu}g^{\mu\alpha}\delta^{\nu}{}_{\beta}\right] - \frac{2}{3}\theta u_{\alpha;\nu}g^{\alpha\nu} + \frac{1}{3}\theta^2 \\ &= \frac{1}{2}\left[u^{\alpha}{}_{;\mu}u_{\alpha;\nu}P^{\mu\nu} + u^{\beta;\alpha}u_{\alpha;\beta}\right] - \frac{1}{3}\theta^2. \end{aligned} \tag{22.7.28}$$

So we know $u_\alpha\sigma^{\alpha\beta}{}_{;\beta} = -\sigma^{\alpha\beta}\sigma_{\alpha\beta}$. Now we have

$$Ts^{\alpha}{}_{;\alpha} = \xi\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} - q^{\alpha}\left(\frac{T_{,\alpha}}{T} + a_\alpha\right). \tag{22.7.29}$$

Here, $\xi\theta^2$ describes entropy generation by dynamic viscous heating, $2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ describes entropy generation by bulk viscous heating, $-q^\alpha T_{,\alpha}/T$ describes entropy generation by temperature increasing, and $-q^\alpha a_\alpha$ is relativistic in origin and associated with the inertia of the flowing heat.

(e) For a fluid at rest in a-stationary gravitational field, we have

$$a_\alpha = u_{\alpha;\beta}u^\beta = u_{\alpha,\beta}u^\beta - \Gamma^{\mu}{}_{\alpha\beta}u_\mu u^\beta = u_{\alpha,0}u^0 + \Gamma^0{}_{\alpha 0}, \tag{22.7.30}$$

i.e.

$$a_0 = 0, \quad a_i = g^{00}\Gamma_{0i0} = \frac{g_{00,i}}{2g_{00}} = \ln\left(\sqrt{-g_{00}}\right)_{,i}. \tag{22.7.31}$$

And also from $P_\alpha{}^\beta = \delta^\beta_\alpha + u^\beta u_\alpha$, we have

$$P_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22.7.32}$$

So

$$q_0 = 0, \quad q_i = -\kappa(T_{,i} + Ta_i) = -\frac{\kappa}{\sqrt{-g_{00}}}(T\sqrt{-g_{00}})_{,i}. \tag{22.7.33}$$

Using the idealized law of heat conduction, we have

$$\begin{aligned} Ts^{\alpha}{}_{;\alpha} &= \xi\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \kappa P^{\alpha\beta}(T_{,\beta} + Ta_\beta)\left(\frac{T_{,\alpha}}{T} + a_\alpha\right) \\ &= \xi\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \kappa T P^{\alpha\beta}(T_{,\alpha} + Ta_\alpha)(T_{,\beta} + Ta_\beta) \\ &\geq 0. \end{aligned} \tag{22.7.34}$$

22.8 THE VECTOR POTENTIAL FOR ELECTRODYNAMICS

Show that in any coordinate frame the connection coefficients cancel out of both equations (22.19a) and (22.17b), so they can be written

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}; \quad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{22.8.1}$$

(In the language of differential forms these equations are $\boldsymbol{F} = \boldsymbol{dA}$, $\boldsymbol{dF} = 0$.) Then use this form of the equations to show that equation (22.19a) implies equation (22.17b), as asserted in the text.

Solution:

It is eays to derive that

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - \Gamma^{\rho}{}_{\nu\mu}A_{\rho} - A_{\mu,\nu} + \Gamma^{\rho}{}_{\mu\nu}A_{\rho} = A_{\nu,\mu} - A_{\mu,\nu} \tag{22.8.2}$$

and

$$\begin{aligned} &F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} \\ &= F_{\alpha\beta,\gamma} - \Gamma^{\rho}{}_{\alpha\gamma}F_{\rho\beta} - \Gamma^{\rho}{}_{\beta\gamma}F_{\alpha\rho} + F_{\beta\gamma,\alpha} - \Gamma^{\rho}{}_{\beta\alpha}F_{\rho\gamma} - \Gamma^{\rho}{}_{\gamma\alpha}F_{\beta\rho} + F_{\gamma\alpha,\beta} - \Gamma^{\rho}{}_{\gamma\beta}F_{\rho\alpha} - \Gamma^{\rho}{}_{\alpha\beta}F_{\gamma\rho} \\ &= F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta}. \end{aligned} \tag{22.8.3}$$

If $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, we have

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = A_{\beta,\alpha\gamma} - A_{\alpha,\beta\gamma} + A_{\gamma,\beta\alpha} - A_{\beta,\gamma\alpha} + A_{\alpha,\gamma\beta} - A_{\gamma,\alpha\beta} = 0. \tag{22.8.4}$$

So if $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$, we have $F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$.

22.9 CHARGE CONSERVATION IN THE PRESENCE OF GRAVITY

Show that Maxwell's equations (22.17a,b) imply the equation of charge conservation (22.18a) when gravity is present, just as they do in special relativity theory.

Solution:

From exercise 16.6, we know

$$S^{\mu\nu}{}_{;\alpha\beta} = S^{\mu\nu}{}_{;\beta\alpha} + R^{\mu}{}_{\rho\beta\alpha}S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}. \tag{22.9.1}$$

So we have

$$4\pi J^{\alpha}{}_{;\alpha} = F^{\alpha\beta}{}_{;\beta\alpha} = F^{\alpha\beta}{}_{;\alpha\beta} - R^{\alpha}{}_{\rho\beta\alpha}F^{\rho\beta} - R^{\beta}{}_{\rho\beta\alpha}F^{\alpha\rho} = F^{\alpha\beta}{}_{;\alpha\beta} + R_{\rho\beta}F^{\rho\beta} - R_{\rho\alpha}F^{\alpha\rho}. \tag{22.9.2}$$

Note that $\boldsymbol{F}$ is anti-symmetric. We have $R_{\rho\beta}F^{\rho\beta} = 0$ and $F^{\alpha\beta}{}_{;\alpha\beta} = -F^{\beta\alpha}{}_{;\alpha\beta} = -F^{\alpha\beta}{}_{;\beta\alpha}$. Now it is easy to see that $J^{\alpha}{}_{;\alpha} = 0$.

22.10 INTERACTING ELECTROMAGNETIC FIELD AND CHARGED MATTER

As in special relativity, so also in the presence of gravity ("equivalence principle"), the stress-energy tensor for an electromagnetic field is

$$T^{(\mathrm{EM})}{}_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\mu}F_{\beta}{}^{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right). \tag{22.10.1}$$

Use Maxwell's equations (22.17 a,b) in the presence of gravity to show that

$$T^{(\mathrm{EM})\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_{\beta}. \tag{22.10.2}$$

But $F^{\alpha\beta}J_{\beta}$ is just the Lorentz 4-force per unit volume with which the electromagnetic field acts on the charged matter; i.e., it is $T^{(\mathrm{MATTER})\alpha\beta}{}_{;\beta}$. Consequently, the above equation can be rewritten in the form (22.18 b) cited in the text.

Solution:

According to the definition of the stress-energy tensor for an electromagnetic field, we have

$$T^{(\mathrm{EM})\alpha\beta}{}_{;\beta} = \frac{1}{4\pi}\left(F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta}\right)_{;\beta}. \tag{22.10.3}$$

Note that

$$(F^{\alpha\mu}F^{\beta}{}_{\mu})_{;\beta} = F^{\alpha\mu;\beta}F_{\beta\mu} + F^{\alpha\mu}F^{\beta}{}_{\mu;\beta} = \frac{1}{2}F^{\alpha\mu;\beta}F_{\beta\mu} + \frac{1}{2}F^{\beta\alpha;\mu}F_{\beta\mu} - 4\pi F^{\alpha\mu}J_{\mu} \tag{22.10.4}$$

and

$$\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta}\right)_{;\beta} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu;\alpha} = \frac{1}{2}F_{\beta\mu}F^{\mu\beta;\alpha} \tag{22.10.5}$$

We have

$$T^{(\mathrm{EM})\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_{\beta}. \tag{22.10.6}$$

22.11 ELECTROMAGNETIC FIELD AND STRESS ENERGY

Derive the equations given in part D of Box 22.4 for $\boldsymbol{F}$, $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{T}$.

Solution:

The definition of vector potential in geometric optics is

$$\boldsymbol{A} \equiv \mathscr{R}\{ae^{i\theta}\boldsymbol{f}\}. \tag{22.11.1}$$

As the reduced wavelength is much smaller than the scale over which a or $\boldsymbol{f}$ vary, we have

$$\boldsymbol{F} \equiv \boldsymbol{d}\boldsymbol{A} = \mathscr{R}\{a(\boldsymbol{d}e^{i\theta}) \wedge \boldsymbol{f}\} = \mathscr{R}\{iae^{i\theta}\boldsymbol{k} \wedge \boldsymbol{f}\}. \tag{22.11.2}$$

Take the Coulomb's gauge and we have $f^0 = 0$. So $E^i = F^{0i} = \mathscr{R}\{iae^{i\theta}k^0 f^i\}$ in local Lorentz frame, i.e.

$$\boldsymbol{E} = \mathscr{R}\{iak^0(\text{projection of } \boldsymbol{f} \text{ perpendicular to } \boldsymbol{k})\}. \tag{22.11.3}$$

And

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \mathscr{R}\{iae^{i\theta}\boldsymbol{k} \times \boldsymbol{f}\} = \frac{\boldsymbol{k}}{k^0} \times \boldsymbol{E} = \boldsymbol{n} \times \boldsymbol{E}. \tag{22.11.4}$$

The stress-energy tensos is

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta}\right). \tag{22.11.5}$$

When averged over a wavelength for $A, B \propto e^{i\theta}$, we have

$$\langle\mathscr{R}(A)\mathscr{R}(B)\rangle = \frac{1}{2}\mathscr{R}(A\bar{B}). \tag{22.11.6}$$

So

$$\langle T^{\alpha\beta}\rangle = \frac{a^2}{8\pi}\left[(k^\alpha f^\mu - f^\alpha k^\mu)(k^\beta \bar{f}_\mu - \bar{f}^\beta k_\mu) - \frac{1}{4}(k^\mu f^\nu - f^\mu k^\nu)(k_\mu \bar{f}_\nu - \bar{f}_\mu k_\nu)\eta^{\alpha\beta}\right]. \tag{22.11.7}$$

Note that $k^\mu k_\mu = 0$, $k^\mu f_\mu = 0$ and $f^\mu f_\mu = 1$, we have

$$\langle T^{\alpha\beta}\rangle = \frac{a^2}{8\pi}k^\alpha k^\beta. \tag{22.11.8}$$

22.12 POLARIZATION

At an event $\mathscr{P}_0$ through which geometric-optics waves are passing, introduce a local Lorentz frame with z-axis along the direction of propagation. Then $\boldsymbol{k} = \omega(\boldsymbol{e}_0 + \boldsymbol{e}_z)$. Since the polarization vector is orthogonal to $\boldsymbol{k}$, it is $\boldsymbol{f} = f^0(\boldsymbol{e}_0 + \boldsymbol{e}_z) + f^1\boldsymbol{e}_x + f^2\boldsymbol{e}_y$; and since $\boldsymbol{f} \cdot \bar{\boldsymbol{f}} = 1$, it has $|f^1|^2 + |f^2|^2 = 1$.

(a) Show that the component f^0 of the polarization vector has no influence on the electric and magnetic fields measured in the given frame; i.e., show that one can add a multiple of $\boldsymbol{k}$ to $\boldsymbol{f}$ without affecting any physical measurements.

(b) Show that the following polarization vectors correspond to the types of polarization:

- $\boldsymbol{f} = \boldsymbol{e}_x$, linear polarization in x direction;
- $\boldsymbol{f} = \boldsymbol{e}_y$, linear polarization in y direction;
- $\boldsymbol{f} = (\boldsymbol{e}_x + i\boldsymbol{e}_y)/\sqrt{2}$, righthand circular polarization;
- $\boldsymbol{f} = (\boldsymbol{e}_x - i\boldsymbol{e}_y)/\sqrt{2}$, lefthand circular polarization;
- $\boldsymbol{f} = \alpha\boldsymbol{e}_x + i(1-\alpha^2)^{1/2}\boldsymbol{e}_y$, righthand elliptical polarization.

(c) Show that the type of polarization (linear; circular; elliptical with given eccentricity of ellipse) is the same as viewed in any local Lorentz frame at any event along a given ray.

Solution:

(a) From 22.11.2 we have $\boldsymbol{F} \propto \boldsymbol{k} \wedge \boldsymbol{f}$. And so one can add a multiple of $\boldsymbol{k}$ to $\boldsymbol{f}$ without affecting any physical measurements.

(b) For $\boldsymbol{f} = \boldsymbol{e}_x$ and $\boldsymbol{f} = \boldsymbol{e}_y$, it is obvious to see that they describes linear polarization in x and y direction respectively. For $\boldsymbol{f} = (\boldsymbol{e}_x + i\boldsymbol{e}_y)/\sqrt{2}$, we have

$$E^x = \frac{a\omega}{\sqrt{2}}\cos\Big(\theta + \frac{\pi}{2}\Big); \quad E^y = \frac{a\omega}{\sqrt{2}}\cos(\theta + \pi), \tag{22.12.1}$$

where $\theta = k_t t + k_z z = \omega(z - t)$. As $(E^x)^2 + (E^y)^2 = a^2\omega^2/2$, it is a circular polarization. And the phase angle of E^x is advanced, so it is righthand. Similarly, $\boldsymbol{f} = (\boldsymbol{e}_x - i\boldsymbol{e}_y)/\sqrt{2}$ describes lefthand circular polarization. As for $\boldsymbol{f} = \alpha\boldsymbol{e}_x + i(1-\alpha^2)^{1/2}\boldsymbol{e}_y$, we have

$$E^x = \alpha a\omega\cos\Big(\theta + \frac{\pi}{2}\Big); \quad E^y = (1-\alpha^2)^{1/2}a\omega\cos(\theta + \pi). \tag{22.12.2}$$

As $(E^x/\alpha)^2 + (E^y/\sqrt{1-\alpha^2})^2 = a^2\omega^2$, it is a elliptical polarization. And the phase angle of E^x is advanced, so it is righthand.

(c) The Lorentz transformation will transform the real and imaginary part of $\boldsymbol{f}$ separately. So it is obvious a linear polarization will remain linear polarized in other Lorentz frames. For a circular or elliptical polarization, the real and imaginary part of $\boldsymbol{f}$ are perpendicular to each other. So in the new frame, they will be orthogonal too. The Lorentz transformation cannot change the norm of a vector. So the ratio between the real and imaginary of $\boldsymbol{f}$ will the same, i.e. the eccentricity of ellipse can not be changed.

22.13 THE AREA OF A BUNDLE OF RAYS

Write equation (22.31) in a coordinate system in which one of the coordinates is chosen to be $x^0 = \theta$, the phase (a retarded time coordinate).

(a) Show that $g^{00} = 0$ and that no derivatives $\partial/\partial\theta$ appear in equation (22.33); so propagation of a can be described within a single $\theta =$ constant hypersurface.

(b) Perform the following construction. Pick a ray $\mathcal{L}_0$ along which a is to be propagated. Pick a bundle of rays, with two-dimensional cross section, that (i) all lie in the same constant-phase surface as $\mathcal{L}_0$, and (ii) surround $\mathcal{L}_0$. (The surface is three-dimensional, so any bundle filling it has a two-dimensional cross section.) At any event $\mathscr{P}_0$ in any local Lorentz frame there, on a "slice of simultaneity" $x^0 =$ constant, measure the cross-sectional area $\mathfrak{a}$ of the bundle.

(Note: the area being measured is perpendicular to $\boldsymbol{k}$ in the three-dimensional Euclidean sense; it can be thought of as the region occupied momentarily by a group of photons propagating along, side by side, in the $\boldsymbol{k}$ direction.)

Show that the area $\mathfrak{a}$ is the same, at a given event $\mathscr{P}_0$ regardless of what Lorentz frame is used to measure it; but the area changes from point to point along the ray $\mathcal{L}_0$ as a result of the rays' divergence away from each other or convergence toward each other:

$$\partial_{\boldsymbol{k}}\mathfrak{a} = (\boldsymbol{\nabla}\cdot\boldsymbol{k})\mathfrak{a}. \tag{22.13.1}$$

Then show that $a^2\mathfrak{a}$ is a constant everywhere along the ray $\mathcal{L}_0$ ("conservation of photon flux").

Solution:

(a) In this coordinate frame, we have

$$k_\alpha = \theta_{,\alpha} = (1,0,0,0), \quad k^\alpha = g^{\alpha 0}. \tag{22.13.2}$$

And so $0 = k^\alpha k^\alpha = g^{00}$, i.e. $g^{00} = 0$ and $k^0 = 0$. So no derivatives $\partial/\partial\theta$ appear in equation (22.33).

(b) Choose a local Lorentz frame at $\mathscr{P}_0$ and let the $\theta = t - z$ around $\mathscr{P}_0$. We further choose a bundle with rectangular cross section lies in the $x - y$ plane. Suppose that the edges of the cross section are $\boldsymbol{v}$ and $\boldsymbol{w}$. Then we have $\boldsymbol{v}\cdot\boldsymbol{w} = 0$, $\boldsymbol{v}\cdot\boldsymbol{k} = \boldsymbol{w}\cdot\boldsymbol{k} = 0$, and $\boldsymbol{v}\cdot\boldsymbol{e}_0 = \boldsymbol{w}\cdot\boldsymbol{e}_0 = 0$.

In a new Lorentz frame, the cross section will cut the rays in other points. So the new edges can be written as $\boldsymbol{v}' = \boldsymbol{v} + \alpha\boldsymbol{k}$ and $\boldsymbol{w}' = \boldsymbol{w} + \beta\boldsymbol{k}$ for some α and β. We have

$$\boldsymbol{v}'\cdot\boldsymbol{v}' = \boldsymbol{v}\cdot\boldsymbol{v}, \quad \boldsymbol{w}'\cdot\boldsymbol{w}' = \boldsymbol{w}\cdot\boldsymbol{w}, \quad \boldsymbol{v}'\cdot\boldsymbol{w}' = 0. \tag{22.13.3}$$

We can conclude that in all Lorentz frames at $\mathscr{P}_0$ the cross section has identical shape and identical area.

In the original Lorentz frame, the light rays in the vicinity of the cross section Σ are

$$x^\alpha = x^\alpha(\Sigma) + k^\alpha(\Sigma)\lambda. \tag{22.13.4}$$

Suppose the four vertexes of the rectangular at Σ are $(0,0,0,0)$, $(dx,0,0,0)$, $(0,dy,0,0)$ and $(dx,dy,0,0)$ respectively. Then along the light ray, the vertexes will be

$$(k^x,k^y,k^z,k^t)\epsilon, \quad (dx/\epsilon + k^x + k^x{}_{,x}dx, k^y, k^z, k^t)\epsilon, \quad (dx, dy/\epsilon + k^y + k^y{}_{,y}dy, k^z, k^t), \cdots \tag{22.13.5}$$

So the area of the cross section is

$$S' = dxdy + \epsilon(k^x{}_{,x} + k^y{}_{,y})dxdy + O(\epsilon^2), \tag{22.13.6}$$

i.e.

$$\partial_{\boldsymbol{k}}\mathfrak{a} = (\boldsymbol{\nabla}\cdot\boldsymbol{k})\mathfrak{a}. \tag{22.13.7}$$

Recall that

$$\partial_{\boldsymbol{k}}a = -\frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{k})a. \tag{22.13.8}$$

We have $\partial_{\boldsymbol{k}}a^2\mathfrak{a} = 0$, i.e. $a^2\mathfrak{a}$ is a constant everywhere along the ray $\mathcal{L}_0$.

22.14 FOCUSING THEOREM

The cross-sectional area $\mathfrak{a}$ of a bundle of rays all lying in the same surface of constant phase changes along the central ray of the bundle at the rate 22.13.7.

(a) Derive the following equation ("focusing equation") for the second derivative of $\mathfrak{a}^{1/2}$:

$$\frac{d^2\mathfrak{a}^{1/2}}{d\lambda^2} = -\left(|\sigma|^2 + \frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta\right)\mathfrak{a}^{1/2}. \tag{22.14.1}$$

where λ is affine parameter along the central ray ($\boldsymbol{k} = d/d\lambda$), and the "magnitude of the shear of the rays", $|\sigma|$, is defined by the equation

$$|\sigma|^2 = \frac{1}{2}k_{\alpha;\beta}k^{\alpha;\beta} - \frac{1}{4}(k^\mu{}_{;\mu})^2. \tag{22.14.2}$$

(b) Show that, in a local Lorentz frame where $\boldsymbol{k} = \omega(\boldsymbol{e}_t + \boldsymbol{e}_z)$ at the origin,

$$|\sigma|^2 = \frac{1}{4}(k_{x,x} - k_{y,y})^2 + (k_{x,y})^2. \tag{22.14.3}$$

(c) Discussion: The quantity $|\sigma|$ is called the shear of the bundle of rays because it measures the extent to which neighboring rays are sliding past each other [see, e.g., Sachs (1964)]. Hence, the focusing equation 22.14.1 says that shear focuses a bundle of rays (makes $d^2\mathfrak{a}^{1/2}d\lambda^2$); and spacetime curvature also focuses it if $R_{\alpha\beta}k^\alpha k^\beta > 0$, but defocuses it if $R_{\alpha\beta}k^\alpha k^\beta < 0$. (When a bundle of toothpicks, originally circular in cross section, is squeezed into an elliptic cross section, it is sheared.)

(d) Assume that the energy density $T_{\hat{0}\hat{0}}$, as measured by any observer anywhere in spacetime, is nonnegative. By combining the focusing equation 22.14.1 with the Einstein field equation, conclude that

$$\frac{d^2\mathfrak{a}^{1/2}}{d\lambda^2} \le 0 \tag{22.14.4}$$

for any bundle of rays, all in the same surface of constant phase, anywhere in spacetime.

Solution:

(a) From equation 22.13.7, we have

$$\frac{d\mathfrak{a}^{1/2}}{d\lambda} = \frac{1}{2}k^\mu{}_{;\mu}\mathfrak{a}^{1/2}. \tag{22.14.5}$$

And so

$$\frac{d^2\mathfrak{a}^{1/2}}{d\lambda^2} = \frac{1}{2}k^\mu{}_{;\mu\nu}k^\nu\mathfrak{a}^{1/2} + \frac{1}{2}k^\mu{}_{;\mu}\frac{d\mathfrak{a}^{1/2}}{d\lambda}. \tag{22.14.6}$$

Since

$$k^\mu{}_{;\mu\nu}k^\nu = (k^\mu{}_{;\nu\mu} - R^\mu{}_{\rho\mu\nu}k^\rho)k^\nu = (k^\mu{}_{;\nu}k^\nu)_{;\mu} - k^\mu{}_{;\nu}k^\nu{}_{;\mu} - R_{\rho\nu}k^\rho k^\nu = -(k^{\mu;\nu}k_{\mu;\nu} + R_{\rho\nu}k^\rho k^\nu) \tag{22.14.7}$$

(noting that $k_\alpha = \theta_{,\alpha}$ and $k^\mu{}_{;\nu}k^\nu = 0$), we have

$$\frac{d^2\mathfrak{a}^{1/2}}{d\lambda^2} = -\left(\frac{1}{2}k_{\alpha;\beta}k^{\alpha;\beta} - \frac{1}{4}(k^\mu{}_{;\mu})^2 + \frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta\right)\mathfrak{a}^{1/2}. \tag{22.14.8}$$

(b) As $k^\mu{}_{;\nu}k^\nu = 0$, we have

$$k_{\mu,t} + k_{\mu,z} = 0. \tag{22.14.9}$$

So the structure of the matrix $k_{\alpha;\beta}$ will be

$$\begin{pmatrix} k_{t,t} & k_{t,x} & k_{t,y} & -k_{t,t} \\ k_{t,x} & k_{x,x} & k_{x,y} & -k_{t,x} \\ k_{t,y} & k_{x,y} & k_{y,y} & -k_{t,y} \\ -k_{t,t} & -k_{t,x} & -k_{t,y} & k_{t,t} \end{pmatrix}. \tag{22.14.10}$$

And so the structure of the matrix $k^{\alpha;\beta}$ is

$$\begin{pmatrix} k_{t,t} & -k_{t,x} & -k_{t,y} & k_{t,t} \\ -k_{t,x} & k_{x,x} & k_{x,y} & -k_{t,x} \\ -k_{t,y} & k_{x,y} & k_{y,y} & -k_{t,y} \\ k_{t,t} & -k_{t,x} & -k_{t,y} & k_{t,t} \end{pmatrix}. \tag{22.14.11}$$

We have

$$k_{\alpha;\beta}k^{\alpha;\beta} = k_{x,x}^2 + k_{y,y}^2 + 2k_{x,y}^2 \tag{22.14.12}$$

and

$$(k^{\mu}{}_{;\mu})^2 = (-k_{t,t} + k_{x,x} + k_{y,y} + k_{z,z})^2 = (k_{x,x} + k_{y,y})^2. \tag{22.14.13}$$

And so

$$|\sigma|^2 = \frac{1}{4}(k_{x,x} - k_{y,y})^2 + (k_{x,y})^2. \tag{22.14.14}$$

(d) Using $g_{\alpha\beta}k^{\alpha}k^{\beta} = 0$, we have

$$R_{\alpha\beta}k^{\alpha}k^{\beta} = G_{\alpha\beta}k^{\alpha}k^{\beta} = 8\pi T_{\alpha\beta}k^{\alpha}k^{\beta}. \tag{22.14.15}$$

The energy density measured by any observer anywhere in spacetime is $T_{\alpha\beta}u^{\alpha}u^{\beta}$ and the direction of $\boldsymbol{u}$ can be made infinitely close to that of $\boldsymbol{k}$. If $T_{\alpha\beta}u^{\alpha}u^{\beta}$ is always nonnegative, $T_{\alpha\beta}k^{\alpha}k^{\beta}$ must not be negative. So we conclude that $d^2\mathfrak{a}^{1/2}/d\lambda^2 \leq 0$ for any bundle of rays, all in the same surface of constant phase, anywhere in spacetime.

22.15 INVERSE SQUARE LAW FOR FLUX

The specific flux of radiation entering a telescope from a given source is defined by

$$F_{\nu} = \int I_{\nu} d\Omega, \tag{22.15.1}$$

where integration is over the total solid angle (assumed $\ll 4\pi$) subtended by the source on the observer's sky. Use the Boltzmann equation (conservation of I_{ν}/ν^3) to show that $F_{\nu} \propto$ (distance from source)$^{-2}$ for observers who are all at rest relative to each other in flat spacetime.

Solution:

In a flat spacetime, ν is conserved in propagation so we have the conservation of I_{ν}. The solid angle extended by the object is propotional to the square of distance in flat spacetime. Thus, we can infer that $F_{\nu} \propto$ (distance from source)$^{-2}$ for observers who are all at rest relative to each other in flat spacetime.

22.16 BRIGHTNESS OF THE SUN

Does the surface of the sun look any brighter to an astronaut standing on Mercury than to a student standing on Earth?

Solution:

The surface brightness is defined as

$$\frac{F_\nu}{\Delta\Omega} = I_\nu. \tag{22.16.1}$$

If we neglect the curvature of spacetime in solar system, I_ν is conserved during propagation. So the surface of the sun looks as bright to an astronaut standing on Mercury as to a student on Earth.

22.17 BLACK BODY RADIATION

An "optically thick" source of black-body radiation (e.g., the surface of a star, or the hot matter filling the universe shortly after the big bang) emits photons isotropically with a specific intensity, as seen by an observer at rest near the source, given (Planck radiation law) by

$$I_\nu = \frac{2h\nu^3}{e^{h\nu/kT} - 1}. \tag{22.17.1}$$

Here T is the temperature of the source. Show that any observer, in any local Lorentz frame, anywhere in the universe, who examines this radiation as it flows past him, will also see a black-body spectrum. Show, further, that if he calculates a temperature by measuring the specific intensity I_ν at anyone frequency, and if he calculates a temperature from the shape of the spectrum, those temperatures will agree. (Radiation remains black body rather than being "diluted" into "grey-body.") Finally, show that the temperature he measures is redshifted by precisely the same factor as the frequency of any given photon is redshifted,

$$\frac{T_{\rm observed}}{T_{\rm emitted}} = \left(\frac{\nu_{\rm observed}}{\nu_{\rm emitted}}\right) \text{ for a given photon.} \tag{22.17.2}$$

[Note that the redshifts can be "Doppler" in origin, "cosmological" in origin, "gravitational" in origin, or some inseparable mixture. All that matters is the fact that the parallel-transport law for a photon's 4-momentum, $\nabla_{\boldsymbol{p}}\boldsymbol{p} = 0$, guarantees that the redshift $\nu_{\rm observed}/\nu_{\rm emitted}$ is independent of frequency emitted.]

Solution:

The quantity I_ν/ν^3 is conserved during propagation, i.e.

$$\frac{I_{\nu_{\rm observed}}}{\nu^3_{\rm observed}} = \frac{I_{\nu_{\rm emitted}}}{\nu^3_{\rm emitted}}. \tag{22.17.3}$$

As $I_{\nu_{\rm observed}} = 2h\nu^3_{\rm observed}/(e^{h\nu_{\rm observed}/kT_{\rm observed}} - 1)$, we have

$$I_{\nu_{\rm emitted}} = \frac{2h\nu^3_{\rm emitted}}{(e^{h\nu_{\rm observed}/kT_{\rm observed}} - 1)}. \tag{22.17.4}$$

The parallel-transport law for a photon's 4-momentum, $\nabla_{\boldsymbol{p}}\boldsymbol{p} = 0$, guarantees that the redshift $\nu_{\rm observed}/\nu_{\rm emitted}$ is independent of frequency emitted. So we can define a quantity $T_{\rm emitted}$, independent of frequency, as

$$T_{\rm emitted} \equiv \frac{\nu_{\rm emitted}}{\nu_{\rm observed}} T_{\rm observed}. \tag{22.17.5}$$

Then we have

$$I_{\nu_{\rm emitted}} = \frac{2h\nu^3_{\rm emitted}}{(e^{h\nu_{\rm emitted}/kT_{\rm emitted}} - 1)}, \tag{22.17.6}$$

for any $\nu_{\rm emitted}$. Now, we have demonstrated all the statements in the exercise.

22.18 STRESS-ENERGY TENSOR

(a) Show that the stress-energy tensor for a swarm of identical particles at an event $\mathscr{P}_0$ can be written as an integral over the mass hyperboloid of the momentum space at $\mathscr{P}_0$:

$$\boldsymbol{T} = \int (\mathscr{N}\boldsymbol{p}\otimes\boldsymbol{p})(d\mathscr{V}_p/p^0), \quad \frac{d\mathscr{V}_p}{p^0} \equiv \frac{dp^x dp^y dp^z}{p^0} \text{ in a local Lorentz frame.} \tag{22.18.1}$$

(b) Verify that the Boltzmann equation, $d\mathscr{N}/d\lambda = 0$, implies $\boldsymbol{\nabla}\cdot\boldsymbol{T} = 0$ for any swarm of identical particles.

Solution:

(a) $d\mathscr{V}_p/p^0$ is a scalar under Lorentz transform. Thus, $\boldsymbol{T}$ so defined is a rank 2 tensor. In local Lorentz frame, we have

$$T^{00} = \int \mathscr{N} p^0 d\mathscr{V}_p = \text{ energy density}, \quad T^{0i} = \int \mathscr{N} p^i d\mathscr{V}_p = \text{ momentum density}, \tag{22.18.2}$$

and

$$T^{ij} = \int \mathscr{N}\frac{p^i p^j}{p^0} d\mathscr{V}_p = \int \mathscr{N} p^i v^j d\mathscr{V}_p = \text{ flux of momentum density.} \tag{22.18.3}$$

So the $\boldsymbol{T}$ defined above is the the stress-energy tensor for a swarm of identical particles at an event $\mathscr{P}_0$.

(b) From Boltzmann equation, we have

$$0 = \frac{d\mathscr{N}}{d\lambda} = \frac{\partial\mathscr{N}}{\partial\tau} + \frac{\partial\mathscr{N}}{\partial x^i}\frac{dx^i}{d\tau} + \frac{\partial\mathscr{N}}{\partial p^i}\frac{dp^i}{d\tau} = p^0\frac{\partial\mathscr{N}}{\partial t} + p^i\frac{\partial\mathscr{N}}{\partial x^i} = p^\beta\frac{\partial\mathscr{N}}{\partial x^\beta}. \tag{22.18.4}$$

in a local Lorentz frame. Thus,

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} = \int \frac{\partial\mathscr{N}}{\partial x^\beta} p^\alpha p^\beta \frac{dp^x dp^y dp^z}{p^0} = 0. \tag{22.18.5}$$

22.19 KINETIC THEORY FOR NONIDENTICAL PARTICLES

For a swarm of particles with a wide distribution of rest masses, define

$$\mathscr{N} = \frac{\Delta N}{\mathscr{V}_x\mathscr{V}_p\Delta m}, \tag{22.19.1}$$

where $\mathscr{V}_x$ and $\mathscr{V}_p$ are spatial and momentum 3-volumes, and ΔN is the number of particles in the region $\mathscr{V}_x\mathscr{V}_p$ with rest masses between $m - \Delta m/2$ and $m + \Delta m/2$. Show the following.

(a) $\mathscr{V}_x\mathscr{V}_p\Delta m$ is independent of Lorentz frame and independent of location on the world tube of a bundle of particles.

(b) $\mathscr{N}$ can be regarded as a function of location $\mathscr{P}$ in spacetime and 4-momentum $\boldsymbol{p}$ inside the future light cone of the tangent space at $\mathscr{P}$:

$$\mathscr{N} = \mathscr{N}(\mathscr{P}, \boldsymbol{p}). \tag{22.19.2}$$

(c) $\mathscr{N}$ satisfies the collisionless Boltzmann equation (kinetic equation)

$$\frac{d\mathscr{N}(\mathscr{P}(\lambda), \boldsymbol{p}(\lambda))}{d\lambda} = 0 \text{ along geodesic trajectory of any particle.} \tag{22.19.3}$$

(d) $\mathscr{N}$ can be rewritten in a local Lorentz frame as

$$\mathscr{N} = \frac{\Delta N}{[(p^0/m)\Delta x\Delta y\Delta z][\Delta p^0\Delta p^x\Delta p^y\Delta p^z]}. \tag{22.19.4}$$

(e) The stress-energy tensor at an event $\mathscr{P}$ can be written as an integral over the interior of the future light cone of momentum space

$$T^{\mu\nu} = \int (\mathscr{N} p^\mu p^\nu) m^{-1} dp^0 dp^1 dp^2 dp^3 \tag{22.19.5}$$

in a local Lorentz frame (Track-l notation for integral; see Box 5.3);

$$\begin{aligned} \boldsymbol{T} &= \int (\mathscr{N}\boldsymbol{p}\otimes\boldsymbol{p}) m^{-1} {}^*1 \text{ in frame-independent notation} \\ &= \int (\mathscr{N}\boldsymbol{p}\otimes\boldsymbol{p}) m^{-1} \boldsymbol{d}p^0\wedge\boldsymbol{d}p^1\wedge\boldsymbol{d}p^2\wedge\boldsymbol{d}p^3 \end{aligned} \tag{22.19.6}$$

in a local Lorentz frame (Track-2 notation; see Box 5.4).

Solution:

(a) From a sarm particles whose rest masses range between $m - \Delta m/2$ and $m + \Delta m/2$, we have on-shell condition

$$(m - \Delta m/2)^2 < -\boldsymbol{p}\cdot\boldsymbol{p} < (m + \Delta m/2)^2. \tag{22.19.7}$$

In the limit that $\Delta m \to 0$, the proof of invariance of phase volume in the case of a swarm of identical particles can be generalized directly here.

(b) As $g_{\alpha\beta}p^\alpha p^\beta = -m^2$, p^0 can be determined once m and p^i is known. So we can get $\mathscr{N}(\mathscr{P}, \boldsymbol{p})$ from $\mathscr{N}(\mathscr{P}, p^i, m)$. And we have $\boldsymbol{p}\cdot\boldsymbol{p} = -m^2 < 0$, i.e. the 4-momentum $\boldsymbol{p}$ is inside the future light cone of the tangent space at $\mathscr{P}$.

(c) The rest mass of the particles are invariant along geodesics. So the collisionless Boltzmann equation in this exercise is the same as that of identical particles.

(d) In local Lorentz frame, we have $p^0\Delta p^0 = m\Delta m$ when p^is are fixed. So we have

$$\mathscr{N} = \frac{\Delta N}{[(p^0/m)\Delta x\Delta y\Delta z][\Delta p^0\Delta p^x\Delta p^y\Delta p^z]}. \tag{22.19.8}$$

(e) In local Lorentz frame, we have

$$T^{\mu\nu} = \int dm \int \mathscr{N} p^\mu p^\nu (d\mathscr{V}_p/p^0) = \int m^{-1}\mathscr{N} p^\mu p^\nu d\mathscr{V}_p dp^0 = \int m^{-1}\mathscr{N} p^\mu p^\nu dp^0 dp^1 dp^2 dp^3. \tag{22.19.9}$$

Chapter 23

SPHERICAL STARS

23.1 ISOTROPIC COORDINATES AND NEWTONIAN LIMIT

An alternative set of coordinates sometimes used for static, spherical systems is the "isotropic coordinate system" $(t, \bar{r}, \theta, \phi)$. The metric in isotropic coordinates has the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\mu}[d\bar{r}^2 + \bar{r}^2 d\Omega^2], \tag{23.1.1}$$

where Φ and μ being functions of $\bar{r}$.

(a) Exhibit the coordinate transformation connecting the Schwarzschild coordinates (23.7) to the isotropic coordinates 23.1.1.

(b) From equation (16.2a) [or equivalently (18.15c)], show that, in the Newtonian limit, the metric coefficient Φ of the isotropic line element becomes the Newtonian potential; and μ becomes equal to $-\Phi$. By combining with part (a), discover that $\Lambda = rd\Phi/dr$ in the Newtonian limit.

Solution:

(a) The Schwarzschild metric is

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2. \tag{23.1.2}$$

By comparision, we have

$$r = \bar{r}e^{\mu(\bar{r})} \tag{23.1.3}$$

and so

$$dr = e^{\mu(\bar{r})}(1 + \mu'\bar{r})d\bar{r}. \tag{23.1.4}$$

Then we can get

$$e^{2\Lambda}(1 + \mu'\bar{r})^2 = 1. \tag{23.1.5}$$

(b) In the Newtonian limit, we have

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 + 2\mu)(d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{23.1.6}$$

From equation (18.15c), we know Φ is Newtonian potential and $\mu = -\Phi$. From equation 23.1.3, we have

$$r = \bar{r}(1 - \Phi(\bar{r})). \tag{23.1.7}$$

From equation 23.1.5, we have

$$\Lambda = \frac{d\Phi}{d\bar{r}}\bar{r}. \tag{23.1.8}$$

Keeping the first order of Φ, we have

$$\Lambda = \frac{d\Phi}{dr} r. \tag{23.1.9}$$

23.2 PROPER REFERENCE FRAMES OF FLUID ELEMENTS

(a) Verify that equations (23.15a,b) define an orthonormal tetrad and its dual basis of 1-forms, at each event in spacetime.

(b) Verify that the components of the fluid 4-velocity relative to these tetrads are given by equations (23.15c). Why do these components guarantee that the tetrads form "proper reference frames" for the fluid elements?

(c) Verify equations (23.15d) for the components of the stress-energy tensor.

Solution:

(a) The Schwarzschild metric is

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{23.2.1}$$

It is easy to verify

$$\boldsymbol{e}_{\hat{\alpha}} \cdot \boldsymbol{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \tag{23.2.2}$$

and

$$\boldsymbol{\omega}^{\hat{\alpha}} \cdot \boldsymbol{\omega}^{\hat{\beta}} = \eta^{\hat{\alpha}\hat{\beta}} \tag{23.2.3}$$

from (23.15a,b) in the textbook. So they define an orthonormal tetrad and its dual basis of 1-forms, at each event in spacetime.

(b) In Schwarzschild metric, $u^{\alpha} = (e^{-\Phi}, 0, 0, 0)$. As $\boldsymbol{e}_{\hat{t}} = e^{-\Phi}\partial_t$, we have $\boldsymbol{u} = \boldsymbol{e}_{\hat{t}}$. Since $u^{\hat{\alpha}} = (1, 0, 0, 0)$ in this frame, we can say it is the proper reference frame for the fluid elements.

(c) Recall that

$$T_{\hat{\alpha}\hat{\beta}} = (\rho + p) u_{\hat{\alpha}} u_{\hat{\beta}} + p\eta_{\hat{\alpha}\hat{\beta}}. \tag{23.2.4}$$

We have

$$T_{\hat{t}\hat{t}} = \rho, \quad T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p \tag{23.2.5}$$

and $T_{\hat{\alpha}\hat{\beta}} = 0$ if $\hat{\alpha} \neq \hat{\beta}$.

23.3 LAW OF LOCAL ENERGY-MOMENTUM CONSERVATION

Evaluate the four components of the equation $T^{\alpha\beta}{}_{;\beta}$ for the stress-energy tensor (23.14) in the Schwarzschild coordinate system of equation (23.7).

Solution:

In Schwarzschild coordinate system, the stress energy tensor is

$$T^{\alpha\beta} = \begin{pmatrix} \rho e^{-2\Phi} & 0 & 0 & 0 \\ 0 & p e^{-2\Lambda} & 0 & 0 \\ 0 & 0 & p r^{-2} & 0 \\ 0 & 0 & 0 & p r^{-2} \sin^{-2}\theta \end{pmatrix}. \tag{23.3.1}$$

The nonvanishing connection coefficients are

$$\begin{aligned}
&\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \Phi';\\
&\Gamma^r{}_{tt} = e^{2\Phi-2\Lambda}\Phi'; \quad \Gamma^r{}_{rr} = \Lambda'; \quad \Gamma^r{}_{\theta\theta} = -re^{-2\Lambda}; \quad \Gamma^r{}_{\phi\phi} = -re^{-2\Lambda}\sin^2\theta;\\
&\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = r^{-1}; \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta;\\
&\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = r^{-1}; \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \sin^{-1}\theta\cos\theta.
\end{aligned} \tag{23.3.2}$$

Note that

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta}T^{\mu\beta} + \Gamma^\beta{}_{\mu\beta}T^{\alpha\mu}. \tag{23.3.3}$$

So the r component gives

$$\begin{aligned}
0 &= T^{rr}{}_{,r} + \Gamma^r{}_{tt}T^{tt} + \Gamma^r{}_{rr}T^{rr} + \Gamma^r{}_{\theta\theta}T^{\theta\theta} + \Gamma^r{}_{\phi\phi}T^{\phi\phi} + (\Gamma^t{}_{rt} + \Gamma^r{}_{rt} + \Gamma^\theta{}_{r\theta} + \Gamma^\phi{}_{r\phi})T^{rr}\\
&= e^{-2\Lambda}(p_{,r} - 2\Lambda' p) + e^{-2\Lambda}\Phi'\rho + e^{-2\Lambda}\Lambda' p - 2e^{-2\Lambda}r^{-1}p + e^{-2\Lambda}(\Phi' + \Lambda' + 2r^{-1})p\\
&= e^{-2\Lambda}p_{,r} + e^{-2\Lambda}\Phi'(\rho + p),
\end{aligned} \tag{23.3.4}$$

i.e.

$$p_{,r} = -(\rho + p)\Phi'. \tag{23.3.5}$$

23.4 EINSTEIN CURVATURE TENSOR

Calculate the components of the Einstein curvature tensor, $G_{\alpha\beta}$in Schwarzschild coordinates. Then perform a transformation to obtain $G_{\hat\alpha\hat\beta}$ the components in the orthonormal frame of equations (23.15a,b).

Solution:

The calculation of Einstein curvature tensor is straight forward. We obtain

$$\begin{aligned}
G_{tt} &= \frac{\left[2\,re^{2\,\Phi}\Lambda' + \left(e^{2\,\Lambda} - 1\right)e^{2\,\Phi}\right]e^{-2\,\Lambda}}{r^2};\\
G_{rr} &= \frac{2\,r\Phi' - e^{2\,\Lambda} + 1}{r^2};\\
G_{\theta\theta} &= \left[r^2\Phi'^2 + r^2\Phi'' - r\Lambda' - \left(r^2\Lambda' - r\right)\Phi'\right]e^{-2\,\Lambda};\\
G_{\phi\phi} &= \left[r^2\Phi'^2 + r^2\Phi'' - r\Lambda' - \left(r^2\Lambda' - r\right)\Phi'\right]e^{-2\,\Lambda}\sin^2\theta.
\end{aligned} \tag{23.4.1}$$

All other terms would vanish. After a transformation to proper frame, we have

$$\begin{aligned}
G_{\hat t\hat t} &= \frac{\left(2\,r\Lambda' + e^{2\,\Lambda} - 1\right)e^{-2\,\Lambda}}{r^2};\\
G_{\hat r\hat r} &= \frac{\left(2\,r\Phi' - e^{2\,\Lambda} + 1\right)e^{-2\,\Lambda}}{r^2};\\
G_{\hat\theta\hat\theta} &= \left[\Phi'^2 + \Phi'' - \frac{1}{r}\Lambda' - \left(\Lambda' - \frac{1}{r}\right)\Phi'\right]e^{-2\,\Lambda};\\
G_{\hat\phi\hat\phi} &= \left[\Phi'^2 + \Phi'' - \frac{1}{r}\Lambda' - \left(\Lambda' - \frac{1}{r}\right)\Phi'\right]e^{-2\,\Lambda}.
\end{aligned} \tag{23.4.2}$$

23.5 TOTAL NUMBER OF BARYONS IN A STAR

Show that, if $r = R$ is the location of the surface of a static star, then the total number of baryons inside the star is

$$A = \int_0^R 4\pi r^2 n e^{\Lambda} dr. \tag{23.5.1}$$

Solution:

The space-part metric in Schwarzschild coordinates are

$$ds^2 = e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{23.5.2}$$

So we have

$$g^{1/2} = e^{\Lambda} r^2 \sin\theta. \tag{23.5.3}$$

Then the total number of baryons inside the star is

$$\int n(r) g^{1/2} dr d\theta d\phi = \int_0^R n e^{\Lambda} r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \int_0^R 4\pi r^2 n e^{\Lambda} dr. \tag{23.5.4}$$

23.6 BUOYANT FORCE IN A STAR

An observer at rest at some point inside a relativistic star measures the radial pressure-buoyant force, $F_{\rm buoy}$, on a small fluid element of volume V. Let him use the usual laboratory techniques. Do not confuse him by telling him he is in a relativistic star. What value will he find for $F_{\rm buoy}$, in terms of ρ, p, m, V, and dp/dr? If he equates this buoyant force to an equal and opposite gravitational force, $F_{\rm grav}$, what will $F_{\rm grav}$ be in terms of ρ, p, m, V, and r? (Use equation 23.22.) How do these results differ from the corresponding Newtonian results?

Solution:

For an observer at rest in Schwarzschild coordinate frame, we have $u^{\alpha} = (e^{-\Phi}, 0, 0, 0)$. The accerleration of the observer is

$$a^0 = \Gamma^0{}_{\alpha\beta} u^{\alpha} u^{\beta} = 0; \quad a^i = \Gamma^i{}_{\alpha\beta} u^{\alpha} u^{\beta} = e^{-2\Phi} \Gamma^i{}_{00}, \tag{23.6.1}$$

i.e. $\boldsymbol{a} = e^{-2\Lambda}\Phi' \partial_r = e^{-\Lambda}\Phi' \boldsymbol{e}_{\hat{r}}$. So

$$\boldsymbol{F}_{\rm buoy} = (\rho + p) V \boldsymbol{a} = -V e^{-\Lambda} p_{,r} \boldsymbol{e}_{\hat{r}}. \tag{23.6.2}$$

Note that $V = \Delta S \Delta r e^{\Lambda}$, we can also obtain this by

$$\boldsymbol{F}_{\rm buoy} = -\Delta S (p(r + \Delta r) - p(r)) \boldsymbol{e}_{\hat{r}} = -\Delta S p_{,r} \Delta r \boldsymbol{e}_{\hat{r}} = -V e^{-\Lambda} p_{,r} \boldsymbol{e}_{\hat{r}}. \tag{23.6.3}$$

And straightforwardly, we have

$$\boldsymbol{F}_{\rm grav} = V e^{-\Lambda} p_{,r} \boldsymbol{e}_{\hat{r}} = -V e^{-\Lambda} \frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} \boldsymbol{e}_{\hat{r}}. \tag{23.6.4}$$

In Newtonian limit, we have

$$\boldsymbol{F}_{\rm buoy} = -\boldsymbol{F}_{\rm grav} = \frac{\rho V m}{r^2} = \boldsymbol{e}_{\hat{r}}. \tag{23.6.5}$$

23.7 GRAVITATIONAL ENERGY OF A NEWTONIAN STAR

Calculate in Newtonian theory the energy one would gain from gravity if one were to construct a star by adding one spherical shell of matter on top of another, working from the inside outward. Use Laplace's equation $(r^2\Phi_{,r})_{,r} = 4\pi r^2\rho$ and the equation of hydrostatic equilibrium $p_{,r} = -\rho\Phi_{,r}$ to put the answer in the following equivalent forms:

$$\begin{aligned}(\text{energy gained from gravity}) &\equiv -(\text{gravitational potential energy})\\ &= \int_0^R (\rho r\Phi_{,r})4\pi r^2 dr = \int_0^R (\rho m/r)4\pi r^2 dr\\ &= -\frac{1}{2}\int_0^R (\rho\Phi)4\pi r^2 dr = \frac{1}{8\pi}\int_0^\infty (\Phi_{,r})^2 4\pi r^2 dr\\ &= 3\int_0^R 4\pi r^2 p dr. \end{aligned} \tag{23.7.1}$$

Solution:

Suppose we already have a sphere with mass m and radius r. If we want to add one spherical shell with thichness dr of matter on top of it, the work that gravity have done is

$$dW = (\rho 4\pi r^2 dr)\times\left(\frac{m}{r}\right) = 4\pi\rho m r dr. \tag{23.7.2}$$

So the energy gained from gravity is

$$W = \int_0^R 4\pi\rho m r dr. \tag{23.7.3}$$

Note that $\Phi_{,r} = m/r^2$ and $(r^2\Phi_{,r})_{,r} = 4\pi r^2\rho$, we have

$$W = \int_0^R r\Phi_{,r}4\pi\rho r^2 dr = \int_0^\infty r\Phi_{,r}(r^2\Phi_{,r})_{,r}dr. \tag{23.7.4}$$

On the one hand, we have

$$W = 2\int_0^\infty r^2(\Phi_{,r})^2 dr + \int_0^\infty r^3\Phi_{,r}\Phi_{,rr}dr. \tag{23.7.5}$$

On the other hand, we have

$$W = r^3(\Phi_{,r})^2|_0^\infty - \int_0^\infty r^2\Phi_{,r}d(r\Phi_{,r}) = -\int_0^\infty r^2(\Phi_{,r})^2 dr - \int_0^\infty r^3\Phi_{,r}\Phi_{,rr}dr. \tag{23.7.6}$$

So we have

$$3\int_0^\infty r^2(\Phi_{,r})^2 dr + 2\int_0^\infty r^3\Phi_{,r}\Phi_{,rr}dr = 0 \tag{23.7.7}$$

and

$$W = \frac{1}{2}\int_0^\infty r^2(\Phi_{,r})^2 dr = \frac{1}{8\pi}\int_0^\infty (\Phi_{,r})^2 4\pi r^2 dr. \tag{23.7.8}$$

Furthermore, we have

$$W = \frac{1}{2}r^2\Phi_{,r}\Phi|_0^\infty - \frac{1}{2}\int_0^\infty \Phi(r^2\Phi_{,r})_{,r}dr = -\frac{1}{2}\int_0^R (\rho\Phi)4\pi r^2 dr \tag{23.7.9}$$

Note that $\rho\Phi_{,r} = -p_{,r}$. We have

$$W = \int_0^R r\Phi_{,r}4\pi\rho r^2 dr = -\int_0^R 4\pi r^3 p_{,r}dr = -4\pi r^3 p|_0^\infty + \int_0^R 4\pi p d(r^3) = 3\int_0^R 4\pi r^2 p dr. \tag{23.7.10}$$

23.8 NEWTONIAN STARS OF UNIFORM DENSITY

Calculate the structures of uniform-density configurations in Newtonian theory. Show that the relativistic configurations of Box 23.2 become identical to the Newtonian configurations in the weak-gravity limit. Also show that there are no mass or radius limits in Newtonian theory.

Solution:

In Newtonian theory, we have

- mass equation:
$$m = \int_0^r 4\pi r^2 \rho dr; \tag{23.8.1}$$
- equation of hydrostatic equilibrium:
$$\frac{dp}{dr} = -\frac{\rho m}{r^2}, \text{ with } p(r=0) = p_c = \text{ central pressure}; \tag{23.8.2}$$
- source equation for Φ:
$$\frac{d\Phi}{dr} = \frac{m}{r^2}. \tag{23.8.3}$$

If the density is uniform, we have $m = 4\pi\rho_0 r^3/3$ for $r < R$, and so

$$\frac{dp}{dr} = -\frac{4\pi}{3}\rho_0^2 r; \quad \frac{d\Phi}{dr} = \frac{4\pi}{3}\rho_0 r. \tag{23.8.4}$$

When $r > R$, we have $m = M = 4\pi\rho_0 R^3/3$, and so

$$\frac{d\Phi}{dr} = \frac{M}{r^2}. \tag{23.8.5}$$

The solution is

$$p(r) = \begin{cases} \frac{2\pi}{3}\rho_0^2(R^2 - r^2) & r < R \\ 0 & r > R \end{cases} \qquad \Phi(r) = \begin{cases} -\frac{M}{R} + \frac{2\pi}{3}\rho_0(r^2 - R^2) & r < R \\ -\frac{M}{r} & r > R \end{cases} \tag{23.8.6}$$

In GR theory, we have

$$p = \rho_0 \left[\frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}} \right] \tag{23.8.7}$$

when $r < R$ and

$$e^{\Phi} = \begin{cases} \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2} & r < R \\ \left(1 - \frac{2M}{r}\right)^{1/2} & r > R \end{cases} \tag{23.8.8}$$

In the limit $M \ll R$, we have

$$p = \rho_0 \frac{1 - \frac{Mr^2}{R^3} - 1 + \frac{M}{R}}{3(1 - \frac{M}{R}) - 1 + \frac{Mr^2}{R^3}} = \frac{\rho_0}{2}\left(\frac{M}{R} - \frac{Mr^2}{R^3}\right) = \frac{2\pi}{3}\rho_0^2(R^2 - r^2) \tag{23.8.9}$$

when $r < R$ and

$$1 + \Phi = \begin{cases} \frac{3}{2}\left(1 - \frac{M}{R}\right) - \frac{1}{2}\left(1 - \frac{Mr^2}{R^3}\right) & r < R \\ \left(1 - \frac{M}{r}\right) & r > R \end{cases} \tag{23.8.10}$$

i.e.

$$\Phi(r) = \begin{cases} -\frac{M}{R} + \frac{2\pi}{3}\rho_0(r^2 - R^2) & r < R \\ -\frac{M}{r} & r > R \end{cases} \tag{23.8.11}$$

So we recover the Newtonian theory. Clearly, p is always finite in Newtonian theory and so there are no mass or radius limits in Newtonian theory.

23.9 GOOD BEHAVIOR OF r EXERCISES

Carry out explicitly the full details of the proof, at the beginning of this section, that $2m/r$ is always less than unity and r is a monotonic function of l.

Solution:

We have OV equation

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} \tag{23.9.1}$$

and boundary condition $p(r = 0) = p_c$ and $p(r = R) = 0$.

23.10 CENTER OF STAR OCCUPIED BY IDEAL FERMI GAS AT EXTREME RELATIVISTIC LIMIT

Opposite to the idealization of a star built from an incompressible fluid is the idealization in which it is built from an ideal Fermi gas at zero temperature, so highly compressed that the particles have relativistic energies, in comparison with which any rest mass they possess is negligible. In this limit, with two particles per occupIed cell of volume h^3 in phase space, one has

$$n = (2/h^3)4\pi \int_0^{p_F} p^2 dp = 8\pi p_F^3/3h^3, \quad \rho = (2/h^3)4\pi \int_0^{p_F} cp \cdot p^2 dp = 2\pi c p_F^4/h^3, \tag{23.10.1}$$

and finally

$$p = -\frac{d(\rho/n)}{d(1/n)} = 2\pi c p_F^4/3h^3 = \rho/3, \tag{23.10.2}$$

as if one were dealing with radiation instead of particles (p_F = Fermi momentum; momentum of highest occupied state).

(a) Write out the relativistic equation of hydrostatic equilibrium for a substance satisfying the equation of state $p = \rho/3$.

(b) Show that there exists a well-defined analytic solution for the limiting case of infinite central density, in which $m(r)/r$ has the value $3/14$.

(c) Find $\rho(r)$, $p(r)$ and $n(r)$.

(d) Show that the number of particles out to any finite r-value is finite, despite the fact that $n(r)$ is infinite at the origin.

(e) Show that the 3-geometry has a "conical singularity" at $r = 0$.

(f) Make an "embedding diagram" for this 3-geometry.

Solution:

(a) Substitute the equation of state into OV equation and we can get

$$\frac{d\rho}{dr} = -\frac{4\rho(3m + 4\pi\rho r^3)}{3r(r - 2m)}. \tag{23.10.3}$$

(b) Try the power law solution $\rho = Ar^n$. We find that

$$m = \int_0^r 4\pi\rho r^2 dr = 4\pi A \frac{r^{n+3}}{n+3} \tag{23.10.4}$$

and

$$nAr^{n-1} = -4Ar^n \frac{12\pi A \frac{r^{n+3}}{n+3} + 4\pi A r^{n+3}}{3r^2 - 24\pi A \frac{r^{n+4}}{n+3}} \tag{23.10.5}$$

We can get

$$n = -2, \quad A = \frac{3}{56\pi}. \tag{23.10.6}$$

Thus

$$m = \frac{3}{14} r. \tag{23.10.7}$$

(c) Obviously, we have

$$\rho(r) = \frac{3}{56\pi} r^{-2}, \quad p(r) = \frac{1}{56\pi} r^{-2}. \tag{23.10.8}$$

From 23.10.1, we can derive that

$$n(r) = \frac{8\pi}{3h^3} \left(\frac{h^3\rho}{2\pi}\right)^{3/4} = \frac{1}{3} \left(\frac{3}{7h}\right)^{3/4} \pi^{-1/2} r^{-3/2}. \tag{23.10.9}$$

(d) The number of particles out to any finite r-value is

$$N = \int_0^r 4\pi n(r') r'^2 dr' = \frac{8}{9} \left(\frac{3}{7h}\right)^{3/4} \pi^{1/2} r^{3/2}, \tag{23.10.10}$$

which is finite.

(e) The 3-geometry is

$$ds^2 = \frac{dr^2}{1-2m/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = \frac{7}{4} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{23.10.11}$$

For a conical surface with half opening angle α, we have

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 = \left(\frac{1}{\tan^2\alpha} + 1\right) dr^2 + r^2 d\phi^2. \tag{23.10.12}$$

Now it is easy to see the 3-geometry has a "conical singularity" at $r = 0$.

(f) The lift is

$$z(r) = \int_0^r \frac{dr'}{(r'/2m - 1)^{1/2}} = \frac{\sqrt{3}}{2} r. \tag{23.10.13}$$

Chapter 25

PARTICLE MOTION IN SCHWARZSCHILD GEOMETRY

25.1 CONSTANT OF MOTION OBTAINED FROM HAMIlTON'S PRINCIPLE

Prove the above theorem of conservation of $p_{\rm K} \equiv \boldsymbol{p}\cdot\boldsymbol{\xi}$ from Hamilton's principle

$$\delta\int \frac{1}{2}g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda = 0 \tag{25.1.1}$$

as applied to geodesic paths. Recall: In this action principle, $g_{\mu\nu}$ is to be regarded as a known function of position, x, along the path; and the path itself, $x^\mu(\lambda)$, is to be varied.

Solution:

We choose λ properly so that $p^\mu = dx^\mu/d\lambda$. The perturbation of the path is $\delta x^\mu(\lambda)$. Then we have

$$\delta L = \int \frac{1}{2}g_{\mu\nu,\alpha}\delta x^\alpha p^\mu p^\nu d\lambda + \int p_\mu \frac{d\delta x^\mu}{d\lambda}d\lambda = p_\mu \delta x^\mu|_{\lambda_1}^{\lambda_2} - \int\left(\frac{dp_\alpha}{d\lambda} - \frac{1}{2}g_{\mu\nu,\alpha}p^\mu p^\nu\right)\delta x^\alpha d\lambda. \tag{25.1.2}$$

The Hamilton's principle ensures that $\delta L = 0$ when $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$. So generally, we have

$$\delta L = p_\mu(\lambda_2)\delta x^\mu(\lambda_2) - p_\mu(\lambda_1)\delta x^\mu(\lambda_1). \tag{25.1.3}$$

If we choose $\delta x^\mu(\lambda) = \epsilon\xi^\mu[x^\mu(\lambda)]$ where ξ^μ is killing vector field, we have

$$\delta L = \epsilon[p_\mu(\lambda_2)\xi^\mu(\lambda_2) - p_\mu(\lambda_1)\xi^\mu(\lambda_1)]. \tag{25.1.4}$$

On the other hand, we can calculate δL as

$$\begin{aligned}
\delta L &= \frac{1}{2}\epsilon\int\left(g_{\mu\nu,\alpha}\xi^\alpha\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\xi^\mu{}_{,\alpha}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\xi^\nu{}_{,\alpha}\frac{dx^\alpha}{d\lambda}\right)d\lambda \\
&= \frac{1}{2}\epsilon\int\left(g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda \\
&= \frac{1}{2}\epsilon\int\left(\xi_{\mu;\nu} + \xi_{\nu;\mu}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda \\
&= 0.
\end{aligned} \tag{25.1.5}$$

Now we find

$$p_\mu(\lambda_2)\xi^\mu(\lambda_2) - p_\mu(\lambda_1)\xi^\mu(\lambda_1) = 0, \tag{25.1.6}$$

i.e. $p_{\rm K} \equiv \boldsymbol{p}\cdot\boldsymbol{\xi}$ is conserved.

25.2 SUPER-HAMilTONIAN FORMALISM FOR GEODESIC MOTION

Show that a set of differential equations in Hamiltonian form results from varying p_μ and x^μ independently in the variational principle $\delta I = 0$, where

$$I = \int (p_\mu dx^\mu - \mathcal{H} d\lambda) \tag{25.2.1}$$

and

$$\mathcal{H} \equiv \frac{1}{2} g_{\mu\nu}(x) p^\mu p^\nu. \tag{25.2.2}$$

Show that the "super-Hamiltonian" $\mathcal{H}$ is a constant of motion, and that the solutions of these equations are geodesics. What do the choices $\mathcal{H} = 1/2$, $\mathcal{H} = 0$, $\mathcal{H} = -1/2\mu^2$, or $\mathcal{H} = -1/2$ mean for the geodesic and its parametrization?

Solution:

The variation of action is given by

$$\begin{aligned}\delta I &= \int \delta p_\mu dx^\mu + p_\mu d\delta x^\mu - \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu d\lambda - \frac{\partial \mathcal{H}}{\partial p_\mu}\delta p_\mu d\lambda \\ &= \int \left(\frac{dx^\mu}{d\lambda} - \frac{\partial \mathcal{H}}{\partial p_\mu}\right)\delta p_\mu d\lambda - \left(\frac{dp_\mu}{d\lambda} + \frac{\partial \mathcal{H}}{\partial x^\mu}\right)\delta x^\mu d\lambda + p_\mu \delta x^\mu|_{\lambda_1}^{\lambda_2}.\end{aligned} \tag{25.2.3}$$

Equation of motion can be obtained from $\delta I = 0$,

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu}; \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu}. \tag{25.2.4}$$

So we have

$$\frac{d\mathcal{H}}{d\lambda} = \frac{\partial \mathcal{H}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} + \frac{\partial \mathcal{H}}{\partial p_\mu}\frac{dp_\mu}{d\lambda} = -\frac{dp_\mu}{d\lambda}\frac{dx^\mu}{d\lambda} + \frac{dx^\mu}{d\lambda}\frac{dp_\mu}{d\lambda} = 0, \tag{25.2.5}$$

i.e. $\mathcal{H}$ is a constant of motion. Note that $\mathcal{H} \equiv g^{\mu\nu}p_\mu p_\nu/2$. We can get

$$\frac{dx^\mu}{d\lambda} = p^\mu; \quad \frac{dp_\mu}{d\lambda} + \frac{1}{2} g^{\alpha\beta}{}_{,\mu} p_\alpha p_\beta = 0, \tag{25.2.6}$$

After some algebra manipulations, we arrive at

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{25.2.7}$$

So the solutions of these equations are geodesics.

- $\mathcal{H} = 1/2$: geodesic is spacelike and λ is proper length.
- $\mathcal{H} = 0$: geodesic is lightlike and λ is so parametrized that $dx^\mu/d\lambda$ is propotional to momentum.
- $\mathcal{H} = -1/2\mu^2$: geodesic is timelike and λ is τ/μ, where τ is proper time.
- $\mathcal{H} = -1/2$: geodesic is timelike and λ is τ.

25.3 KILLING VECTORS IN FLAT SPACETIME

Find ten Killing vectors in flat Minkowski spacetime that are linearly independent. (Restrict attention to linear relationships with constant coefficients).

Solution:

We have four translation killing vectors

$$(1,0,0,0), \quad (0,1,0,0), \quad (0,0,1,0), \quad (0,0,0,1) \tag{25.3.1}$$

and six rotation killing vectors

$$(x,-t,0,0), \quad (y,0,-t,0), \quad (z,0,0,-t), \quad (0,y,-x,0), \quad (0,z,0,-x), \quad (0,0,z,-y). \tag{25.3.2}$$

25.4 POISSON BRACKET AS KEY TO CONSTANTS OF MOTION

If $\boldsymbol{\xi}$ is a Killing vector, show that $p_{\rm K} \equiv \xi^\mu p_\mu$ commutes (has vanishing Poisson bracket) with the Hamiltonian $\mathcal{H}$ of the previous problem, $[\mathcal{H}, p_{\rm K}] = 0$, so $dp_{\rm K}/d\lambda = 0$. (Hint: Use a convenient coordinate system.)

Solution:

In local Lorentz frame, we have

$$\begin{aligned}[\mathcal{H}, p_{\rm K}] &= \frac{1}{2}[\eta^{\mu\nu}p_\mu p_\nu, \xi^\alpha(x)p_\alpha] = \frac{1}{2}\eta^{\mu\nu}p_\alpha[p_\mu p_\nu, \xi^\alpha(x)] \\ &= p^\mu p_\alpha[p_\mu, \xi^\alpha(x)] = -p^\mu p_\alpha \xi^\alpha{}_{,\mu} \\ &= -\frac{1}{2}p^\mu p^\alpha(\xi_{\alpha,\mu} + \xi_{\mu,\alpha}) = 0. \end{aligned} \tag{25.4.1}$$

So $dp_{\rm K}/d\lambda = 0$.

25.5 COMMUTATOR OF Killing VECTORS IS A Killing VECTOR

Consider two Killing vectors, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, which happen not to commute [as differential operators; i.e., the commutator of equations (8.13) does not vanish; consider rotations about perpendicular directions as a case in point]; thus,

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] \equiv \boldsymbol{\zeta} \neq 0. \tag{25.5.1}$$

(a) Show that no single coordinate system can be simultaneously adapted, in the sense of equation (25.1), to both the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ symmetries (see exercise 9.9).

(b) Let $p_\xi \equiv p_\mu \xi^\mu$, $p_\eta \equiv p_\mu \eta^\mu$, and $p_\zeta \equiv p_\mu \zeta^\mu$, and derive the Poisson-bracket relationship $[p_\xi, p_\eta] = -p_\zeta$. In a geometry, the symmetries of which are related in this way, show that p_ζ is also a constant of motion.

(c) In a coordinate system where $\boldsymbol{\zeta} = (\partial/\partial x^{\rm K})$, define $\mathcal{H}$ as in (25.10) and show from $[\mathcal{H}, p_\zeta] = 0$ that $\boldsymbol{\zeta}$ is a Killing vector.

Thus the commutator of two Killing vectors is itself a Killing vector.

Solution:

(a) If there is a single coordinate system which can be simultaneously adapted to both the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ symmetries, then we have $\boldsymbol{\xi} \equiv (\partial/\partial x^{\rm K_1})$, $\boldsymbol{\eta} \equiv (\partial/\partial x^{\rm K_2})$. It is easy to see that $[\boldsymbol{\xi}, \boldsymbol{\eta}] = [\partial/\partial x^{\rm K_1}, \partial/\partial x^{\rm K_2}] = 0$. So if $[\boldsymbol{\xi}, \boldsymbol{\eta}] \neq 0$, no single coordinate system can be simultaneously adapted to both the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ symmetries.

(b) Direct calculations show that

$$\begin{aligned}[p_\xi, p_\eta] &= [p_\mu \xi^\mu, p_\nu \eta^\nu] \\ &= \frac{\partial p_\mu \xi^\mu}{\partial x^\alpha}\frac{\partial p_\nu \eta^\nu}{\partial p_\alpha} - \frac{\partial p_\mu \xi^\mu}{\partial p_\alpha}\frac{\partial p_\nu \eta^\nu}{\partial x^\alpha} \\ &= p_\mu \xi^\mu{}_{,\nu}\eta^\nu - p_\nu \xi^\mu \eta^\nu{}_{,\mu} \\ &= -p_\mu(\xi^\nu \eta^\mu{}_{,\nu} - \xi^\mu{}_{,\nu}\eta^\nu) \\ &= -p_\mu \zeta^\mu \\ &= -p_\zeta. \end{aligned} \tag{25.5.2}$$

Since $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is Killing vectors, we have $[\mathcal{H}, p_\xi] = [\mathcal{H}, p_\eta] = 0$. Thus we find that

$$[\mathcal{H}, p_\zeta] = -[\mathcal{H}, [p_\xi, p_\eta]] = [p_\xi, [p_\eta, \mathcal{H}]] + [p_\eta, [\mathcal{H}, p_\xi]] = 0, \tag{25.5.3}$$

i.e. p_ζ is also a constant of motion.

(c) In this coordinate, we have

$$0 = [\mathcal{H}, p_\zeta] = \frac{1}{2}[g^{\alpha\beta}p_\alpha p_\beta, p_{\mathrm{K}}] = \frac{1}{2}[g^{\alpha\beta}, p_{\mathrm{K}}]p_\alpha p_\beta = \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^{\mathrm{K}}}p_\alpha p_\beta \tag{25.5.4}$$

for any p_α. So we know $\partial g^{\alpha\beta}/\partial x^{\mathrm{K}} = 0$, i.e. $\boldsymbol{\zeta}$ is a Killing vector.

25.6 EIGENVALUE PROBLEM FOR KIllING VECTORS

Show that any Killing vector satisfies $\xi^\mu{}_{;\mu} = 0$, and is an eigenvector with eigenvalue $\kappa = 0$ of the equation

$$\xi^{\mu;\nu}{}_{;\nu} + R^\mu{}_\nu \xi^\nu = \kappa \xi^\mu. \tag{25.6.1}$$

Find a variational principle (Raleigh-Ritz type) for this eigenvalue equation.

Solution:

For killing vector ξ^μ, we have

$$\xi^{\mu;\nu}{}_{;\nu} = -\xi^{\nu;\mu}{}_{;\nu} = -\xi_\nu{}^{;\mu\nu}. \tag{25.6.2}$$

Recall that

$$A^\alpha{}_{;\gamma\beta} - A^\alpha{}_{;\beta\gamma} = R^\alpha{}_{\mu\beta\gamma}A^\mu. \tag{25.6.3}$$

We find

$$\xi_\nu{}^{;\mu\nu} - \xi_\nu{}^{;\nu\mu} = R_{\nu\rho}{}^{\nu\mu}\xi^\rho = R^\mu{}_\rho \xi^\rho. \tag{25.6.4}$$

As $\xi^\mu{}_{;\mu} = 0$, we can get

$$\xi^{\mu;\nu}{}_{;\nu} + R^\mu{}_\nu \xi^\nu = 0. \tag{25.6.5}$$

Define $f_{\mu\nu} \equiv \xi_{\nu;\mu} - \xi_{\mu;\nu} = -2\xi_{\mu;\nu}$. Recall the Lagrange formulation of EM field. We have

$$\delta\left(\int -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}\sqrt{-g}d^4x\right) = -\int f^{\mu\nu}{}_{;\nu}\delta\xi_\mu\sqrt{-g}d^4x = 2\int \xi^{\mu;\nu}{}_{;\nu}\delta\xi_\mu\sqrt{-g}d^4x. \tag{25.6.6}$$

So the required variational principle for this eigenvalue equation is

$$S = \int \mathcal{L}\sqrt{-g}d^4x \tag{25.6.7}$$

where

$$\mathcal{L} = -\xi^{\mu;\nu}\xi_{\mu;\nu} + R_{\mu\nu}\xi^\mu\xi^\nu. \tag{25.6.8}$$

25.7 RADIAL VELOCITY OF A TEST PARTICLE

Obtain a formula for the radial component of velocity $v_{\hat{r}}$ that an observer at r would measure [see (25.20) for $v_{\hat{r}}$]. Express $E_{\rm local}$, $v_{\hat{r}}$ and $v_{\hat{\phi}}$ in terms of r and the constants E, p_ϕ.

Solution:

From equation (25.21) and (25.22) of the textbook, we know

$$E_{\rm local} = (1-2M/r)^{-1/2}E, \quad v_{\hat{\phi}} = \frac{(1-2M/r)^{1/2}p_\phi}{rE} \tag{25.7.1}$$

The radial component of velocity can be obtained through

$$\begin{aligned} v_{\hat{r}} &\equiv \frac{p^{\hat{r}}}{p^{\hat{0}}} \equiv \frac{\langle \boldsymbol{\omega}^{\hat{r}}, \boldsymbol{p}\rangle}{E_{\rm local}} = \frac{\langle |g_{rr}|^{1/2}\boldsymbol{d}r, d/d\lambda\rangle}{E_{\rm local}} = \frac{dr/d\lambda}{(1-2M/r)^{-1/2}E_{\rm local}} \\ &= \frac{1}{E}\sqrt{E^2-(1-2M/r)(L^2/r^2+\mu^2)} \\ &= \sqrt{1-\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2E^2}+\frac{\mu^2}{E^2}\right)}. \end{aligned} \tag{25.7.2}$$

25.8 ROTATIONAL KILLING VECTORS FOR SCHWARZSCHILD GEOMETRY

(a) Show that in the isotropic coordinates of exercise 23.1, the metric for the Schwarzschild geometry takes the form

$$ds^2 = -(1-M/2\bar{r})^2(1+M/2\bar{r})^{-2}dt^2 + (1+M/2\bar{r})^4(d\bar{r}^2+\bar{r}^2d\Omega^2). \tag{25.8.1}$$

(b) Exhibit a coordinate transformation that brings this into the form

$$ds^2 = -(1-M/2\bar{r})^2(1+M/2\bar{r})^{-2}dt^2 + (1+M/2\bar{r})^4(dx^2+dy^2+dz^2), \tag{25.8.2}$$

with $\bar{r} = (x^2+y^2+z^2)^{1/2}$.

(c) Show that $\boldsymbol{\xi}_x = y(\partial/\partial z) - z(\partial/\partial y)$ and similar vectors $\boldsymbol{\xi}_y$ and $\boldsymbol{\xi}_z$ are each Killing vectors, by verifying (see exercise 25.5c) that the Poisson brackets $[\mathcal{H}, L_{\rm K}]$ vanish for each $L_{\rm K} = \boldsymbol{p}\cdot\boldsymbol{\xi}_K$, $K = x, y, z$.

(d) Show that $\boldsymbol{\xi}_z = (\partial/\partial\phi)_{t,r,\theta}$; and show that for orbits in the equatorial plane $L_z = p_\phi$, $L_x = L_y = 0$.

Solution:

(a) In Schwarzschild coordinate system, we have

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2d\Omega^2. \tag{25.8.3}$$

In isotropic coordinate system, we have

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\mu}(d\bar{r}^2+\bar{r}^2d\Omega^2). \tag{25.8.4}$$

By comparison, we find

$$e^{\Lambda}dr = e^{\mu}d\bar{r} \tag{25.8.5}$$

and

$$e^{-\mu}r = \bar{r}. \tag{25.8.6}$$

So we have

$$e^{\Lambda}dr = e^{\mu}\left(-e^{-\mu}\frac{d\mu}{dr}r + e^{-\mu}\right)dr \tag{25.8.7}$$

i.e.

$$1 - r\frac{d\mu}{dr} = e^{\Lambda} = \frac{1}{\sqrt{1-\frac{2M}{r}}}. \tag{25.8.8}$$

The solution is

$$\mu = \ln\left(\frac{r}{2M}\right) - 2\ln\left(\sqrt{\frac{r}{2M}} + \sqrt{\frac{r}{2M}-1}\right) + \ln C \tag{25.8.9}$$

i.e.

$$e^{\mu} = \frac{Cr}{(\sqrt{r}+\sqrt{r-2M})^2} = \frac{C}{\left(1+\sqrt{1-\frac{2M}{r}}\right)^2}. \tag{25.8.10}$$

Now we find

$$\bar{r} = \left(1+\sqrt{1-\frac{2M}{r}}\right)^2\frac{r}{C}. \tag{25.8.11}$$

Assume that $r = \bar{r}$ when $r \to \infty$. We have

$$\bar{r} = \left(1+\sqrt{1-\frac{2M}{r}}\right)^2\frac{r}{4} \tag{25.8.12}$$

and

$$r = \frac{1}{\bar{r}}\left(\frac{M}{2}+\bar{r}\right)^2. \tag{25.8.13}$$

Particularly, we have

$$\sqrt{1-\frac{2M}{r}} = \frac{1-M/2\bar{r}}{1+M/2\bar{r}}. \tag{25.8.14}$$

We can obtain

$$e^{\mu} = (1+M/2\bar{r})^2, \quad e^{\Phi} = \frac{1-M/2\bar{r}}{1+M/2\bar{r}}. \tag{25.8.15}$$

The metric is

$$ds^2 = -(1-M/2\bar{r})^2(1+M/2\bar{r})^{-2}dt^2 + (1+M/2\bar{r})^4(d\bar{r}^2+\bar{r}^2d\Omega^2). \tag{25.8.16}$$

(b) The coordinate transformation is

$$\begin{cases} x = \bar{r}\sin\theta\cos\phi \\ y = \bar{r}\sin\theta\sin\phi \\ z = \bar{r}\cos\theta \end{cases}. \tag{25.8.17}$$

(c) We have

$$\mathcal{H} = -\frac{1}{2}(1-M/2\bar{r})^{-2}(1+M/2\bar{r})^2p_0^2 + \frac{1}{2}(1+M/2\bar{r})^{-4}(p_x^2+p_y^2+p_z^2) \tag{25.8.18}$$

and

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \tag{25.8.19}$$

The calculation shows that

$$\begin{aligned}[\mathcal{H}, L_x] &= \sum_{y,z}\frac{\partial\mathcal{H}}{\partial x^i}\frac{\partial L_x}{\partial p_i} - \sum_{y,z}\frac{\partial\mathcal{H}}{\partial p_i}\frac{\partial L_x}{\partial x^i} \\ &= \frac{\partial\mathcal{H}}{\partial\bar{r}}\frac{y}{\bar{r}}(-z) + \frac{\partial\mathcal{H}}{\partial\bar{r}}\frac{z}{\bar{r}}y - (1+M/2\bar{r})^{-4}p_yp_z - (1+M/2\bar{r})^{-4}p_z(-p_y) \\ &= 0.\end{aligned} \tag{25.8.20}$$

Similarly, we can figure out that $[\mathcal{H}, L_y] = [\mathcal{H}, L_z] = 0$.

(d) The transformation matrix for vectors is

$$\begin{pmatrix} A^t \\ A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \bar{r}\cos\theta\cos\phi & -\bar{r}\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & \bar{r}\cos\theta\sin\phi & \bar{r}\sin\theta\cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A^t \\ A^{\bar{r}} \\ A^\theta \\ A^\phi \end{pmatrix}. \tag{25.8.21}$$

For $(A^t, A^{\bar{r}}, A^\theta, A^\phi) = (0, 0, 0, 1)$, we find $(A^t, A^x, A^y, A^z) = (0, -\bar{r}\sin\theta\sin\phi, \bar{r}\sin\theta\cos\phi, 0) = (0, -y, x, 0)$. So $\boldsymbol{\xi}_z = (\partial/\partial\phi)_{t,r,\theta}$. And for orbits in the equatorial plane, $L_z = \boldsymbol{\xi}_z \cdot \boldsymbol{p} = p_\phi$. Note that $z = p_z = 0$, we have $L_x = L_y = 0$.

25.9 CONSERVATION OF TOTAL ANGULAR MOMENTUM OF A TEST PARTICLE

Prove by a Poisson-bracket calculation that the total angular momentum squared, $p_\theta^2 + (\sin\theta)^{-2}p_\phi^2$ is a constant of motion for any Schwarzschild geodesic.

Solution:

The Hamiltonian is

$$2\mathcal{H} = -(1-2M/r)^{-1}p_t^2 + (1-2M/r)p_r^2 + r^{-2}p_\theta^2 + r^{-2}(\sin\theta)^{-2}p_\phi^2 \tag{25.9.1}$$

So we have

$$\begin{aligned}[\mathcal{H}, p_\theta^2 + (\sin\theta)^{-2}p_\phi^2] &= \frac{1}{2}[-(1-2M/r)^{-1}p_t^2 + (1-2M/r)p_r^2, p_\theta^2 + (\sin\theta)^{-2}p_\phi^2] \\ &\quad + \frac{1}{2}[r^{-2}(p_\theta^2 + (\sin\theta)^{-2}p_\phi^2), p_\theta^2 + (\sin\theta)^{-2}p_\phi^2] \\ &= 0. \end{aligned} \tag{25.9.2}$$

Thus $p_\theta^2 + (\sin\theta)^{-2}p_\phi^2$ is a constant of motion for any Schwarzschild geodesic.

25.10 SELECTING EQUATION BY SELECTING WHAT IS VARIED

Write out the integral that is varied in (25.8) for the special case of the Schwarzschild metric (25.12). What equation results from the demand $\delta I = 0$ if only $\phi(\lambda)$ is varied? If only $t(\lambda)$?

Solution:

From equation 25.1.2, if only $\phi(\lambda)$ is varied, we have

$$\frac{dp_\phi}{d\lambda} - \frac{1}{2}g_{\mu\nu,\phi}p^\mu p^\nu = 0. \tag{25.10.1}$$

In Schwarzschild metric, we have $dp_\phi/d\lambda = 0$. Similarly, if $t(\lambda)$ is varied, we have $dp_t/d\lambda = 0$.

25.11 MOTION DERIVED FROM SUPER-HAMILTONIAN

Write out the super-Hamiltonian (25.10) for the special case of the Schwarzschild metric. Deduce from its form that p_ϕ and p_t are constants of motion. Derive (25.14), (25.17), and (25.18) from this super-Hamiltonian formalism.

Solution:

The super-Hamiltonian for the special case of the Schwarzschild metric is

$$\mathcal{H} = \frac{1}{2}\left[-(1-2M/r)^{-1}p_t^2 + (1-2M/r)p_r^2 + r^{-2}p_\theta^2 + r^{-2}(\sin\theta)^{-2}p_\phi^2\right]. \tag{25.11.1}$$

As $\partial\mathcal{H}/\partial t = \partial\mathcal{H}/\partial\phi = 0$, p_ϕ and p_t are constants of motion. And we have

$$\frac{d\phi}{d\tau} = \frac{1}{\mu}\frac{d\phi}{d\lambda} = \frac{1}{\mu}\frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2\sin^2\theta}. \tag{25.11.2}$$

Demand $\theta = \pi/2$ and define $\tilde{L} \equiv p_\phi/\mu$. We have

$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2}. \tag{25.11.3}$$

Similarly,

$$\frac{dt}{d\tau} = \frac{1}{\mu}\frac{dt}{d\lambda} = \frac{1}{\mu}\frac{\partial\mathcal{H}}{\partial p_t} = \frac{-p_t}{\mu(1-2M/r)} = \frac{\tilde{E}}{1-2M/r}, \tag{25.11.4}$$

where $\tilde{E} \equiv -p_t/\mu$. And in the case of massive particles, $\mathcal{H} = -\frac{1}{2}\mu^2$. Note that

$$\frac{dr}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_r} = (1-2M/r)p_r. \tag{25.11.5}$$

We have

$$-\frac{E^2}{1-2M/r} + \frac{1}{1-2M/r}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} + \mu^2 = 0, \tag{25.11.6}$$

where $E \equiv -p_t$, $L = -p_\phi$ and θ is assumed to be $\pi/2$.

25.12 REDSHIFT BY TIMED PULSES

Derive expression (25.26) for the photon redshift by considering two pulses of light emitted successively by an atom at rest at radius r.

Solution:

The geodesics equation for photons with constant θ and ϕ satisfies that

$$\frac{dr}{dt} = 1 - \frac{2M}{r} \tag{25.12.1}$$

So we have $\Delta t_{\rm em} = \Delta t_{\rm rec}$. As a result,

$$\frac{\Delta\tau_{\rm em}}{\left(1-\frac{2M}{r_{\rm em}}\right)^{1/2}} = \frac{\Delta\tau_{\rm rec}}{\left(1-\frac{2M}{r_{\rm rec}}\right)^{1/2}}. \tag{25.12.2}$$

Thus, if $r_{\rm rec}=\infty$ and $r_{\rm em}=r$, we have

$$\frac{1}{1+z}=\frac{\lambda_{\rm em}}{\lambda_{\rm rec}}=\frac{\Delta\tau_{\rm em}}{\Delta\tau_{\rm rec}}=\frac{\left(1-\frac{2M}{r_{\rm em}}\right)^{1/2}}{\left(1-\frac{2M}{r_{\rm rec}}\right)^{1/2}}=\left(1-\frac{2M}{r}\right)^{1/2}, \tag{25.12.3}$$

i.e.

$$z=\left(1-\frac{2M}{r}\right)^{-1/2}-1. \tag{25.12.4}$$

25.13 QUALITATIVE FORMS OF PARTICLE ORBITS

Verify the statements about particle orbits made in part C of Box 25.6.

The proof for each statements about particle orbits made in part C of Box 25.6 is trivial. I will not write down detailed demonstration for all of them here.

25.14 IMPACT PARAMETER

For a scattering orbit (i.e., unbound orbit), show that $\tilde{L}=\tilde{E}v_\infty b$, where b is the impact parameter and v_∞ the asymptotic ordinary velocity; also show that

$$b=\tilde{L}/(\tilde{E}^2-1)^{1/2}. \tag{25.14.1}$$

Draw a picture illustrating the physical significance of the impact parameter.

Solution:

For a particle at infinity, we have

$$\frac{d\phi}{d\tau}=\frac{\tilde{L}}{r^2},\quad \frac{dt}{d\tau}=\tilde{E},\quad -\tilde{E}^2+\left(\frac{dr}{d\tau}\right)^2+\frac{\tilde{L}^2}{r^2}+1=0. \tag{25.14.2}$$

The definition of impact parameter is

$$b\equiv r\frac{v_{\hat{\phi}}}{v_\infty}=r^2\frac{d\phi}{dt}\frac{1}{v_\infty}=\frac{\tilde{L}}{\tilde{E}v_\infty}. \tag{25.14.3}$$

So we have $\tilde{L}=\tilde{E}v_\infty b$. Note that

$$v_\infty=\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2\right]^{1/2}=\frac{1}{\tilde{E}}\left(\tilde{E}^2-1-\frac{\tilde{L}^2}{r^2}+\frac{\tilde{L}^2}{r^2}\right)^{1/2}=\frac{(\tilde{E}^2-1)^{1/2}}{\tilde{E}}. \tag{25.14.4}$$

So we have

$$b=\frac{\tilde{L}}{(\tilde{E}^2-1)^{1/2}}. \tag{25.14.5}$$

25.15 TIME TO FALL TO $r = 2M$

Show from equation (25.16) and the first picture in Box 25.6 that orbits (general $\tilde{L}$ value!) which approach $r = 2M$ do so in a finite proper time, but (equation 25.32) an infinite coordinate time t. For equilibrium stars, which must have radii $R > 2M$, the coordinate time t to fall to the surface is finite, of course.

Solution:

If $\tilde{E}^2 > \tilde{V}^2_{\max}$, then we have

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r) \geq \tilde{E}^2 - \tilde{V}^2_{\max} > 0. \tag{25.15.1}$$

The proper time to fall to $r = 2M$ is

$$\Delta\tau = \int_{2M}^{R} \frac{d\tau}{dr} dr \leq \left(\tilde{E}^2 - \tilde{V}^2_{\max}\right)^{-1/2} (R - 2M), \tag{25.15.2}$$

which is finite.

On the other hand, we also have

$$\left(\tilde{E}\frac{dr^*}{dt}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r^*) < \tilde{E}^2. \tag{25.15.3}$$

The coordinate time to fall to $r = 2M$ ($r^* = -\infty$) is

$$\Delta t = \int_{\infty}^{R^*} \frac{dt}{dr^*} dr^* \geq \int_{\infty}^{R^*} dr^* \to \infty. \tag{25.15.4}$$

25.16 PERIASTRON SHIFT FOR NEARLY CIRCULAR ORBITS

Rewrite equation (25.42) in the form

$$(du/d\phi)^2 + (1 - 6u_0)(u - u_0)^2 - 2(u - u_0)^3 = (\tilde{E}^2 - \tilde{E}_0^2)/L^{\dagger 2} \tag{25.16.1}$$

Express the constant $u_0 \equiv M/r_0$ in terms of $L^\dagger \equiv \tilde{L}/M$, and express $\tilde{E}_0$ in terms of u_0. Show for a nearly circular orbit of radius r_0 that the angle swept out between two successive periastra (points of closest approach to the star) is

$$\Delta\phi = 2\pi(1 - 6M/r_0)^{-1/2}. \tag{25.16.2}$$

Sketch the shape of the orbit for $r_0 = 8M$.

Solution:

First, we have equation (25.42)

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2 - (1 - 2u)(1 + L^{\dagger 2}u^2)}{L^{\dagger 2}}. \tag{25.16.3}$$

Take the derivative of both sides and we have

$$L^{\dagger 2}\frac{d^2u}{d\phi^2} = (1 + L^{\dagger 2}u^2) + (2u - 1)L^{\dagger 2}u. \tag{25.16.4}$$

So for circular orbits, we know

$$\tilde{E}_0^2 - (1-2u_0)(1+L^{\dagger 2}u_0^2) = 0 \tag{25.16.5}$$

and

$$3L^{\dagger 2}u_0^2 - L^{\dagger 2}u_0 + 1 = 0. \tag{25.16.6}$$

Thus, we find

$$u_0 = \frac{L^\dagger - \sqrt{L^{\dagger 2} - 12}}{6L^\dagger} \tag{25.16.7}$$

and

$$\tilde{E} = \frac{1-2u_0}{\sqrt{1-3u_0}} \tag{25.16.8}$$

We can verify that

$$\begin{aligned}
\frac{\tilde{E}^2 - (1-2u)(1+L^{\dagger 2}u^2)}{L^{\dagger 2}} &= \frac{\tilde{E}^2 - \tilde{E}_0^2}{L^{\dagger 2}} + \frac{(1-2u_0)(1+L^{\dagger 2}u_0^2) - (1-2u)(1+L^{\dagger 2}u^2)}{L^{\dagger 2}} \\
&= \frac{\tilde{E}^2 - \tilde{E}_0^2}{L^{\dagger 2}} + \frac{2(u-u_0)}{L^{\dagger 2}} + u_0^2 - u^2 + 2(u^3 - u_0^3) \\
&= \frac{\tilde{E}^2 - \tilde{E}_0^2}{L^{\dagger 2}} + 2(u_0 - 3u_0^2)(u-u_0) + u_0^2 - u^2 + 2(u^3 - u_0^3) \\
&= \frac{\tilde{E}^2 - \tilde{E}_0^2}{L^{\dagger 2}} + 2u^3 - u^2 + 2u_0(1-3u_0)u + 4u_0^3 - u_0^2 \\
&= \frac{\tilde{E}^2 - \tilde{E}_0^2}{L^{\dagger 2}} + 2(u-u_0)^3 - (1-6u_0)(u-u_0)^2,
\end{aligned} \tag{25.16.9}$$

i.e.

$$(du/d\phi)^2 + (1-6u_0)(u-u_0)^2 - 2(u-u_0)^3 = (\tilde{E}^2 - \tilde{E}_0^2)/L^{\dagger 2}. \tag{25.16.10}$$

Take the derivative of both sides, we would arrive at

$$\frac{d^2u}{d\phi^2} + (1-6u_0)(u-u_0) - 3(u-u_0)^2 = 0. \tag{25.16.11}$$

If $|u-u_0| \ll 1$, we have

$$\frac{d^2(u-u_0)}{d\phi^2} + (1-6u_0)(u-u_0) = 0. \tag{25.16.12}$$

The solution is

$$u = u_0 + \epsilon \sin(\omega t + \phi_0), \tag{25.16.13}$$

where $\omega^2 = 1 - 6u_0$. Thus the angle swept out between two successive periastra is

$$\Delta\phi = 2\pi(1 - 6M/r_0)^{-1/2}. \tag{25.16.14}$$

25.17 ANGULAR MOTION DURING INFALL

From equation (25.42), deduce that the total angle $\Delta\phi$ swept out on a trajectory falling into $r = 0$ is finite. The computation is straightforward; but the interpretation, in view of the behavior of $t(\lambda)$ on the same trajectory (equation 25.32 and exercise 25.15), is not. The interpretation will be elucidated in Chapter 31.

Solution:

If the particle can fall into $r = 0$, then we have $\tilde{E}^2 > \tilde{V}_{\max}^2$, i.e.

$$\left(\frac{du}{d\phi}\right)^2 \geq \frac{\tilde{E}^2 - \tilde{V}_{\max}(u)}{L^{\dagger 2}}. \tag{25.17.1}$$

And when $u > 1$, we have

$$-(1-2u)(1+L^{\dagger 2}u^2) > L^{\dagger 2}u^3 \tag{25.17.2}$$

i.e.

$$\left(\frac{du}{d\phi}\right)^2 > u^3 \tag{25.17.3}$$

So

$$\Delta\phi = \int_{\mu_0}^{\infty}\frac{d\phi}{du}du < \int_1^{\infty}u^{-3/2}du + \int_{\mu_0}^{1}L^{\dagger}[\tilde{E}^2-\tilde{V}_{\max}(u)]^{-1/2}du = 2 + L^{\dagger}[\tilde{E}^2-\tilde{V}_{\max}(u)]^{-1/2}(1-\mu_0), \tag{25.17.4}$$

i.e. the total angle $\Delta\phi$ swept out on a trajectory falling into $r = 0$ is finite.

25.18 MAXIMUM AND MINIMUM OF EFFECTIVE POTENTIAL

Derive the expressions given in the caption of Figure 25.2 for the locations of the maximum and the minimum of the effective potential as a function of angular momentum. Determine also the limiting form of the dependence of barrier height on angular momentum in the limit in which $\tilde{L}$ is very large compared to M.

Solution:

From the expression of $\tilde{V}(r)$, we have

$$\frac{d\tilde{V}^2}{dr} = \frac{2M}{r^2}\left(1+\frac{\tilde{L}^2}{r^2}\right) - 2\left(1-\frac{2M}{r}\right)\frac{\tilde{L}^2}{r^3} = \frac{2Mr^2 - 2\tilde{L}^2 r + 6M\tilde{L}^2}{r^4}. \tag{25.18.1}$$

So the $r_{\rm extrem}$s satisfy that

$$Mr_{\rm extrem}^2 - \tilde{L}^2 r_{\rm extrem} + 3M\tilde{L}^2 = 0. \tag{25.18.2}$$

The solution is

$$r_{\rm extrem} = \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M} = \frac{1\pm\sqrt{1-12L^{\dagger -2}}}{2}ML^{\dagger 2} = \frac{6M}{1\mp\sqrt{1-12L^{\dagger -2}}}. \tag{25.18.3}$$

Further more, the second derivative of $\tilde{V}^2(r)$ at $r_{\rm extrem}$ is

$$\left.\frac{d^2\tilde{V}^2}{dr^2}\right|_{r_{\rm extrem}} = \frac{4Mr_{\rm extrem} - 2L^{\dagger 2}}{r_{\rm extrem}^4} = \frac{\pm 2\sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{r_{\rm extrem}^4}. \tag{25.18.4}$$

So the maximum of the $\tilde{V}(r)$ is at

$$r_{\max} = \frac{\tilde{L}^2 - \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M} = \frac{6M}{1+\sqrt{1-12L^{\dagger -2}}}, \tag{25.18.5}$$

while the minimum of the $\tilde{V}(r)$ is at

$$r_{\min} = \frac{6M}{1-\sqrt{1+12L^{\dagger -2}}}. \tag{25.18.6}$$

The barrier height is

$$\begin{aligned}\tilde{V}_{\max}^2 &= \left(1-\frac{2M}{r_{\max}}\right)\left(1+\frac{\tilde{L}^2}{r_{\max}^2}\right)\\ &= \left(1-\frac{1+\sqrt{1-12L^{\dagger -2}}}{3}\right)\left(1+\frac{1-6L^{\dagger -2}+\sqrt{1-12L^{\dagger -2}}}{18}L^{\dagger 2}\right)\\ &= \frac{L^{\dagger 2}+36+(L^{\dagger 2}-12)\sqrt{1-12L^{\dagger -2}}}{54}.\end{aligned} \tag{25.18.7}$$

In the limit $L^\dagger \to \infty$, we have

$$\tilde{V}^2_{\max} \approx \frac{L^{\dagger 2} + 36 + (L^{\dagger 2} - 12)\left(1 - 6L^{\dagger -2} - 18L^{\dagger -4}\right)}{54} \approx \frac{L^{\dagger 2} + 9 + 27L^{\dagger -2}}{27}. \tag{25.18.8}$$

25.19 KEPLER LAW VALID FOR CIRCULAR ORBITS

From $d\phi/d\tau$ of (25.17) and $dt/d\tau$ of (25.18), deduce an expression for the circular frequency of revolution as seen by a faraway observer; and from the results of exercise 25.18 (or otherwise) show that it fulfills exactly the Kepler relation

$$\omega^2 r^3 = M \tag{25.19.1}$$

for any circular orbit of Schwarzschild r-value equal to r, whether stable (potential minimum) or unstable (potential maximum).

Solution:

From equation (25.17) and (25.18), we have

$$\omega = \frac{d\phi}{dt} = \frac{\tilde{L}}{r^2 \tilde{E}}\left(1 - \frac{2M}{r}\right) = \frac{1}{M}\frac{L^\dagger}{x^2 \tilde{E}}\left(1 - \frac{2}{x}\right), \tag{25.19.2}$$

where $x \equiv r/M$. For circular orbits, we have

$$\tilde{E}^2 = \left(1 - \frac{2}{x}\right)\left(1 + \frac{L^{\dagger 2}}{x^2}\right) \tag{25.19.3}$$

and

$$x^2 - L^{\dagger 2}x + 3L^{\dagger 2} = 0, \tag{25.19.4}$$

i.e.

$$x^2 + L^{\dagger 2} = L^{\dagger 2}x - 2L^{\dagger 2}. \tag{25.19.5}$$

Thus we find

$$\omega^2 r^3 = \omega^2 x^3 M^3 = \frac{L^{\dagger 2}(x-2)}{x^2 + L^{\dagger 2}} M = M. \tag{25.19.6}$$

25.20 HAMILTON-JACOBI FUNCTION

Construct the locus in the r, θ diagram of points of constant dynamic phase $\tilde{S}(t, r, \theta) = 0$ for $t = 0$ and for values $\tilde{L} = 4M$, $\tilde{E} = 1$ (or for $\tilde{L} = 2\sqrt{3}M$, $\tilde{E} = (8/9)^{1/2}$, or for some other equally simple set of values for these two parameters). Show that the whole set of surfaces of constant $\tilde{S}$ can be obtained by rotating the foregoing locus through one angle, then another and another, and recopying or retracing. Interpret physically the principal features of the resulting pattern of curves.

The meaning of the exercise is not very clear. Any help is appreciated.

25.21 DEFLECTION BY GRAVITY CONTRASTED WITH DEFLECTION BY ELECTRIC FORCE

A test particle of arbitrary velocity β flies past a mass M at an impact parameter b so great that the deflection is small. Show that the deflection is

$$\theta = \frac{2M}{b\beta^2}(1+\beta^2) \tag{25.21.1}$$

Derive the deflection according to Newtonian mechanics for a particle moving with the speed of light. Show that (25.49) in the limit $\beta \to 1$ is twice the Newtonian deflection. Derive also (flat-space analysis) the contrasting formula for the deflection of a fast particle of rest mass μ and charge e by a nucleus of charge Ze,

$$\theta = \frac{2Ze^2}{\mu b\beta^2}(1-\beta^2)^{1/2}. \tag{25.21.2}$$

How feasible is it to rule out a "vector" theory of gravitation [see, for example, Brillouin (1970)], patterned after electromagnetism, by observations on the bending of light by the sun?

Solution:

In Schwarzschild metric, the orbit of a test particle satisfies the equation

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2 - (1-2u)(1+L^{\dagger 2}u^2)}{L^{\dagger 2}}, \tag{25.21.3}$$

where $u \equiv M/r$. Differentiate once with respect to ϕ to convert into a second-order equation. Rearrange to put on the left all those terms that would be there in the absence of gravity, and on the right all those that originate from the $-2u$ term (gravitation) in the factor $(1-2u)$. Now we have

$$\frac{d^2u}{d\phi^2} + u = 3u^2 + \frac{1}{L^{\dagger 2}}. \tag{25.21.4}$$

Neglecting the right-hand side of the equation, we have the straight line solution

$$u = A\sin(\phi - \phi_0). \tag{25.21.5}$$

By redefine the starting point of ϕ, the solution can be simplified into $u = A\sin\phi$. When $\phi = 0$ or π, we find that $u = 0$ and $du/d\phi = \pm A$. From equation 25.21.3, we also find that $(du/d\phi)^2 = (\tilde{E}^2 - 1)/L^{\dagger 2}$. According to equation 25.14.1, we also have $L^\dagger = \sqrt{\tilde{E}^2 - 1}b/M$. So the straight line solution is

$$u = \frac{M}{b}\sin\phi. \tag{25.21.6}$$

Evaluate the perturbing term Δ (of the order b^{-2}) on the right as a function of ϕ by inserting in it the unperturbed expression for $u(\phi)$. We have

$$\frac{d^2\Delta}{d\phi^2} + \Delta = \frac{3M^2}{b^2}\sin^2\phi + \frac{M^2}{b^2(\tilde{E}^2-1)}. \tag{25.21.7}$$

The solution is

$$\Delta = \frac{M^2}{b^2}(2 - \sin^2\phi) + \frac{M^2}{b^2(\tilde{E}^2-1)}. \tag{25.21.8}$$

At the infinity, we have $u = 0$ and so

$$\frac{M}{b}\sin\phi + \frac{M^2}{b^2}(2-\sin^2\phi) + \frac{M^2}{b^2(\tilde{E}^2-1)} = 0. \tag{25.21.9}$$

The solution of $\sin\phi$ to the order of b^{-1} is

$$\sin\phi = -\frac{2M}{b} - \frac{M}{b(\tilde{E}^2-1)} = -\frac{M(1+\beta^2)}{b\beta^2}. \tag{25.21.10}$$

So $\phi = -M(1+\beta^2)/b\beta^2$ and $\pi + M(1+\beta^2)/b\beta^2$. And the deflection angle is

$$\Delta\phi = \frac{2M(1+\beta^2)}{b\beta^2}. \tag{25.21.11}$$

In the limit of $\beta \to 1$, we have

$$\Delta\phi = \frac{4M}{b}. \tag{25.21.12}$$

In Newtonian mechanics, the orbit is hyperbolic, i.e.

$$\frac{p}{r} = 1 + e\sin\phi, \tag{25.21.13}$$

where $p = b^2\beta^2/M$ and $e = (1 + b^2\beta^4/M^2)^{1/2}$. At the infinity, we have

$$\sin\phi = -\frac{1}{e} = -\frac{M}{b\beta^2} + O(b^{-2}). \tag{25.21.14}$$

And the deflection angle is

$$\Delta\phi = \frac{2M}{b\beta^2}. \tag{25.21.15}$$

In the limit of $\beta \to 1$, we have

$$\Delta\phi = \frac{2M}{b}. \tag{25.21.16}$$

To the zeroth order, the orbit of a fast particle of rest mass μ and charge e in the field of a nucleus of charge Ze is a straight line. When it passes the nucleus, it will gain momentum in the direction perpendicular to the line. And we have

$$\Delta p_y = \int_{-\infty}^{\infty} \frac{dp_y}{d\tau}\frac{d\tau}{dx}dx = \int_{-\infty}^{\infty} \frac{\gamma Ze^2 b}{(b^2+x^2)^{3/2}}\frac{\mu}{p_x}dx = \frac{2\gamma Ze^2\mu}{bp_x}. \tag{25.21.17}$$

So the deflection angle is

$$\Delta\phi = \frac{p_y}{p_x} = \frac{2\gamma Ze^2\mu}{bp_x^2} = \frac{2Ze^2}{\mu b\beta^2}(1-\beta^2)^{1/2}. \tag{25.21.18}$$

In the limit of $\beta \to 1$, we have $\Delta\phi = 0$, i.e. the orbit of the particle is not bent. So we can rule out a "vector" theory of gravitation patterned after electromagnetism, by observations on the bending of light by the sun.

25.22 CAPTURE BY A BLACK HOLE

Over and above any scattering of particles by a black hole, there is direct capture into the black hole. Show that the cross section for capture is $\pi b_{\rm crit}^2$ with the critical impact parameter $b_{\rm crit}$ given by $L_{\rm crit}/(E^2-\mu^2)^{1/2}$. From the formulas in the caption of Fig. 25.2 or otherwise, show that for high-energy particles this cross section varies with energy as

$$\sigma_{\rm capt} = 27\pi M^2\left(1 + \frac{2}{3\tilde{E}^2} + \cdots\right) \tag{25.22.1}$$

(photon limit for $\tilde{E} \to \infty$) and for low energies as

$$\sigma_{\rm capt} = 16\pi M^2/\beta^2, \tag{25.22.2}$$

where β is the velocity relative to the velocity of light [Bogorodsky (1962)].

Solution:

From equation 25.18.7, the barrier height of the potential is

$$\tilde{V}_{\max}^2 = \frac{L^{\dagger 2} + 36 + (L^{\dagger 2} - 12)\sqrt{1 - 12L^{\dagger -2}}}{54}. \tag{25.22.3}$$

The energy of incident particles must satisfy that $\tilde{E} > 1$. If the particle is captured by black hole, we have

$$\tilde{V}_{\max}^2 = \frac{L^{\dagger 2} + 36 + (L^{\dagger 2} - 12)\sqrt{1 - 12L^{\dagger -2}}}{54} \leq \tilde{E}^2 \tag{25.22.4}$$

Define $L^{\dagger}_{\rm crit}$ by $L^{\dagger 2}_{\rm crit} + 36 + (L^{\dagger 2}_{\rm crit} - 12)\sqrt{1 - 12L^{\dagger -2}_{\rm crit}} = 54\tilde{E}^2$. If $L^{\dagger} < L^{\dagger}_{\rm crit}$, i.e. $b < L^{\dagger}_{\rm crit} M/(\tilde{E}^2 - 1) \equiv b_{\rm crit}$, the particle will be captured by black hole. The cross section is

$$\sigma_{\rm crit} = \pi b_{\rm crit}^2 = \frac{L^{\dagger 2}_{\rm crit}}{\tilde{E}^2 - 1}\pi M^2. \tag{25.22.5}$$

And

$$(\tilde{E}^2 - 1)x + 36 + [(\tilde{E}^2 - 1)x - 12]\sqrt{1 - \frac{12}{(\tilde{E}^2 - 1)x}} = 54\tilde{E}^2, \tag{25.22.6}$$

where $x \equiv \sigma_{\rm crit}/\pi M^2$. In the limit of $\tilde{E} \to \infty$, the zeroth order approximation of the x is 27. To linear order, we have

$$\tilde{E}^2 x - 27 + 36 + (\tilde{E}^2 x - 27 - 12)\sqrt{1 - \frac{12}{27\tilde{E}^2}} = 54\tilde{E}^2, \tag{25.22.7}$$

The solution is

$$x = 27 + \frac{18}{\tilde{E}^2} + O\left(\frac{1}{\tilde{E}^4}\right), \tag{25.22.8}$$

i.e.

$$\sigma_{\rm crit} = 27\pi M^2 \left(1 + \frac{2}{3\tilde{E}^2} + \cdots\right). \tag{25.22.9}$$

In the low energy limit, we have

$$y + (y - 12)\sqrt{1 - \frac{12}{y}} = 18, \tag{25.22.10}$$

where $y \equiv (\tilde{E}^2 - 1)x$. The solution is $y = 16$. As $\tilde{E}^2 - 1 = \gamma^2 - 1 = 1/(1 - \beta^2) - 1 \approx \beta^2$, we have

$$\sigma_{\rm capt} = 16\pi M^2/\beta^2. \tag{25.22.11}$$

25.23 QUALITATIVE FORMS OF PHOTON ORBITS

Verify all the statements about orbits for particles of zero rest mass made in Box 25.7.

The proof for each statements about photon orbits made in part Box 25.7 is trivial. I will not write down detailed demonstration for all of them here.

25.24 LIGHT DEFLECTION

Using the dimensionless variable $u = M/r$ in place of r itself, and $u_b = M/b$ in place of the impact parameter, transform (25.55) into the first-order equation

$$\left(\frac{du}{d\phi}\right)^2 + (1-2u)u^2 = u_b^2 \tag{25.24.1}$$

and thence, by differentiation, into

$$\frac{d^2u}{d\phi^2} + u = 3u^2. \tag{25.24.2}$$

(a) In the large-impact-parameter or small-u approximation, in which the term on the right is neglected, show that the solution of 25.24.2 yields elementary rectilinear motion (zero deflection).

(b) Insert this zero-order solution into the perturbation term $3u^2$ on the righthand side of 25.24.2, and solve anew for u ("rectilinear motion plus first-order correction"). In this way, verify the formula for the bending of light by the sun given by putting $\beta = 1$ in equation 25.21.1.

Solution:

(a) Neglecting the right-hand side of the equation 25.24.2, we have the straight line solution

$$u = A\sin(\phi - \phi_0). \tag{25.24.3}$$

By redefine the starting point of ϕ, the solution can be simplified into $u = A\sin\phi$. When $\phi = 0$ or π, we find that $u = 0$ and $du/d\phi = \pm A$. According to equation 25.24.1, we also have $du/d\phi = u_b$. So the straight line solution is

$$u = u_b \sin\phi. \tag{25.24.4}$$

(b) Inserting the zero-order solution into the perturbation term $3u^2$ on the righthand side of 25.24.2, we have

$$\frac{d^2\Delta}{d\phi^2} + \Delta = 3u_b^2\sin^2\phi, \tag{25.24.5}$$

where $\Delta = u - u_b\sin\phi$. The solution is

$$\Delta = u_b^2(2 - \sin^2\phi), \tag{25.24.6}$$

i.e. $u = u_b\sin\phi + u_b^2(2-\sin^2\phi)$. When $u = 0$, we have $\sin\phi \approx -2u_b$, and so $\phi \approx -2u_b$ or $\pi + 2u_b$. The deflection angle is $4u_b = 4M/b$.

25.25 CAPTURE OF LIGHT BY A BLACK HOLE

Show that a Schwarzschild black hole presents a cross section $\sigma_{\text{capt}} = 27\pi M^2$ for capture of light.

Solution:

From equation (25.66),

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{1-2M/r}{r^2} = \frac{1}{b^2}, \tag{25.25.1}$$

the effective potential is

$$\tilde{V}^2(r) = \frac{1-2M/r}{r^2}. \tag{25.25.2}$$

The maximum of $\tilde{V}^2(r)$ is at $r = 3M$ and equals $1/27M^2$. If the photon is captured by the black hole, we must have $1/b^2 > 1/27M^2$, i.e. $b < 3^{3/2}M$. So the cross section is $27\pi M^2$.

25.26 RETURN OF LIGHT FROM A BLACK HOLE

Show that flashing a powerful pulse of light onto a black hole leads in principle to a return from rings of brightness located at $b/M - 3^{3/2} = 0.151,\ 0.00028,\ \cdots$ How can one evaluate the difference in time delays of these distinct returns? Show that the intensity I of the return $(\mathrm{erg/cm^2})$ as a function of the energy $E_0(\mathrm{erg/steradian})$ of the original pulse, the mass $M(\mathrm{cm})$ of the black hole, the distance R to it, and the lateral distance r from the "flashlight" to the receptor of returned radiation is

$$I = \frac{E_0}{R^3 r} \sum_{\Theta=(2N+1)\pi} \left| \frac{2bdb}{d\Theta} \right| = \frac{E_0 M^2}{R^3 r}(1.75 + 0.0029 + 0.0000055 + \cdots) \tag{25.26.1}$$

under conditions where diffraction can be neglected.

Solution:

From equation 25.24.1, the deflection angle of the photon is

$$\Theta = 2\int_0^{u_0} \frac{d\phi}{du}du - \pi = 2\int_0^{u_0} \frac{1}{\sqrt{u_b^2 - (1-2u)u^2}} - \pi, \tag{25.26.2}$$

where $(1-2u_0)u_0^2 = u_b^2$. With the help of WolframAlpha, we have

$$\int_0^{u_0} \frac{1}{\sqrt{u_b^2 - (1-2u)u^2}} = 2\sqrt{\frac{2}{v_0 + 6u_0 - 1}} F\left(\sin^{-1}\sqrt{\frac{2u_0 + 4u + v_0 - 1}{2v_0}} \middle| \frac{2v_0}{6u_0 + v_0 - 1} \right)\Bigg|_0^{u_0} \tag{25.26.3}$$

where $v_0 \equiv \sqrt{-12u_0^2 + 4u_0 + 1}$ and $F(\phi|m)$ is the incomplete elliptic integral of the first kind. When $b/M - 3^{3/2}$ is small, $u_b^2 - 1/27$ is also small, as well as $1/3 - u_0$, $v_0 - 1$, $1 - (6u_0 + v_0 - 1)/2v_0$ and $\pi/2 - \sin^{-1}\sqrt{(6u_0 + v_0 - 1)/(2v_0)}$. Note that $F(\pi/2, 1)$ is infinite. So we have the approximation

$$\frac{\Theta + \pi}{2} \approx 2F\left(\sin^{-1}(1-\delta) \middle| \frac{1}{(1-\delta)^2} \right) - 2F\left(\sin^{-1}\sqrt{\frac{1}{3}} \middle| 1 \right), \tag{25.26.4}$$

where $\delta = 1 - \sqrt{(6u_0 + v_0 - 1)/(2v_0)}$. Using the expansion

$$F\left(\sin^{-1}(1-\delta) \middle| \frac{1}{(1-\delta)^2} \right) = -\frac{1}{2}\ln\left(\frac{\delta}{8}\right) + O(\delta), \tag{25.26.5}$$

we have

$$\delta \approx 8\exp\left(-\frac{\Theta + \pi}{2} - 2F\left(\sin^{-1}\sqrt{\frac{1}{3}} \middle| 1 \right) \right) \approx 0.44561\exp\left(-\frac{\Theta}{2}\right) \tag{25.26.6}$$

On the other hand, we have the expansion

$$1 - \delta = \sqrt{\frac{6u_0 + v_0 - 1}{2v_0}} = 1 + 2\left(u_0 - \frac{1}{3}\right) + O\left[\left(u_0 - \frac{1}{3}\right)^2\right] \tag{25.26.7}$$

and

$$\frac{b}{M} = \frac{1}{u_b} = \frac{1}{\sqrt{(1-2u_0)u_0^2}} = 3^{3/2} + \frac{3^{9/2}}{2}\left(u_0 - \frac{1}{3}\right)^2 + O\left[\left(u_0 - \frac{1}{3}\right)^3\right]. \tag{25.26.8}$$

So we have

$$\frac{b}{M} \approx 3^{3/2} + \frac{3^{9/2}}{8}\delta^2 \approx 3^{3/2} + 3.482e^{-\Theta} \tag{25.26.9}$$

If the photon returns, we have $\Theta = (2N+1)\pi$. And so

$$\frac{b}{M} - 3^{3/2} = 0.150, 2.81\times10^{-4}, 5.25\times10^{-7}, \cdots \tag{25.26.10}$$

I do not know how to answer the second and third question. Any help is appreciated.

25.27 ISOTROPIC STAR CLUSTER

For a cluster with distribution function independent of angular momentum, derive properties B.1 to B.6 of Box 25.8.

Solution:

The statement 1 and 2 is trivial. If $F = F(E, \mu)$, we have

$$\begin{aligned}\rho &= 4\pi \int F(e^{\phi} p^{\hat{0}}, \mu) \frac{(p^{\hat{0}})^2 p^{\hat{T}}}{p^{\hat{r}}} dp^{\hat{T}} dp^{\hat{0}} d\mu \\ &= 4\pi \int F(e^{\phi} p^{\hat{0}}, \mu)(p^{\hat{0}})^2 dp^{\hat{0}} d\mu \int \frac{p^{\hat{T}}}{\sqrt{(p^{\hat{0}})^2 - \mu^2 - (p^{\hat{T}})^2}} dp^{\hat{T}} \\ &= 4\pi \int F(e^{\phi} p^{\hat{0}}, \mu)(p^{\hat{0}})^2 \sqrt{(p^{\hat{0}})^2 - \mu^2} dp^{\hat{0}} d\mu; \end{aligned} \tag{25.27.1}$$

$$\begin{aligned}P_r &= 4\pi \int F(e^{\phi} p^{\hat{0}}, \mu)(p^{\hat{r}} p^{\hat{T}}) dp^{\hat{T}} dp^{\hat{0}} d\mu \\ &= 4\pi \int F(e^{\phi} p^{\hat{0}}, \mu) dp^{\hat{0}} d\mu \int \sqrt{(p^{\hat{0}})^2 - \mu^2 - (p^{\hat{T}})^2} p^{\hat{T}} dp^{\hat{T}} \\ &= \frac{4\pi}{3} \int F(e^{\phi} p^{\hat{0}}, \mu)((p^{\hat{0}})^2 - \mu^2)^{3/2} dp^{\hat{0}} d\mu; \end{aligned} \tag{25.27.2}$$

and

$$\begin{aligned}P_T &= 2\pi \int F(e^{\phi} p^{\hat{0}}, \mu) \frac{(p^{\hat{T}})^3}{p^{\hat{r}}} dp^{\hat{T}} dp^{\hat{0}} d\mu \\ &= 2\pi \int F(e^{\phi} p^{\hat{0}}, \mu) dp^{\hat{0}} d\mu \int \frac{(p^{\hat{T}})^3}{\sqrt{(p^{\hat{0}})^2 - \mu^2 - (p^{\hat{T}})^2}} dp^{\hat{T}} \\ &= \frac{4\pi}{3} \int F(e^{\phi} p^{\hat{0}}, \mu)((p^{\hat{0}})^2 - \mu^2)^{3/2} dp^{\hat{0}} d\mu. \end{aligned} \tag{25.27.3}$$

So statement 3 and 4 is true. Finally, note that

$$T^{\hat{\alpha}\hat{\beta}} = \mathrm{diag}(\rho, P, P, P) \tag{25.27.4}$$

and

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{25.27.5}$$

The form of metric and energy momentum tensor of cluster is the same as that of gas sphere. So the statement 5 and 6 must be true.

25.28 SELF-SIMILAR CLUSTER

(a) Find a solution to the equations of structure for a spherical star of infinite central density, with the equation of state $P = \gamma\rho$, where γ is a constant ($0 < \gamma < 1/3$).

(b) Find an isotropic distribution function $F(E, L)$ that leads to a star cluster with the same distributions of ρ, P, m, and Φ as in the gas sphere of part (a).

Solution:

Substitute the equation of state into OV equation. Then we have

$$\frac{d\ln\rho}{dr} = -\frac{(\gamma+1)(m+4\pi\gamma r^3\rho)}{\gamma r(r-2m)}. \tag{25.28.1}$$

Try power-law solution $\rho = Cr^\alpha$. We have

$$m = \int_0^r 4\pi C r^{2+\alpha} dr = \frac{4\pi C}{3+\alpha} r^{3+\alpha}. \tag{25.28.2}$$

And so

$$\frac{\alpha}{r} = -\frac{(\gamma+1)(\frac{4\pi C}{3+\alpha}r^{3+\alpha} + 4\pi C\gamma r^{3+\alpha})}{\gamma r(r - \frac{8\pi C}{3+\alpha}r^{3+\alpha})}. \tag{25.28.3}$$

The solution is $\alpha = -2$ and $C = \gamma/2\pi(\gamma^2+6\gamma+1)$, i.e.

$$\rho = \frac{\gamma^2}{\gamma^2+6\gamma+1}\frac{1}{2\pi r^2}. \tag{25.28.4}$$

And so

$$e^{2\Lambda} = \frac{1}{1-\frac{2m}{r}} = \frac{1}{1-8\pi C} = \frac{\gamma^2+6\gamma+1}{(1+\gamma)^2}. \tag{25.28.5}$$

From

$$\frac{dP}{dr} = -(\rho+P)\frac{d\Phi}{dr}, \tag{25.28.6}$$

we can get

$$e^{2\Phi} = Br^{\frac{4\gamma}{\gamma+1}}, \quad B = \text{const.} \tag{25.28.7}$$

(b) From equation 25.27.1, we have

$$Cr^{-2} = 4\pi \int F(B^{1/2}r^{\frac{2\gamma}{\gamma+1}}E_{\text{local}}, \mu)[E_{\text{local}}^2 - \mu^2]^{1/2}E_{\text{local}}^2 dE_{\text{local}} d\mu. \tag{25.28.8}$$

From equation 25.27.2, we have

$$\gamma C r^{-2} = \frac{4\pi}{3}\int F(B^{1/2}r^{\frac{2\gamma}{\gamma+1}}E_{\text{local}}, \mu)[E_{\text{local}}^2 - \mu^2]^{3/2} dE_{\text{local}} d\mu. \tag{25.28.9}$$

So we should let

$$F(B^{1/2}r^{\frac{2\gamma}{\gamma+1}}E_{\text{local}}, \mu) = A\delta(\mu-\mu_0)[B^{1/2}r^{\frac{2\gamma}{\gamma+1}}E_{\text{local}}]^{-\frac{\gamma}{\gamma+1}}. \tag{25.28.10}$$

A and μ_0 are constants determined by normalization condition from two equations above.

25.29 CLUSTER WITH CIRCULAR ORBITS

What must be the form of the distribution function to guarantee that all stars move in circular orbits? Specialize the equations of structure to this case. Analyze the stability of the orbits of individual stars in the cluster, using an effective-potential diagram. What conditions must the distribution function satisfy if all orbits are to be stable?

The description of the question is vague and I do not know how to solve it. Any help is appreciated. Please read the paper *Can the Redshifts of Quasi-Stellar Objects Be Gravitational? (Zapolsky, 1968)* for more information on a spherical cluster composed of circular moving stars and its stability.

Chapter 26

STELLAR PULSATIONS

26.1 DRAGGING OF INERTIAL FRAMES BY A SLOWLY ROTATING STAR

A fluid sphere rotates very slowly. Analyze its rotation using perturbation theory; keep only effects and terms linear in the angular velocity of rotation.

Solution:

The answer is taken from papers *Variational Principle for the Equilibrium of a Relativistic, Rotating Star (Hartle & Sharp, 1967)* and *Slowly Rotating Relativistic Stars. I. Equations of Structure (Hartle, 1967)*.

For the description of an axially symmetric equilibrium configuration of a rotating fluid, two coordinates are conveniently taken to be the time $t = x^0$ and an angle $\phi = x^3$. The stationary nature of the configuration and its axial symmetry are then expressed by requiring that the metric coefficients be independent of t and ϕ,

$$g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2). \tag{26.1.1}$$

The metric remains a function of x^1 and x^2 alone under coordinate transformations of the form

$$t = t' + f_1(x^1, x^2), \quad \phi = \phi' + f_2(x^1, x^2). \tag{26.1.2}$$

By an appropriate choice of f_1 and f_2, one can always find a coordinate system in which

$$g_{13} = g_{23} = 0. \tag{26.1.3}$$

In addition to the independence of the metric on the angle, the assumption of axial symmetry also entails that the velocity distribution have only an angular component, so that the four-velocity may be written as

$$u^\mu = u^0(1, 0, 0, \Omega). \tag{26.1.4}$$

This means that the source $T_{\mu\nu}$ of the gravitational field is left unchanged by the simultaneous inversion of the time and the azimuthal angle. This symmetry must be reflected in the solutions of the field equations, and consequently we must require the line element to be left unchanged under this inversion. This requires us to set

$$g_{01} = g_{02} = 0. \tag{26.1.5}$$

Now the line element must have the general form:

$$ds^2 = g_{00}(dt)^2 + 2g_{03}dtd\phi + g_{33}(d\phi)^2 + g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2. \tag{26.1.6}$$

This form of the line element remains unchanged under coordinate transformations of the form

$$x^1 = x^2(x^{1'}, x^{2'}), \quad x^2 = x^2(x^{1'}, x^{2'}). \tag{26.1.7}$$

By an appropriate coordinate transformations, the line element can be written as

$$ds^2 = g_{00}(dt)^2 + 2g_{03}dtd\phi + g_{33}(d\phi)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2. \tag{26.1.8}$$

g_{00}, g_{11}, g_{22} and g_{33} are invariant under $\phi \to -\phi$, while g_{03} changes sign. So g_{00}, g_{11}, g_{22} and g_{33} are even functions of Ω while g_{03} is odd functions of Ω. In the case of a slowly rotating star, Ω is small and metric coefficients can be expanded as series of Ω. And so g_{00}, g_{11}, g_{22} and g_{33} contain only even powers of Ω while g_{03} contains odd powers. If we keep only the terms linear in the angular velocity of rotation, we have

$$ds^2 = -e^{2\Phi_0}dt^2 + e^{2\Lambda_0}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2(r^2\sin^2\theta)\omega(r,\theta)d\phi dt, \tag{26.1.9}$$

where ω is linear in Ω, and Φ_0 and Λ_0 are unperturbed potential in non rotating star. Then we can get the Einstein tensor from the metric. To the first order, all components of the Einstein tensor are the same as unperturbed ones, except that

$$\begin{aligned} G^t_\phi = &-\frac{1}{2}[r^2\omega'' - (r^2\Lambda_0' + r^2\Phi_0' - 4r)\omega']\sin^2\theta e^{-2\Lambda_0 - 2\Phi_0} \\ &- \frac{1}{2}\left[3\sin\theta\cos\theta\frac{\partial\omega}{\partial\theta} + \sin^2\theta\frac{\partial^2\omega}{\partial\theta^2}\right]e^{-2\Phi_0} \end{aligned} \tag{26.1.10}$$

The four velocity of the fluid satisfies that

$$(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)(u^t)^2 = -1. \tag{26.1.11}$$

To the first order, we have

$$u^t = e^{-\Phi_0}, u^\phi = \Omega e^{-\Phi_0} \tag{26.1.12}$$

and so

$$u_t = -e^{\Phi_0}, u_\phi = r^2\sin^2\theta(\Omega - \omega)e^{-\Phi_0}. \tag{26.1.13}$$

The $t\phi$ component of stress energy tensor is

$$T^t_\phi = (\rho + p)u^t u_\phi \approx (\rho_0 + p_0)e^{-2\Phi_0}r^2\sin^2\theta(\Omega - \omega). \tag{26.1.14}$$

Define $\bar{\omega} \equiv \Omega - \omega$. The Einstein equation $G^t_\phi = 8\pi T^t_\phi$ gives

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 e^{-\Phi_0 - \Lambda_0}\frac{\partial\bar{\omega}}{\partial r}\right] + \frac{e^{\Lambda_0 - \Phi_0}}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right] = 16\pi(\rho_0 + p_0)e^{\Lambda_0 - \Phi_0}\bar{\omega}. \tag{26.1.15}$$

Defining $j(r) \equiv \exp(-\Lambda_0 - \Phi_0)$ and using equations of structure of non-rotating spherical stars, we have the equation

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\omega}}{\partial r}\right) + \frac{4}{r}\frac{dj}{dr}\bar{\omega} + \frac{e^{\Lambda_0 - \Phi_0}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right] = 0. \tag{26.1.16}$$

Using the expansion

$$\bar{\omega}(r,\theta) = \sum_{l=1}^{\infty}\bar{\omega}_l(r)\left(-\frac{1}{\sin\theta}\frac{dP_l}{d\theta}\right), \tag{26.1.17}$$

we have

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j\frac{d\bar{\omega}_l}{dr}\right) + \left[\frac{4}{r}\frac{dj}{dr} - e^{\Lambda_0 - \Phi_0}\frac{l(l+1) - 2}{r^2}\right]\bar{\omega}_l = 0. \tag{26.1.18}$$

At small r, $j(r)$ is a regular function so that the differential equation admits a small r behavior of the form

$$\begin{aligned} \bar{\omega}_l(r) &\to \text{const.}\, r^{S_+} + \text{const.}\, r^{S_-}, \quad r \to 0, \\ S_\pm &= -\frac{3}{2} \pm \left[\frac{9}{4} + e^{2\Lambda_0}(l^2 + l - 1)\right]. \end{aligned} \tag{26.1.19}$$

If the geometry is to be regular at the origin we must demand that the coefficient of r^{S-} vanish. At large r, $j(r)$ becomes unity and $\bar{\omega}$ has the form

$$\bar{\omega}_l(r) \to \text{const.}\, r^{-l-2} + \text{const.}\, r^{l-1}, \quad r \to 0, \tag{26.1.20}$$

The behavior has already been fixed at the origin. Consequently the ratio of the two constants in equation above is determined and neither will vanish unless they both do. If space is to be flat at large r, $\bar{\omega}_l(r)$ must decrease faster than r^{-3}, so that $\bar{\omega}(r) = \Omega - \omega$ approaches Ω. Thus all coefficients in the Legendre expansion of $\bar{\omega}$ vanish except $l = 1$. Consequently $\bar{\omega}$ is a function of r alone. It obeys the differential equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\bar{\omega} = 0. \tag{26.1.21}$$

This equation is to be integrated outward starting with $\omega(0) = \text{const.}$ Outside the star $j(r) = 1$ and the solution has the form

$$\bar{\omega} = \Omega - \frac{2J}{r^3}, \quad r > a, \tag{26.1.22}$$

i.e. $\omega(r, \theta) = 2J/r^3$. The constant J can be identified with the total angular momentum of the star. And the metric far from the star is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4J\sin^2\theta}{r}d\phi dt. \tag{26.1.23}$$

In Cartesian coordinates, we have

$$-\frac{4J\sin^2\theta}{r}d\phi dt = -\frac{4J}{r^3}(-ydxdt + xdydt), \tag{26.1.24}$$

or generally,

$$-\frac{4J\sin^2\theta}{r}d\phi dt = -4\epsilon_{jkl}S^k\frac{x^l}{r^3}dtdx^j. \tag{26.1.25}$$

Chapter 27

IDEALIZED COSMOLOGIES

27.1 ISOTROPY IMPLIES HOMOGENEITY

Use elementary thought experiments to show that isotropy of the universe implies homogeneity.

Solution:

For any two points in space, we can find a third point whose distance to these two points are equal. If the universe is isotropy from the point of view of any points, then these two points must be equivalent. Now we can say the universe is homogeneous.

27.2 SYNCHRONOUS COORDINATES IN GENERAL

In an arbitrary spacetime manifold (not necessarily homogeneous or isotropic), pick an initial spacelike hypersurface $\mathcal{S}_I$, place an arbitrary coordinate grid on it, eject geodesic world lines orthogonal to it, and give these world lines the coordinates

$$(x^1, x^2, x^3) = \text{constant}, \quad x^0 \equiv t = t_I + \tau, \tag{27.2.1}$$

where τ is proper time along the world line, beginning with τ on $\mathcal{S}_I$. Show that in this coordinate system the metric takes on the synchronous (Gaussian normal) form (27.14).

Solution:

For any point in surface $t = t_I$, we have $g_{0i} = 0$ as geodesics 27.2.1 are orthogonal to the surface. For geodesics 27.2.1, we have $u^\mu = dx^\mu/d\tau = (1, 0, 0, 0)$. We can get $g_{00} = -1$ from $g_{\alpha\beta}u^\alpha u^\beta = -1$. If we substitute geodesics 27.2.1 into geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0, \tag{27.2.2}$$

we have $\Gamma^\alpha{}_{00} = 0$ and so $\Gamma_{\alpha 00} = 0$. And so

$$g_{00,0} = 2\Gamma_{000} = 0. \tag{27.2.3}$$

Now we know $g_{00} = -1$ throughout the whole spacetime. Note that

$$\Gamma_{i00} = g_{i0,0} - \frac{1}{2}g_{00,i}. \tag{27.2.4}$$

As $g_{00,i} = 0$, we have $g_{i0,0} = 0$. And we know $g_{0i} = 0$ throughout the whole spacetime. In this coordinate system the metric takes on the synchronous (Gaussian normal) form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j. \tag{27.2.5}$$

27.3 ARBITRARINESS IN THE EXPANSION FACTOR

How much arbitrariness is there in the definition of the expansion factor $a(t)$? Civilization A started long ago at time t_A. For it, the expansion factor is

$$\frac{\text{proper distance betweee two particles of the "cosmological fluid" at time } t}{\text{proper distance between the same two particles at time } t_A} = a_A(t). \tag{27.3.1}$$

Subsequently men planted civilization B at time t_B on a planet in a nearby galaxy. [At this time, the expansion factor a_A had the value $a_A(t_B)$]. Civilization B defines the expansion factor relative to the time of its own beginning:

$$\frac{\text{proper distance betweee two particles of the "cosmological fluid" at time } t}{\text{proper distance between the same two particles at time } t_B} = a_B(t). \tag{27.3.2}$$

At two subsequent events, C and D, of which both civilizations are aware, they assign to the universe in their bookkeeping by no means identical expansion factors,

$$a_A(t_C) \neq a_B(t_C), \quad a_A(t_D) \neq a_B(t_D). \tag{27.3.3}$$

Show that the relative expansion of the model universe in passing from stage C to stage D in its evolution is nevertheless the same in the two systems of bookkeeping:

$$\frac{a_A(t_D)}{a_A(t_C)} = \frac{a_B(t_D)}{a_B(t_C)}. \tag{27.3.4}$$

Solution:

$$\begin{aligned}\frac{a_A(t_D)}{a_A(t_C)} &= \frac{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_D}{\text{proper distance between the same two particles at time } t_A} \\ &\times \frac{\text{proper distance between the same two particles at time } t_A}{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_C} \\ &= \frac{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_D}{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_C}.\end{aligned} \tag{27.3.5}$$

Similarly, we have

$$\frac{a_B(t_D)}{a_B(t_C)} = \frac{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_D}{\text{proper distance betweee two particles of the "cosmological fluid" at time } t_C}. \tag{27.3.6}$$

So the relative expansion of the model universe in passing from stage C to stage D in its evolution is the same in the two systems of bookkeeping.

27.4 UNIQUENESS OF METRIC FOR 3-SURFACE OF CONSTANT CURVATURE

Let γ_{ij} and $\gamma_{i'j'}$ be two sets of metric coefficients, in coordinate systems $\{x^i\}$ and $\{x^{i'}\}$, that have Riemann curvature tensors of the constant-curvature type (27.21). Let it be given in addition that the curvature parameters K and K' are equal. Show that γ_{ij} and $\gamma_{i'j'}$ are related by a coordinate transformation.

Solution:

γ_{ij} and $\gamma_{i'j'}$ are equivalent if we have a set of independent solutions $x^i = x^i(x^{i'})$ for $i = 1, \cdots, n$ that

$$\gamma_{i'j'} = \gamma_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}. \tag{27.4.1}$$

If we put

$$\frac{\partial x^i}{\partial x^{i'}} = p^i{}_{i'}(x^{1'}, \cdots, x^{n'}), \tag{27.4.2}$$

the transformation law for affine connections gives

$$\frac{\partial p^i{}_{j'}}{\partial x^{k'}} = \Gamma^{i'}{}_{j'k'} p^i{}_{i'} - \Gamma^i{}_{jk} p^j{}_{j'} p^k{}_{k'}. \tag{27.4.3}$$

Hence the problem reduces to the determination of $n(n+1)$ functions x^i, $p^i{}_{i'}$, satisfying differential equations 27.4.2 and 27.4.3 and also the $n(n+1)/2$ finite equations

$$\gamma_{i'j'} - \gamma_{ij} p^i{}_{i'} p^j{}_{j'} = 0. \tag{27.4.4}$$

The conditions of integrability of 27.4.2 are satisfied identically in consequence of 27.4.3. If 27.4.3 holds, it is easily to induce that

$$\frac{\partial p^i{}_{j'}}{\partial x^{k'}} = \frac{\partial p^i{}_{k'}}{\partial x^{j'}} \tag{27.4.5}$$

Notice that

$$\begin{aligned}
\frac{\partial p^i{}_{j'}}{\partial x^{k'} \partial x^{l'}} &= \Gamma^{i'}{}_{j'k',l'} p^i{}_{i'} - \Gamma^i{}_{jk,l'} p^j{}_{j'} p^k{}_{k'} + \Gamma^{i'}{}_{j'k'} p^i{}_{i',l'} - \Gamma^i{}_{jk} p^j{}_{j',l'} p^k{}_{k'} - \Gamma^i{}_{jk} p^j{}_{j',l'} p^k{}_{k',l'} \\
&= \Gamma^{i'}{}_{j'k',l'} p^i{}_{i'} - \Gamma^i{}_{jk,l} p^j{}_{j'} p^k{}_{k'} p^l{}_{l'} - \Gamma^i{}_{jk} p^j{}_{j',l'} p^k{}_{k',l'} \\
&+ \Gamma^{i'}{}_{j'k'} (\Gamma^{m'}{}_{i'l'} p^i{}_{m'} - \Gamma^i{}_{jk} p^j{}_{i'} p^k{}_{l'}) - \Gamma^i{}_{jk} (\Gamma^{m'}{}_{j'l'} p^j{}_{m'} - \Gamma^i{}_{mn} p^m{}_{j'} p^n{}_{l'}) p^k{}_{k'} \\
&= (\Gamma^{i'}{}_{j'k',l'} + \Gamma^{i'}{}_{m'l'} \Gamma^{m'}{}_{j'k'}) p^i{}_{i'} - (\Gamma^i{}_{jk,l} - \Gamma^i{}_{mk} \Gamma^m{}_{jl}) p^j{}_{j'} p^k{}_{k'} p^l{}_{l'} \\
&- \Gamma^{i'}{}_{j'k'} \Gamma^i{}_{jk} p^j{}_{i'} p^k{}_{l'} - \Gamma^{i'}{}_{j'l'} \Gamma^i{}_{jk} p^j{}_{i'} p^k{}_{k'} - \Gamma^i{}_{jk} p^j{}_{j',l'} p^k{}_{k',l'}.
\end{aligned} \tag{27.4.6}$$

So we have

$$\frac{\partial p^i{}_{j'}}{\partial x^{k'} \partial x^{l'}} - \frac{\partial p^i{}_{j'}}{\partial x^{l'} \partial x^{k'}} = R^{i'}{}_{j'k'l'} p^i{}_{i'} - R^i{}_{jkl} p^j{}_{j'} p^k{}_{k'} p^l{}_{l'}. \tag{27.4.7}$$

The conditions of integrability of 27.4.3 are

$$R_{i'j'k'l'} = R_{ijkl} p^i{}_{i'} p^j{}_{j'} p^k{}_{k'} p^l{}_{l'}. \tag{27.4.8}$$

If Riemann curvature tensors in two coordinates have the form

$$R_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \quad R_{i'j'k'l'} = K(\gamma_{i'k'}\gamma_{j'l'} - \gamma_{i'l'}\gamma_{j'k'}), \tag{27.4.9}$$

and equation 27.4.4 holds, equation 27.4.8 will be satisfied identically. Moreover, the left hand of 27.4.4 is constant if 27.4.2 and 27.4.3 hold. So we can adjust the value of $p^i{}_{i'}$ at one point to make 27.4.4 holds at that point. Then we can solve the differential equations 27.4.2 and 27.4.3 whose conditions of integrability are already satisfied consistently. And condition 27.4.4 will be satisfied automatically. So we prove that γ_{ij} and $\gamma_{i'j'}$ are related by a coordinate transformation.

27.5 METRIC FOR 3-SURFACE OF CONSTANT CURVATURE

(a) Show that the following metric has expression (27.21) as its curvature tensor

$$\gamma_{ij} = \left(1 + \frac{1}{4}K\delta_{kl}x^k x^l\right)^{-2}\delta_{ij}. \tag{27.5.1}$$

(b) By transforming to spherical coordinates (R, θ, ϕ) and then changing to a Schwarzschild radial coordinate ($2\pi r$ = "proper circumference"), transform this metric into the form

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{27.5.2}$$

(c) Find a further change of radial coordinate that brings the metric into the form (27.22).

Solution:

(a) The derivative of the metric is

$$\gamma_{ij,k} = -K\left(1 + \frac{1}{4}KR^2\right)^{-3}\delta_{ij}x_k, \tag{27.5.3}$$

where $R^2 = \delta_{kl}x^k x^l$. So the connection is

$$\Gamma^i{}_{jk} = -\frac{K}{2}\left(1 + \frac{1}{4}KR^2\right)^{-1}(\delta_{ij}x_k + \delta_{ik}x_j - \delta_{jk}x_i). \tag{27.5.4}$$

The curvature is

$$\begin{aligned}
R^i{}_{jkl} &= \Gamma^i{}_{jl,k} - \Gamma^i{}_{jk,l} + \Gamma^i{}_{km}\Gamma^m{}_{jl} - \Gamma^i{}_{lm}\Gamma^m{}_{jk} \\
&= \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}x_k(\delta_{ij}x_l + \delta_{il}x_j - \delta_{jl}x_i) - \frac{K}{2}\left(1 + \frac{1}{4}KR^2\right)^{-1}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}) \\
&- \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}x_l(\delta_{ij}x_k + \delta_{ik}x_j - \delta_{jk}x_i) + \frac{K}{2}\left(1 + \frac{1}{4}KR^2\right)^{-1}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \\
&+ \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}(\delta_{ik}x_m + \delta_{im}x_k - \delta_{mk}x_i)(\delta_{mj}x_l + \delta_{ml}x_j - \delta_{jl}x_m) \\
&- \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}(\delta_{il}x_m + \delta_{im}x_l - \delta_{ml}x_i)(\delta_{mj}x_k + \delta_{mk}x_j - \delta_{jk}x_m) \\
&= \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}(\delta_{il}x_j x_k - \delta_{jl}x_i x_k - \delta_{ik}x_j x_l + \delta_{jk}x_i x_l) - K\left(1 + \frac{1}{4}KR^2\right)^{-1}(\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}) \\
&+ \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}(\delta_{ik}x_l x_j - \delta_{ik}\delta_{jl}R^2 + \delta_{il}\delta_{jk}R^2 - \delta_{il}x_k x_j - \delta_{kj}x_i x_l + \delta_{lj}x_i x_k) \\
&= \frac{K^2}{4}\left(1 + \frac{1}{4}KR^2\right)^{-2}(-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})R^2 - K\left(1 + \frac{1}{4}KR^2\right)^{-1}(\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}) \\
&= K\left(1 + \frac{1}{4}KR^2\right)^{-2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),
\end{aligned} \tag{27.5.5}$$

or equivalently,

$$R^i{}_{jkl} = K\left(1 + \frac{1}{4}KR^2\right)^{-4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{27.5.6}$$

(b) In spherical coordinates, we have

$$d\sigma^2 = \left(1 + \frac{1}{4}KR^2\right)^{-2} [dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{27.5.7}$$

Define

$$r \equiv \left(1 + \frac{1}{4}KR^2\right)^{-1} R. \tag{27.5.8}$$

We have

$$dr = \frac{1 - KR^2/4}{(1 + KR^2/4)^2} dR. \tag{27.5.9}$$

Thus

$$\left(1 + \frac{1}{4}KR^2\right)^{-2} dR^2 = \frac{(1 + KR^2/4)^2}{(1 - KR^2/4)^2} dr^2 = \frac{(1 + KR^2/4)^2}{(1 + KR^2/4)^2 - KR^2} dr^2 = \frac{1}{1 - Kr^2} dr^2. \tag{27.5.10}$$

The metric becomes

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{27.5.11}$$

(c) If $K > 0$, define

$$d\chi \equiv \sqrt{\frac{K}{1 - Kr^2}} dr. \tag{27.5.12}$$

So we have $\chi = \arcsin\sqrt{K}r$. The metric becomes

$$d\sigma^2 = K^{-1}[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{27.5.13}$$

If $K < 0$, define

$$d\chi \equiv \sqrt{\frac{-K}{1 + (-K)r^2}} dr. \tag{27.5.14}$$

So we have $\chi = \sinh^{-1}\sqrt{-K}r$. The metric becomes

$$d\sigma^2 = (-K)^{-1}[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{27.5.15}$$

If $K = 0$, define $\chi = r$, and we have

$$d\sigma^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{27.5.16}$$

27.6 PROPERTIES OF THE 3-SURFACES

Verify all statements made in Box 27.2.

Each statements made in section Box 27.2 are intuitive and easy to verify. I will not write down detailed demonstration for all of them here.

27.7 ISOTROPY IMPLIES HOMOGENEITY

Use the contracted Bianchi identity ${}^{(3)}G^{ik}{}_{|k} = 0$ (where the stroke indicates a covariant derivative based on the 3-geometry alone) to show (1) that ${}^{(3)}\nabla K = 0$ in equation (27.21), and therefore to show (2)

that direction-independence of the curvature [isotropy; curvature of form (27.21)] implies and demands homogeneity (K constant in space).

Solution:

Firstly, we have

$$R_{bd} = \gamma^{ac} R_{abcd} = K\gamma^{ac}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}) = 2K\gamma_{bd}. \tag{27.7.1}$$

So the curvature scalar is $R = \gamma^{bd} R_{bd} = 6K$. The Einstein tensor is therefore

$$G_{bd} = R_{bd} - \frac{1}{2}\gamma_{bd} R = -K\gamma_{bd}. \tag{27.7.2}$$

And so we have

$$0 = {}^{(3)}G^{bd}{}_{|d} = -(K\gamma^{bd})_{|d} = -K^{|b}, \tag{27.7.3}$$

i.e. ${}^{(3)}\nabla K = 0$.

27.8 MATTER-DOMINATED AND RADIATION-DOMINATED REGIMES Of FRIEDMANN COSMOLOGY

Derive the results listed in the last two columns of Box 27.3, except for the focusing properties of the curved space.

The results listed in the last two columns of Box 27.3 are straightforward and easy to verify. I will not write down detailed demonstration for all of them here.

27.9 TRANSITION FROM RADIATION-DOMINATED REGIME TO MATTER-DOMINATED REGIME

Including both the radiation and the matter terms in equation (27.51), restate the equation in terms of the arc parameter η (with $d\eta = dt/a$) as independent variable, and integrate to find

$$\begin{aligned} a &= a_{\max}/2 - [(a_{\max}/2)^2 + a^{*2}]^{1/2}\cos(\eta + \delta) \\ t &= (a_{\max}/2)\eta - [(a_{\max}/2)^2 + a^{*2}]^{1/2}[\sin(\eta + \delta) - \sin\delta], \end{aligned} \tag{27.9.1}$$

where

$$\delta = \arctan[a^*/(a_{\max}/2)]. \tag{27.9.2}$$

(a) Verify that under appropriate conditions these expressions reduce at early times to a "circle" relation between radius and time and to a "cycloid" relation later.

(b) Assign to a^{*2} the value $a_0 a_{\max}/10000$ (why?) and construct curves for the dimensionless measures of density,

$$\log_{10}\left[\frac{8\pi a_{\max}^2}{3}\begin{Bmatrix}\rho_m \\ \rho_r \\ \rho_m + \rho_r,\end{Bmatrix}\right], \tag{27.9.3}$$

as a function of the dimensionless measure of time,

$$\log_{10}(t/a_{\max}). \tag{27.9.4}$$

What conclusions emerge from inspecting the logarithmic slope of these curves?

Solution:

(a) When $\eta \ll 1$, we have

$$a \approx a_{\max}/2 - [(a_{\max}/2)^2 + a^{*2}]^{1/2}[\cos\delta - \eta\sin\delta] = a^*\eta \approx a^*\sin\eta, \tag{27.9.5}$$

and

$$t \approx (a_{\max}/2)\eta - [(a_{\max}/2)^2 + a^{*2}]^{1/2}[\sin\delta + \eta\cos\delta - \sin\delta] = 0 \approx a^*(1-\cos\eta) \tag{27.9.6}$$

to the first order of η. When $\eta + \delta - \pi \ll 1$, we have

$$\begin{aligned} a \approx & \quad a_{\max}/2 - [(a_{\max}/2)^2 + a^{*2}]^{1/2}(-1) \\ &= a_{\max}/2 + [(a_{\max}/2)^2 + a^{*2}]^{1/2} \\ &= \frac{a_{\max}/2 + [(a_{\max}/2)^2 + a^{*2}]^{1/2}}{2}[1-\cos(\eta+\delta)] \end{aligned} \tag{27.9.7}$$

and

$$\begin{aligned} t \approx & \quad (a_{\max}/2)(\pi-\delta) + (a_{\max}/2)(\eta+\delta-\pi) - [(a_{\max} \\ &= (a_{\max}/2)(\pi-\delta) + a^* + \{a_{\max}/2 + [(a_{\max}/2)^2 + a^{*2}]^{1/2}\}(\eta+\delta-\pi) \\ &\approx \quad \underline{a_{\max}/2 + [(a_{\max}} \\ &+ (a_{\max}/2)(\pi-\delta) + a^* - \frac{a_{\max}/2 + [(a_{\max}/2)^2 + a^{*2}]^{1/2}}{2}\pi \end{aligned} \tag{27.9.8}$$

to the first order of $\eta + \delta - \pi$. If we shift the t with a constant, we will have a "cycloid" relation between radius and time.

(b) The ratio of matter density and radiation density now is

$$\frac{\rho_{m0}}{\rho_{r0}} = \frac{a_{\max}/a_0^3}{a^{*2}/a_0^4} = \frac{a_0 a_{\max}}{a^{*2}}. \tag{27.9.9}$$

Assigning to a^{*2} the value $a_0 a_{\max}/10000$ means the ratio of matter density and radiation density now is about 10000, which may be compatible with the observation in 1970s. (The observation todya, i.e. 2010s, gives $H_0 = 69.6\mathrm{km/s/Mpc}$, $\Omega_{\mathrm{m}} = 0.286$, $T_r = 2.72528\mathrm{K}$, the ratio of matter density and radiation density now is about 5600.) Firstly, we have

$$\rho_m = \frac{3}{8\pi}\frac{a_{\max}}{a^3}, \quad \rho_r = \frac{3}{8\pi}\frac{a^{*2}}{a^4}. \tag{27.9.10}$$

Given $a^{*2} = a_0 a_{\max}/10000$, we have

$$\frac{8\pi a_{\max}^2}{3}\rho_m = \frac{a_{\max}^3}{a^3} = \frac{1}{\left\{\frac{1}{2} - \left[\left(\frac{1}{2}\right)^2 + \frac{a_0}{10000 a_{\max}}\right]^{1/2}\cos(\eta+\delta)\right\}^3} \tag{27.9.11}$$

$$\frac{8\pi a_{\max}^2}{3}\rho_r = \frac{a_{\max}^2 a^{*2}}{a^4} = \frac{1}{\left\{\frac{1}{2} - \left[\left(\frac{1}{2}\right)^2 + \frac{a_0}{10000 a_{\max}}\right]^{1/2}\cos(\eta+\delta)\right\}^4}\frac{a_0}{10000 a_{\max}} \tag{27.9.12}$$

$$\frac{t}{a_{\max}} = \frac{1}{2}\eta - \left[\left(\frac{1}{2}\right)^2 + \frac{a_0}{10000 a_{\max}}\right]^{1/2}[\sin(\eta+\delta) - \sin\delta], \tag{27.9.13}$$

and

$$\delta = \arctan 2\sqrt{\frac{a_0}{10000 a_{\max}}}. \tag{27.9.14}$$

Take the value from Box 27.4, we have $a_0/10000 a_{\max} = 2.55 \times 10^{-21}$. Curves are shown in Fig 1. When $\rho_r \gg \rho_m$, the logarithmic slope of ρ_m and ρ_r are -1.5 and -2 respectively, meaning $t \propto a^2$. When $\rho_r \ll \rho_m$, the logarithmic slope of ρ_m and ρ_r are -2 and -2.67 respectively, meaning $t \propto a^{3/2}$. The logarithmic slope of $\rho_m + \rho_r$ is -2 before the universe contracts.

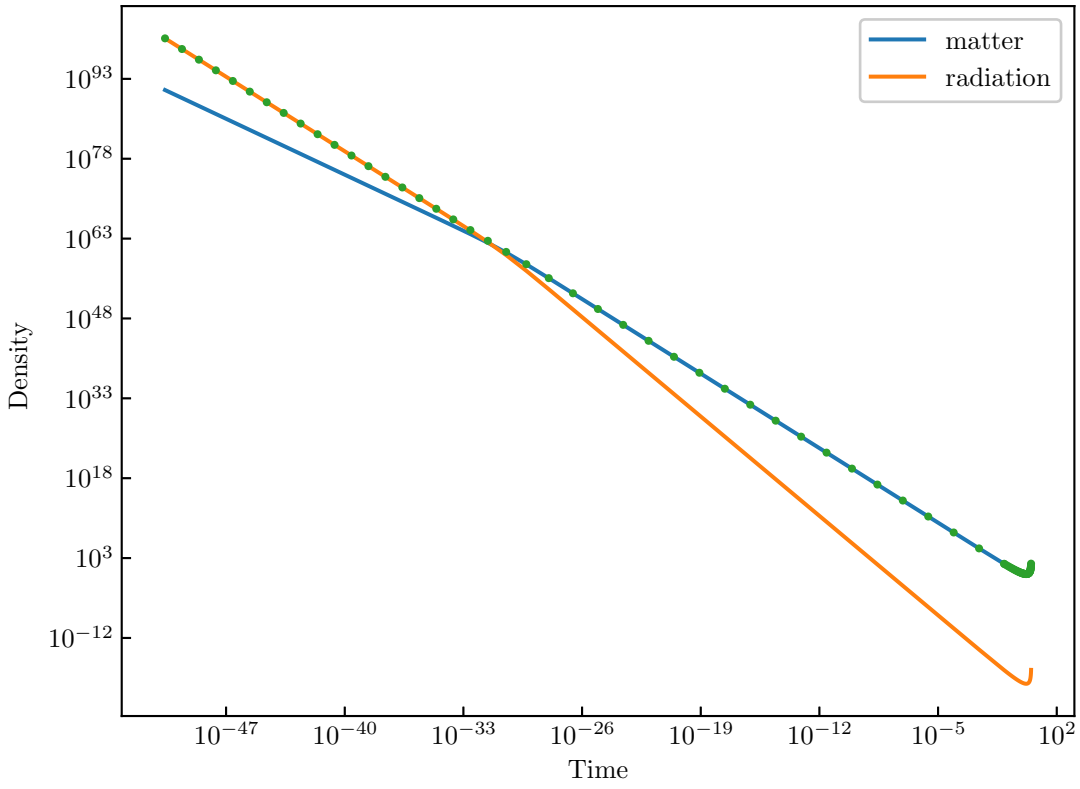


Figure 27.1: Curves for the dimensionless measures of density as a function of the dimensionless measure of time.

27.10 THE EXPANDING AND RECONTRACTING SPHERICAL WAVE FRONT

An explosion takes place at the "N-pole" of the matter-dominated Friedmann model universe at the value of the "arc parameter time" $\eta = \pi/3$, when the radius of the universe has reached half its peak value. The photons from the explosion race out on a spherical wave front. Through what fraction of the "cosmological fluid" has this wave front penetrated at that instant when the wave front has its largest proper surface area?

Solution:

In a matter dominated universe, we have

$$a(\eta) = \frac{a_{\max}}{2}(1 - \cos\eta). \tag{27.10.1}$$

Note when $\eta = \pi/3$, the radius of the universe reaches one quarter of its peak value. The statements in exercise is wrong. The world line of the light is

$$\chi = \eta - \pi/3. \tag{27.10.2}$$

The proper surface area of the wave front is

$$A = 4\pi a^2 \sin^2\chi = \pi a_{\max}^2 (1 - \cos\eta)^2 \sin^2(\eta - \pi/3). \tag{27.10.3}$$

When $\eta = 8\pi/9$, A reaches its maximum value. At that instant, this wave front has penetrated

$$\frac{-\sin(16\pi/9)/4 + 4\pi/9}{\pi/2} \approx 99.1\% \tag{27.10.4}$$

27.11 ON SEEING THE BACK OF ONE'S HEAD

Can a being at rest relative to the "cosmological fluid" ever see the back of his head by means of photons that travel all the way around a closed model universe that obeys the Friedmann cosmology and has a non-zero cosmological constant?

Solution:

In a matter dominated and close universe, if the value of cosmological constant is much smaller than the critical value $\Lambda_{\rm circ}$, the universe will contract before cosmological constant takes a significant role. So the evolution of scale factor would be essentially identical to that in a matter dominated and close universe without cosmological constant. From Penrose diagram for a matter dominated and close universe without cosmological constant, seeing the back of one's head is possible So we will state that it is also possible to see the back of one's head in a matter dominated and close universe with a non-zero cosmological constant.

27.12 DO THE CONSERVATION LAWS FORBID THE PRODUCTION OF PARTICLE-ANTIPARTICLE PAIRS OUT OF EMPTY SPACE BY TIDAL GRAVITATION FORCES?

Find out what is wrong with the following argument: "The classical equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{27.12.1}$$

are not compatible with the production of pairs, since they lead to the identity $T_{\alpha}{}^{\beta}{}_{;\beta} = 0$. Let the initial state be vacuum, and let $T_{\alpha\beta}$ and its derivative be equal to zero on the hypersurface $t =$ const or $t = -\infty$. It then follows from , that the vacuum is always $T_{\alpha}{}^{\beta}{}_{;\beta} = 0$, that the vacuum is always conserved.

Solution:

The argument is wrong because there may exist singularities in the solution of Einstein equation.

27.13 TURN-AROUND UNIVERSE MODEL NEGLECTING MATTER DENSITY

If turn-around (minimum radius) occurs far to the right (large a) of the maximum of the potential $V(a)$ in equation (27.75), the matter terms will be negligible. Let $\rho_{m0} = \rho_{r0} = 0$. Then, solve to show that $\Lambda = 3(a_{\rm min})^{-2}$, $H = (a_{\rm min})^{-1}\tanh(t/a_{\rm min})$ near turnaround ($t = 0$) and that the deceleration parameter $q \equiv -(1/H^2 a)(d^2a/dt^2)$ has the value

$$q = -a^2(a^2 - a_{\rm min}^{-2})^{-1} < -1. \tag{27.13.1}$$

Solution:

The Friedmann equation in this case is

$$H^2 + \frac{1}{a^2} = \frac{1}{3}\Lambda. \tag{27.13.2}$$

We have $H = 0$ when $a = a_{\min}$. And so $\Lambda = 3(a_{\min})^{-2}$. The equation can now be written as

$$\left(\frac{da}{dt}\right)^2 + 1 = \frac{a^2}{a_{\min}^2} \tag{27.13.3}$$

The solution of the equation is

$$\frac{a}{a_{\min}} = \cosh\frac{t}{a_{\min}}. \tag{27.13.4}$$

It is easy to verify that

$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a_{\min}}\tanh\frac{t}{a_{\min}} \tag{27.13.5}$$

and

$$q = -\frac{a_{\min}\cosh(t/a_{\min})}{\sinh^2(t/a_{\min})} \times \frac{1}{a_{\min}}\cosh\frac{t}{a_{\min}} = -\frac{\cosh^2(t/a_{\min})}{\cosh^2(t/a_{\min}) - 1} = -\frac{a^2}{a^2 - a_{\min}^2} < -1. \tag{27.13.6}$$

27.14 "HESITATION" UNIVERSE

Neglect radiation in equation (27.75) but assume K_0 and Λ to be chosen so that the universe spent a very long time with $a(t)$ near a_h (a_h measures location of highest point of the barrier, or the size of the universe at which the universe is most sluggish). Choose $a_h = a_0/3$ to produce an abnormally great number of quasar redshifts near $z = 2$ [as Burbidge and Burbidge (1969a,b) believe to be the case, though their opinion is not shared by all observers]. Show (a) that the density of matter now would account for only 10 per cent of the value of $H_0^2 = (\dot{a}/a)^2_{\text{now}}$ in equation (27.75) ["missing matter", i.e., K_0 and Λ terms, account for 90 per cent], (b) that $a_h \approx 20^{1/2}H_0^{-1}$, and (c) that the deceleration parameter defined in the previous exercise, as evaluated "today", has the value $q_0 = -13/10$.

Solution:

The generalized Friedmann equation now becomes

$$\left[\frac{d}{dt}\left(\frac{a(t)}{a_0}\right)\right]^2 + V\left(\frac{a}{a_0}\right) = -K_0. \tag{27.14.1}$$

and

$$V\left(\frac{a}{a_0}\right) = -\frac{8\pi}{3}\rho_{m0}\frac{a_0}{a} - \frac{1}{3}\Lambda\left(\frac{a}{a_0}\right)^2 \tag{27.14.2}$$

The location of highest point of the barrier satisfies that

$$\frac{8\pi}{3}\rho_{m0}\frac{a_0}{a_h^2} - \frac{2}{3}\Lambda\frac{a_h}{a_0^2} = 0. \tag{27.14.3}$$

The solution is

$$a_h = \left(\frac{4\pi\rho_{m0}}{\Lambda}\right)^{1/3} a_0 \tag{27.14.4}$$

and we demand that $a_h = a_0/3$. Then we have $4\pi\rho_{m0}/\Lambda = 1/27$ and

$$V = -\frac{1}{3}\Lambda\left[\frac{2}{27}\frac{a_0}{a} + \left(\frac{a}{a_0}\right)^2\right]. \tag{27.14.5}$$

When $a = a_0/3$, we have $V_{\max} = -\Lambda/9$. In the case of hesitation, we have $K_0 = \Lambda/9$. So

$$H_0^2 = -K_0 - V(1) = \frac{20}{81}\Lambda \tag{27.14.6}$$

while $8\pi\rho_{m0}/3 = 2\Lambda/81$. So the density of matter now would account for only 10 per cent of the value of H_0^2.

As $K_0 = 1/a_0^2 = \Lambda/9$, we have $a_h = a_0/3 = \Lambda^{-1/2} = (81H_0^2/20)^{-1/2} = 20^{1/2}/9H_0$.

The acceleration of a satisfies that

$$\frac{2}{a_0^2}\frac{d^2a}{dt^2}\frac{da}{dt} + V'\frac{da}{dt} = 0. \tag{27.14.7}$$

So we have

$$\frac{d^2a}{dt^2} = \frac{1}{3}\Lambda\left(a - \frac{1}{27}\frac{a_0^3}{a^2}\right) = \tag{27.14.8}$$

The deceleration parameter is

$$q_0 = -\frac{1}{a_0 H_0^2}\frac{1}{3}\Lambda\left(a_0 - \frac{1}{27}a_0\right) = -\frac{13}{10}. \tag{27.14.9}$$

27.15 UNIVERSE OPAQUE TO BLACK-BODY RADIATION AT A NON-SINGULAR PAST TURN-AROUND REQUIRES IMPOSSIBLE PARAMETERS

From a plot like that in Box 27.5, construct a model of the universe that contains 2.7K black-body radiation at the present, but, with $k = +1$ and $\Lambda > 0$, had a past turn-around (minimum radius) at which the blackbody temperature reached 3000K where hydrogen would be ionized. Try to use values of H_0^{-1} and ρ_{m0} that are as little as possible smaller than presently accepted values.

Solution:

The conditions above implies that

$$H_0^2 + \frac{1}{a_0^2} = \frac{8\pi}{3}\rho_{m0} + \frac{8\pi}{3}\rho_{r0} + \frac{\Lambda}{3}, \tag{27.15.1}$$

$$\frac{1}{a_0^2} = \frac{8\pi}{3}\rho_{m0}\frac{a_0}{a_t} + \frac{8\pi}{3}\rho_{r0}\frac{a_0^2}{a_t^2} + \frac{\Lambda}{3}\frac{a_t^2}{a_0^2}, \tag{27.15.2}$$

and

$$\frac{a_0}{a_t} > \frac{3000}{2.7} = \frac{10000}{9}. \tag{27.15.3}$$

In order to make H_0^{-1} and ρ_{m0} as little as possible, we can take $\rho_{m0} = 0$, then we have

$$H_0^2 = \frac{8\pi}{3}\rho_{r0}(1 - x^2) + \frac{\Lambda}{3}\left(1 - \frac{1}{x^2}\right), \tag{27.15.4}$$

where $x > 10000/9$. If we make Λ as large as possible, we can get H_0 as little as possible.

Chapter 28

EVOLUTION OF THE UNIVERSE INTO ITS PRESENT STATE

28.1 UNCERTAINTY IN EVOLUTION

Current observations, plus the assumption of complete homogeneity and isotropy at the beginning of expansion, plus the assumption that the excess of leptons over antileptons is less than or of the order of the excess of baryons over antibaryons, place the following limits on the cosmological parameters today:

- Matter density today $= \rho_{m0}$ between 10^{-28} and $2 \times 10^{-31}\mathrm{g/cm^3}$;
- $k = 0$ or $+1$ or -1;
- temperature of electromagnetic radiation today $= 2.7 \pm 0.1\mathrm{K}$;
- Total radiation density [observed photons, plus neutrinos and gravitons that presumably originated in big bang in thermal equilibrium with photons] $= \rho_{r0}$ between 0.7×10^{-33} and $1.2 \times 10^{-33}\mathrm{g/cm^3}$.

Use the equations in section 27.10 to calculate the uncertainties in the evolutionary history (Figure 28.1) caused by these uncertainties in the present state of the universe.

Solution:

Define

$$\rho_{c0} \equiv \frac{3H_0^2}{8\pi} \tag{28.1.1}$$

and

$$\Omega_{r0} \equiv \frac{\rho_{r0}}{\rho_{c0}}, \quad \Omega_{m0} \equiv \frac{\rho_{r0}}{\rho_{m0}}, \quad \Omega_{K0} \equiv -\frac{3k/8\pi a_0^2}{\rho_{c0}} = 1 - \Omega_{r0} - \Omega_{m0}. \tag{28.1.2}$$

The evolution of scale factor is given by

$$\eta(z) = \int_z^\infty \frac{dz'}{H_0} \frac{1}{\sqrt{\Omega_{r0}(1+z')^4 + \Omega_{m0}(1+z')^3 + \Omega_{K0}(1+z')^2}} \tag{28.1.3}$$

where $dt = a d\eta$ and $z + 1 = a_0/a$. $a(t)$ can then be obtained by one more integration. The evolution of ρ_r, ρ_m and T is given by

$$\rho_r = \rho_{r0}\left(\frac{a}{a_0}\right)^{-4}, \quad \rho_m = \rho_{m0}\left(\frac{a}{a_0}\right)^{-3}, \quad T = T_0\frac{a_0}{a}. \tag{28.1.4}$$

The numeric calculation of the uncertainties in the evolutionary history can be performed by Monte Carlo method and is eliminated here. A useful website for calculation is Cosmology Calculator.

Chapter 29

PRESENT STATE AND FUTURE EVOLUTION OF THE UNIVERSE

29.1 IMPLICATIONS OF PARAMETER VALUES

Derive the results quoted in Box 29.1.

Each statements in Box 29.1 are straightforward and trivial to prove. I will not write down detailed demonstration for all of them here.

29.2 ALTERNATIVE DERIVATION OF REDSHIFT

Notice that the only part of the line element that is relevant for the light ray is

$$ds^2 = -dt^2 + a^2(t)d\chi^2, \tag{29.2.1}$$

since $d\theta = d\phi = 0$ along its world line (spherical symmetry!). Regard the light ray as made of photons with 4-momenta $\boldsymbol{p}$. From the geodesic equation (or, for the reader who has studied chapter 25, from arguments about Killing vectors), show that

$$p_\chi \equiv \boldsymbol{p} \cdot (\partial/\partial\chi) \tag{29.2.2}$$

is conserved along the photon's world line. Use this fact, the fact that a photon's 4-momentum is null, $\boldsymbol{p}\cdot\boldsymbol{p} = 0$, and the equation $\boldsymbol{E} = -\boldsymbol{p}\cdot\boldsymbol{u}$ for the energy measured by an observer with 4-velocity $\boldsymbol{u}$, to derive the redshift equation (29.11).

Solution:

In this coordinates, $\Gamma_{\chi\mu\nu}$ is not zero for

$$\Gamma_{\chi\chi t} = \Gamma_{\chi t\chi} = \frac{1}{2}g_{\chi\chi,t}. \tag{29.2.3}$$

For vector $\chi^\mu = (0, 1)$, $\chi_\mu = (0, a^2)$, we have

$$\chi_{\mu;\nu} = \chi_{\mu,\nu} - \Gamma_{\alpha\mu\nu}\chi^\alpha = \chi_{\mu,\nu} - \Gamma_{\chi\mu\nu}. \tag{29.2.4}$$

So

$$\chi_{\chi;t} = \chi_{\chi,t} - \Gamma_{\chi\chi t} = (a^2)_{,t} - \frac{1}{2}g_{\chi\chi,t} = \frac{1}{2}g_{tt}, \chi_{t;\chi} = -\Gamma_{\chi t\chi} = -\frac{1}{2}g_{tt}. \tag{29.2.5}$$

All other components vanish. We now have $\chi_{\mu;\nu}+\chi_{\nu;\mu}=0$ and so $\boldsymbol{\chi}$ is a killing vector. Thus p_χ is conserved along the photon's world line. From $\boldsymbol{p}\cdot\boldsymbol{p}=0$, we have $p_t^2-a^{-2}p_\chi^2=0$. So $ap_t=p_\chi$ is a constant. Recall that $E=-p_t$. We have

$$\frac{\lambda_1}{\lambda_2}=\frac{E_2}{E_1}=\frac{a_1}{a_2}. \tag{29.2.6}$$

29.3 REDSHIFT OF PARTICLE DE BROGLIE WAVELENGTHS

A particle of finite rest mass μ moves along a geodesic world line through the expanding cosmological fluid. Let

$$p\equiv\frac{\mu v}{(1-v^2)^{1/2}} \tag{29.3.1}$$

be the spatial 4-momentum of the particle as measured by observers at rest in the fluid. (The ordinary velocity they measure in their proper reference frames is v.) The associated "de Broglie wavelength" of the particle is $\lambda\equiv h/p$.

(a) Show that this de Broglie wavelength is redshifted in precisely the same manner as a photon wavelength:

$$\lambda/a=\text{constant}. \tag{29.3.2}$$

(b) Employing this result, show that, for the molecules of an ideal gas that fills the universe, their mean kinetic energy decreases in inverse proportion to a^2 when the gas is nonrelativistic and (like photon energies) in inverse proportion to a when the gas is highly relativistic.

Solution:

(a) Similar to the case in exercise 29.2, we neglect coordinate θ and ϕ. So p_χ is still a constant along geodesic. And we also have the normalization condition $p_t^2-a^{-2}p_\chi^2=\mu^2$, leading to $p=a^{-1}p_\chi$. Note that $\lambda\propto 1/p$. We have

$$\lambda/a=\text{constant}. \tag{29.3.3}$$

(b) The energy distribution of particles is characterized by $\exp(-(E-\mu)/T)$, where T is the temperature of particles. When the gas is nonrelativistic, we have $E\approx\mu+p^2/2\mu$, and so $T\propto p^2\propto a^{-2}$. When the gas is relativistic, we have $E\approx p\gg\mu$, and so $T\propto p\propto a^{-1}$.

29.4 $m(z)$ DERIVED USING STATISTICAL PHYSICS

Derive the magnitude-redshift relation using a statistical description of the photon distribution [cf. eq. (22.49) and associated discussion].

Solution:

Equation 22.49 in the textbook states that the phase space density of photon gas is

$$\mathscr{N}=h^{-4}(I_\nu/\nu^3), \tag{29.4.1}$$

where

$$I_\nu\equiv\frac{d(\text{energy})}{d(\text{time})d(\text{area})d(\text{frequency})d(\text{solid angle})}. \tag{29.4.2}$$

And $\mathscr{N}$ is constant along geodesic. The flux can be calculated as

$$F = \int I_\nu d\nu \Delta\Omega, \tag{29.4.3}$$

where $\Delta\Omega$ is the angle extended by the object,

$$\Delta\Omega = \frac{A}{[a(t_e)\Sigma(\chi_r - \chi_e)]^2}. \tag{29.4.4}$$

So we have

$$F = \Delta\Omega \int I_{\nu_r} d\nu_r \propto \frac{1}{a(t_e)^2\Sigma^2(\chi_r - \chi_e)} \int I_{\nu_e} \frac{\nu_r^3}{\nu_e^3}\frac{\nu_r}{\nu_e} d\nu_e = \frac{1}{(1+z)^4 a(t_e)^2\Sigma^2(\chi_r - \chi_e)} \int I_e d\nu_e. \tag{29.4.5}$$

Note that $(1+z)a_{t_e} = a(t_r)$ and $R \equiv a(t_r)\Sigma(\chi_r - \chi_e)$. We obtain

$$F \propto \frac{1}{(1+z)^2 R^2}. \tag{29.4.6}$$

29.5 DOPPLER SHIFT VERSUS COSMOLOGICAL REDSHIFT

(a) Consider, in flat spacetime, a galaxy moving away from the Earth with velocity v, and emitting light that is received at Earth. Let the distance between Earth and galaxy, as measured in the Earth's Lorentz frame at some specific moment of emission, be r; and let the Doppler shift of the radiation when it is eventually received be $z = \Delta\lambda/\lambda$. Show that the flux of energy S received at the Earth is related to the galaxy's intrinsic luminosity L by

$$S = \frac{L}{4\pi r^2 (1+z)^4}. \tag{29.5.1}$$

(b) Compare this formula for the flux with formula (29.27), where the redshift is of cosmological origin. Why is the number of factors of $1+z$ different for the two formulas?

Solution:

(a) Suppose there is an observer located at the same place with earth and stationary relative to the galaxy. The flux of energy S received by the observer is

$$S' = \frac{L}{4\pi r'^2}. \tag{29.5.2}$$

The projected area of the galaxy in two frames are equivalent. Thus the solid angle extended by the galaxy in two frames are related by

$$r^2 \Delta\Omega = r'^2 \Delta\Omega'. \tag{29.5.3}$$

From equation (22.49), the intensity of light in two frames are related by

$$\frac{I_\nu}{\nu^3} = \frac{I_{\nu'}}{\nu'^3}. \tag{29.5.4}$$

Now, we have

$$S = \Delta\Omega \int I_\nu d\nu = \frac{r'^2}{r^2}\Delta\Omega' \int \frac{\nu^3}{\nu'^3} I_{\nu'} \frac{\nu}{\nu'} d\nu' = \frac{r'^2}{(1+z)^4 r^2} S' = \frac{L}{4\pi r^2 (1+z)^4}. \tag{29.5.5}$$

(b) Because there is a beaming effect in the case of Doppler redshift.

29.6 SOURCE COUNTS

Suppose that one could find (which one cannot) a family of light or radio sources that (1) are all identical with intrinsic luminosities L, (2) are distributed uniformly throughout the universe, and (3) are born at the same rate as they die so that the number in a unit comoving coordinate volume is forever fixed.

(a) Show that the number of such sources $N(z)$ with redshifts less than z, as observed from Earth today, would be

$$N(z) = (\text{constant}) \cdot z^3 \left[1 - \frac{3}{2}(1+q_0)z + O(z^2)\right]. \tag{29.6.1}$$

(b) Show that the number of sources $N(S)$ with fluxes greater than S as observed at Earth today would be

$$N(S) = (\text{constant}) \cdot \left(\frac{LH_0^2}{4\pi S}\right)^{3/2} \left[1 - 3\left(\frac{LH_0^2}{4\pi S}\right)^{1/2} + O\left(\frac{LH_0^2}{4\pi S}\right)\right]. \tag{29.6.2}$$

Solution:

(a) The number of sources is proportional to the comoving volume

$$V = \int_0^\chi 4\pi\Sigma(\chi')d\chi' = \begin{cases} \frac{1}{2}\chi - \frac{1}{4}\sin(2\chi) \\ \frac{1}{3}\chi^3 \\ -\frac{1}{2}\chi + \frac{1}{4}\sinh(2\chi) \end{cases} = \frac{1}{3}\chi^3 + O(\chi^5). \tag{29.6.3}$$

Note the equation (29.12) and (29.16) in the textbook:

$$\ell = a_0\chi, \quad z = H_0\ell + \frac{1}{2}(H_0\ell)^2 + O([H_0\ell]^3). \tag{29.6.4}$$

We can get

$$\chi = \frac{z}{a_0 H_0}\left[1 - \frac{1+q_0}{2}z + O(z^2)\right]. \tag{29.6.5}$$

Now we have

$$N \propto \chi^3 + O(\chi^5) \propto z^3\left[1 - \frac{3}{2}(1+q_0)z + O(z^2)\right]. \tag{29.6.6}$$

(b) The flux of the source received by us is

$$\frac{L}{4\pi(1+z)^2\Sigma^2 a_0^2}. \tag{29.6.7}$$

If the flux must be larger that S, the maximum redshift of the object would satisfy that

$$(1+z)\Sigma = \left(\frac{L}{4\pi a_0^2 S}\right)^{1/2}. \tag{29.6.8}$$

Note that $\Sigma = \chi + O(\chi^3)$ and equation 29.6.5. We have

$$(1+z)\Sigma = \frac{z}{a_0 H_0}\left[1 + \frac{1-q_0}{2}z + O(z^2)\right]. \tag{29.6.9}$$

Now we arrive at the equation

$$z\left[1 + \frac{1-q_0}{2}z + O(z^2)\right] = \left(\frac{LH_0^2}{4\pi S}\right)^{1/2}. \tag{29.6.10}$$

The solution for z is

$$z = \left(\frac{LH_0^2}{4\pi S}\right)^{1/2} \left[1 + \frac{q_0 - 1}{2}\left(\frac{LH_0^2}{4\pi S}\right)^{1/2} + O\left(\frac{LH_0^2}{4\pi S}\right)\right]. \tag{29.6.11}$$

Substitute 29.6.11 into 29.6.1. We can obtain

$$N(S) = (\text{constant}) \cdot \left(\frac{LH_0^2}{4\pi S}\right)^{3/2} \left[1 - 3\left(\frac{LH_0^2}{4\pi S}\right)^{1/2} + O\left(\frac{LH_0^2}{4\pi S}\right)\right]. \tag{29.6.12}$$

29.7 COSMIC-RAY DENSITY

Suppose the universe has contained the same number of galaxies indefinitely into the past. Suppose further that the cosmic rays in the universe were created in galaxies and that a negligible fraction of them have been degraded or lost since formation. Derive an expression for the average density of energy in cosmic rays in the universe today in terms of: (1) the number density of galaxies, N_0, today; and (2) the nonconstant rate, dE/dz, at which the average galaxy created cosmic-ray energy during the past history of the universe.

Solution:

At redshift z in range dz, the average galaxy liberates energy $(dE/dz)dz$ into cosmic rays. Taking the redshift of cosmic ray into account, the average density of energy in cosmic rays in the universe today is

$$\epsilon = N_0 \int_0^{z_{\max}} \frac{1}{1+z}\frac{dE}{dz}dz. \tag{29.7.1}$$

29.8 FRACTION OF SKY COVERED BY GALAXIES

Assume that the redshifts of quasars are cosmological. Let the number of galaxies per unit physical volume in the universe today be N_0, and assume that no galaxies have been created or destroyed since a redshift of > 7. Let D be the average angular diameter of a galaxy. Calculate the probability that the light from a quasar at redshift z, has passed through at least one intervening galaxy during its travel to Earth.

Solution:

I think it is the physical diameter of a galaxy that matters and can be averaged. So I will assume the D to be the average physical diameter of a galaxy.

Suppose we sit at $\chi = 0$, then the light at redshift z will be at

$$\chi(z) = \frac{0}{z}\frac{dz'}{H_0}\frac{1}{\sqrt{\Omega_{r0}(1+z')^4 + \Omega_{m0}(1+z')^3 + \Omega_{K0}(1+z')^2 + \Omega_\Lambda}}. \tag{29.8.1}$$

The physical distance from $z + dz$ to z is

$$dl(z) = a\frac{d\chi}{dz}dz \tag{29.8.2}$$

The probability that the light from a quasar at redshift z, has passed through at least one intervening galaxy during its travel to earth is therefore

$$P = \int_0^z N_0 D dl(z). \tag{29.8.3}$$

Chapter 31

SCHWARZSCHILD GEOMETRY

31.1 TIDAL FORCES ON IN FALLING EXPLORER

(a) Carry out the details of the derivation of the Riemann tensor components (31.6).

(b) Calculate, roughly, the critical mass $M_{\rm cric}$ such that, if $M > M_{\rm cric}$ the explorer's body (a human body made of normal flesh and bones) can withstand the tidal forces at $r = 2M$, but if $M < M_{\rm cric}$ his body is mutilated by them.

Solution:

(a) Focusing on the $\hat{t}$ and $\hat{r}$, the transformation matrix is

$$\Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \tag{31.1.1}$$

where $\beta = v^{\hat{r}}$ and $\gamma = (1-\beta^2)^{-1/2}$. So

$$\begin{aligned} R_{\hat{\tau}\hat{\rho}\hat{\tau}\hat{\rho}} &= \Lambda_{\hat{\tau}}{}^{\hat{\alpha}}\Lambda_{\hat{\rho}}{}^{\hat{\beta}}\Lambda_{\hat{\tau}}{}^{\hat{\gamma}}\Lambda_{\hat{\rho}}{}^{\hat{\delta}}R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \\ &= \Lambda_{\hat{\tau}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{r}}\Lambda_{\hat{\tau}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{r}}R_{\hat{t}\hat{r}\hat{t}\hat{r}} + \Lambda_{\hat{\tau}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{t}}\Lambda_{\hat{\tau}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{r}}R_{\hat{r}\hat{t}\hat{t}\hat{r}} + \Lambda_{\hat{\tau}}{}^{\hat{t}}\Lambda_{\hat{\rho}}{}^{\hat{r}}\Lambda_{\hat{\tau}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{t}}R_{\hat{t}\hat{r}\hat{r}\hat{t}} + \Lambda_{\hat{\tau}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{t}}\Lambda_{\hat{\tau}}{}^{\hat{r}}\Lambda_{\hat{\rho}}{}^{\hat{t}}R_{\hat{r}\hat{t}\hat{r}\hat{t}} \\ &= \gamma^4(1-2\beta^2+\beta^4)R_{\hat{t}\hat{r}\hat{t}\hat{r}} \\ &= R_{\hat{t}\hat{r}\hat{t}\hat{r}}. \end{aligned} \tag{31.1.2}$$

Other equations in (31.6) can be verified by the same procedure.

(b) The tidal force per unit mass near the event horizon is of the order

$$g \sim \frac{M}{r^3}l \sim \frac{l}{M^2}, \tag{31.1.3}$$

where l is the length scale of the human. If we take $g_{\rm cric} = 10^2{\rm m/s^2}$ and $l = 1m$, we can get

$$M_{\rm cric} \sim \frac{c^3}{G}\sqrt{\frac{l}{g}} \approx 4\times 10^{34}{\rm kg} \approx 2\times 10^4 M_{\rm sun}. \tag{31.1.4}$$

31.2 NONRADIAL LIGHT CONES

Show that the world line of a photon traveling nonradially makes an angle less than 45 degrees with the vertical v-axis of a Kruskal-Szekeres coordinate diagram. From this, infer that particles with finite rest mass, traveling nonradially or radially, must always move "generally upward" (angle less than 45 degrees with vertical v-axis).

Solution:

The metric in Kruskal-Szekeres coordinate is

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{31.2.1}$$

For photons, we have

$$\frac{32M^3}{r}e^{-r/2M}[-(p^v)^2 + (p^u)^2] + r^2[(p^\theta)^2 + (p^\phi)^2] = 0. \tag{31.2.2}$$

Thus, we must have $p^v > |p^u|$, i.e. the world line of a photon traveling nonradially makes an angle less than 45 degrees with the vertical v-axis of a Kruskal-Szekeres coordinate diagram. And also, articles with finite rest mass, traveling nonradially or radially, must always move "generally upward".

31.3 THE CRACK OF DOOM

Use a Kruskal diagram to show the following.

(a) If a man allows himself to fall through the gravitational radius $r = 2M$, there is no way whatsoever for him to avoid hitting (and being killed in) the singularity at $r = O$.

(b) Once a man has fallen inward through $r = 2M$, there is no way whatsoever that he can send messages out to his friends at $r > 2M$, but he can still receive messages from them.

Solution:

In Kruskal diagram, all particles, traveling nonradially or radially, must always move "generally upward", i.e. angle less than 45 degrees with vertical v-axis. The horizon $r = 2M$ is $v = \pm u$ in the diagram. By simple geometrical analysis, we can verify two statements above.

31.4 HOW LONG TO LIVE?

Show that once a man falling inward reaches the gravitational radius, no matter what he does subsequently (no matter in what directions, how long, and how hard he blasts his rocket engines), he will be pulled into the singularity and killed In a proper time of

$$\tau < \tau_{\max} = \pi M = 1.54 \times 10^{-5}\frac{M}{M_\odot}\text{seconds}. \tag{31.4.1}$$

Solution:

The trajectory of longest proper time lapse must be a geodesic. The radial motion of a free particle is described by

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) \tag{31.4.2}$$

In order to make τ as large as possible, we must have $\tilde{E} = 0$ and $\tilde{L} = 0$, i.e.

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r} - 1 \tag{31.4.3}$$

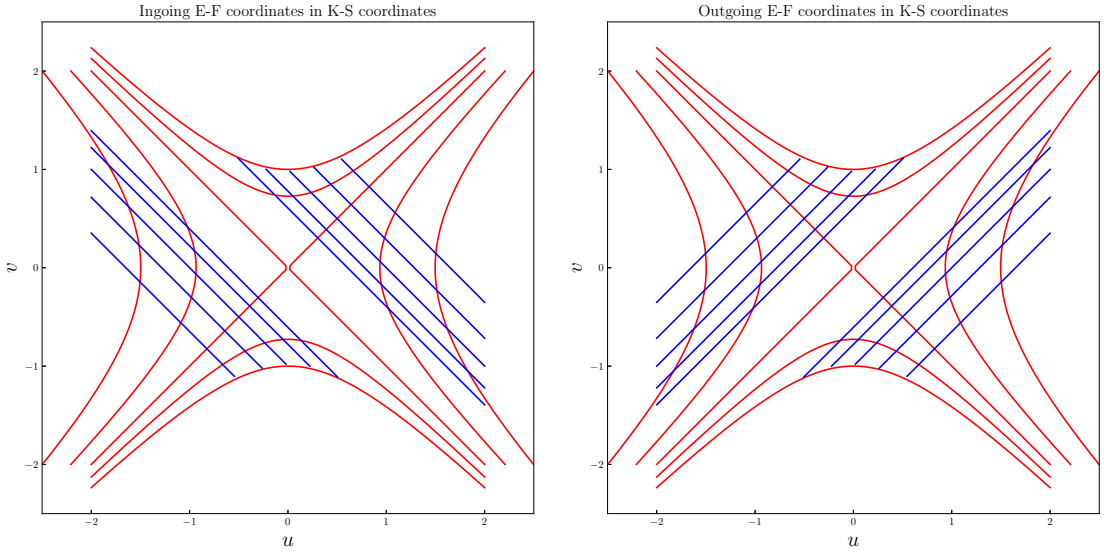
So

$$\tau_{\max} = \int_0^{2M} \frac{dr}{\sqrt{\frac{2M}{r} - 1}} = \pi M = \pi \frac{GM_\odot}{c^3} \frac{M}{M_\odot} = 1.54 \times 10^{-5} \frac{M}{M_\odot} \text{seconds}. \tag{31.4.4}$$

31.5 EDDINGTON-FINKELSTEIN AND KRUSKAL-SZEKERES COMPARED

Use coordinate diagrams to compare the ingoing and outgoing Eddington-Finkelstein coordinates of Box 31.2 with the Kruskal-Szekeres coordinates. Pattern the comparison after that between Schwarzschild and Kruskal-Szekeres in Figures 31.3 and 31.4.

Solution:



The red lines are constant r, while the blue lines are constant $\tilde{V}$ in the left and constant $\tilde{U}$ in the right.

31.6 ANOTHER COORDINATE SYSTEM

Construct a coordinate diagram for the $\tilde{U}$, $\tilde{V}$, θ, ϕ coordinate system of Box 31.2. Show such features as (1) the relationship to Schwarzschild and to Kruskal-Szekeres coordinates; (2) the location of $r = 2M$; and (3) radial geodesics.

Solution:

(1) The relationship to Schwarzschild coordinates is

$$\tilde{U} = t - r - 2M \ln|r/2M - 1|, \quad \tilde{V} = t + r + 2M \ln|r/2M - 1|. \tag{31.6.1}$$

The relationship to Kruskal-Szekeres coordinates is

$$\tilde{U} = -4M \ln(u - v), \quad \tilde{V} = 4M \ln(u + v). \tag{31.6.2}$$

(2) The location of $r = 2M$ is $\tilde{V} = -\infty$ and $\tilde{U} = \infty$.

(3) The line elements for constant θ and ϕ is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{U} d\tilde{V}. \tag{31.6.3}$$

Connection in this coordinates is

$$\Gamma_{\tilde{V}\tilde{U}\tilde{U}} = g_{\tilde{V}\tilde{U},\tilde{U}}, \quad \Gamma_{\tilde{U}\tilde{V}\tilde{V}} = g_{\tilde{V}\tilde{U},\tilde{V}}. \tag{31.6.4}$$

All other components vanish. And so

$$\Gamma^{\tilde{U}}{}_{\tilde{U}\tilde{U}} = \frac{g_{\tilde{V}\tilde{U},\tilde{U}}}{g_{\tilde{V}\tilde{U}}}, \quad \Gamma^{\tilde{V}}{}_{\tilde{V}\tilde{V}} = \frac{g_{\tilde{V}\tilde{U},\tilde{V}}}{g_{\tilde{V}\tilde{U}}}. \tag{31.6.5}$$

The radial geodesics for particles with finite mass are

$$\frac{d^2\tilde{U}}{d\tau^2} + \Gamma^{\tilde{U}}{}_{\tilde{U}\tilde{U}} \left(\frac{d\tilde{U}}{d\tau}\right)^2 = 0 \tag{31.6.6}$$

and

$$\frac{d^2\tilde{V}}{d\tau^2} + \Gamma^{\tilde{V}}{}_{\tilde{V}\tilde{V}} \left(\frac{d\tilde{V}}{d\tau}\right)^2 = 0. \tag{31.6.7}$$

And we also have normalization

$$g_{\tilde{U}\tilde{V}} \frac{d\tilde{U}}{d\tau} \frac{d\tilde{V}}{d\tau} = 1. \tag{31.6.8}$$

While the radial geodesics for photons are $\tilde{U} =$ constant or $\tilde{V} =$ constant.

31.7 SCHWARZSCHILD METRIC IN ISOTROPIC COORDINATES

(a) Show that, rewritten in the isotropic coordinates of Exercise 23.1, the Schwarzschild metric reads

$$ds^2 = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 [d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2)]; \tag{31.7.1}$$

and derive the transformation

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2 \tag{31.7.2}$$

between the two radial coordinates.

(b) regions of spacetime (I, II, III, IV; see Figure 31.3) are covered by the isotropic coordinate patch, and which are not?

(c) Calculate and construct an embedding diagram for the spacelike hypersurface $t = 0$, $0 < \bar{r} < \infty$.

(d) Find a coordinate transformation that interchanges the region near $r = 0$ with the region near $\bar{r} = \infty$, while leaving the metric coefficients in their original form.

Solution:

(a) The metric in Schwarzschild coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{31.7.3}$$

The metric in isotropic coordinates must be the form of

$$ds^2 = -f^2(\bar{r})dt^2 + g^2(\bar{r})(d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{31.7.4}$$

So we have

$$g(\bar{r})\bar{r} = r, \quad g(\bar{r})d\bar{r} = \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \tag{31.7.5}$$

i.e.

$$\frac{d\bar{r}}{\bar{r}} = \frac{dr}{r\sqrt{1 - 2M/r}}. \tag{31.7.6}$$

By choosing $\bar{r} = M/2$ when $r = 2M$, we have

$$\frac{2\bar{r}}{M} = \frac{1 + \sqrt{1 - 2M/r}}{1 - \sqrt{1 - 2M/r}}. \tag{31.7.7}$$

Now it is easy to derive

$$\sqrt{1 - 2M/r} = \frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}, \quad r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2. \tag{31.7.8}$$

And the metric in isotropic coordinates is

$$ds^2 = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{31.7.9}$$

(b) When $\bar{r} = M/2$,$r(\bar{r})$ has minimum $2M$. When $\bar{r} \to 0$ or $\bar{r} \to \infty$, $r \to \infty$. So region I and III are covered by isotropic coordinate patch.

(c) The embedding function $z(\bar{r})$ is given by

$$dr^2 + dz^2 = \left(1 + \frac{M}{2\bar{r}}\right)^4 d\bar{r}^2, \quad r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r}. \tag{31.7.10}$$

The solution is

$$z = 4M\sqrt{r/2M - 1} = 4M\sqrt{\frac{\bar{r}}{2M} + \frac{M}{8\bar{r}} - \frac{1}{2}}. \tag{31.7.11}$$

(d) The transformation $\bar{r} \to \frac{M^2}{4\bar{r}}$ leaves $z(\bar{r})$ and $r(\bar{r})$ unchanged. So it will also leave metric unchanged.

(e) Similarly explore the spacetime geometry for $Q = M$.

31.8 REISSNER-NORDSTROM GEOMETRY

(a) Solve the Einstein field equations for a spherically symmetric, static gravitational field

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{31.8.1}$$

with no matter present, but with a radial electric field $\boldsymbol{B} = 0$, $\boldsymbol{E} = f(r)\boldsymbol{e}_{\hat{r}}$ in the static orthonormal frame

$$\boldsymbol{\omega}^{\hat{t}} = e^{\Phi}\boldsymbol{d}t, \quad \boldsymbol{\omega}^{\hat{r}} = e^{\Lambda}\boldsymbol{d}r, \quad \boldsymbol{\omega}^{\hat{\theta}} = r\boldsymbol{d}\theta, \quad \boldsymbol{\omega}^{\hat{\phi}} = r\sin\theta\boldsymbol{d}\phi. \tag{31.8.2}$$

Use as a source in the Einstein field equations the stress-energy of the electric field.

(b) Show that the constant Q is the total charge as measured by a distant observer ($r \gg 2M$ and $r \gg Q$), who uses a Gaussian flux integral, or who studies the coulomb-force-dominated orbits of test charges with charge-to-mass ratio $e/\mu \gg M/Q$. What is the charge-to-mass ratio, in dimensionless units, for an electron? Show that the constant M is the total mass as measured by a distant observer using the Keplerian orbits of electrically neutral particles.

(c) Show that for $Q > M$, the Reissner-Nordstrom coordinate system is well behaved from $r = \infty$ down to $r = 0$, where there is a physical singularity and infinite tidal forces.

(d) Explore the nature of the spacetime geometry for $Q < M$, using all the techniques of this chapter (coordinate transformations, Kruskal-like coordinates, studies of particle orbits, embedding diagrams, etc.).

(e) Similarly explore the spacetime geometry for $Q = M$.

(f) For the case of a large ratio of charge to mass, show that the region near $r = 0$ is unphysical. More precisely, show that any spherically symmetric distribution of charged stressed matter that gives rise to the fields $\boldsymbol{E} = Q/r^2\boldsymbol{e}_r$ outside its boundary must modify these fields for $r < r_0 = Q^2/2M$.

Solution:

(a) In the static orthonormal frame, the stress energy tensor of the EM field is

$$\boldsymbol{T}_{\hat{t}\hat{t}} = \frac{f^2}{8\pi}, \quad \boldsymbol{T}_{\hat{r}\hat{r}} = -\frac{f^2}{8\pi}, \quad \boldsymbol{T}_{\hat{\theta}\hat{\theta}} = \frac{f^2}{8\pi}, \quad \boldsymbol{T}_{\hat{\phi}\hat{\phi}} = \frac{f^2}{8\pi}. \tag{31.8.3}$$

All other terms vanish. On the other hand, we have

$$\begin{aligned} &G_{\hat{t}\hat{t}} = r^{-2}(1 - e^{-2\Lambda}) + 2r^{-1}e^{-2\Lambda}\Lambda', \quad G_{\hat{r}\hat{r}} = -r^{-2}(1 - e^{-2\Lambda}) + 2r^{-1}e^{-2\Lambda}\Phi' \\ &G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = e^{-2\Lambda}\left[r^{-1}(\Phi' - \Lambda') + \Phi'' + \Phi'^2 - \Phi'\Lambda'\right]. \end{aligned} \tag{31.8.4}$$

Using Einstein equation, we have $G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = 0$, from which we can obtain $\Lambda' + \Phi' = 0$. By $G_{\hat{t}\hat{t}} = G_{\hat{\theta}\hat{\theta}}$, we can get $(r^2e^{-2\Lambda})'' = 2$. The general solution is $r^2e^{-2\Lambda} = r^2 - 2Mr + Q^2$. And so $e^{2\Phi} = C(1 - 2M/r + Q^2/r^2)$. By rescaling t coordinates, we can get

$$f(r) = \frac{Q}{r^2}, \quad e^{2\Phi} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad e^{2\Lambda} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}. \tag{31.8.5}$$

The metric is therefore

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{31.8.6}$$

(b) Using Gaussian theorem, we have

$$4\pi Q_{\text{total}} = \oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \int \frac{Q}{r^2} \cdot r^2 \sin\theta d\theta d\phi = 4\pi Q, \tag{31.8.7}$$

i.e. Q is the total charge as measured by a distant observer. For electron, the charge-to-mass ratio in dimensionless units is

$$\frac{e}{m_e} = \frac{\sqrt{k_e G} e/c^2}{G m_e/c^2} = \sqrt{\frac{k_e}{G}} \frac{e}{m_e} = 2.04 \times 10^{21}. \tag{31.8.8}$$

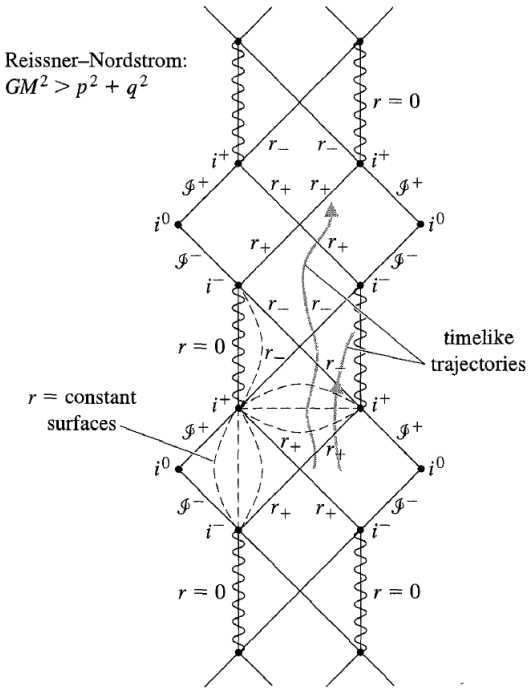
For electrically neutral particles, there orbits will follow geodesics of the metric. For a distant observer, the r^{-2} term in the metric can be neglected. So the orbit of a electrically neutral particle will be essentially the same as that in Schwarzschild metric. Thus M can be viewed as the total mass of the system.

(c) If $Q > M$, then we have

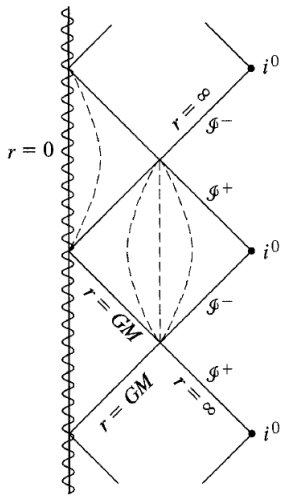
$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-M)^2 + Q^2 - M^2}{r^2} > 0. \tag{31.8.9}$$

So the Reissner-Nordstrom coordinate system is well behaved from $r = \infty$ down to $r = 0$, where there is a physical singularity and infinite tidal forces.

(d) If $Q < M$, there will be two horizons $r_\pm = M \pm \sqrt{M^2 - Q^2}$. In the region $r > r_+$ and $r < r_-$, the t direction (constant r) is time like. While In the region $r_- < r < r_+$, the t direction is time like. The conformal diagram for the whole spacetime is given by the figure below.



(e) There is only one horizon $r = M$ in this case. And the t direction is spacelike when $r < M$ and $r > M$ ($r = M$ is null space). The conformal diagram for the whole spacetime is given by the figure below.



(f) For the case of a large ratio of charge to mass, $r = 0$ is a naked singularity which is observable from the outside. I known little about the naked singularities and their properties. Any help is appreciated. Please refer to Gravitational collapse (Penrose, R.) and Generalized strong curvature singularities and weak cosmic censorship in cosmological space-times (W. Rudnicki et al.) for more information.

Chapter 32

GRAVITATIONAL COLLAPSE

32.1 UNIQUENESS OF REISSNER-NORDSTROM GEOMETRY

Prove the following generalization of Birkhoff's theorem. Let the geometry of a given region of spacetime (1) be spherically symmetric, and (2) be a solution to the Einstein field equations with an electromagnetic field as source. Then that geometry is necessarily a piece of the Reissner-Nordstrom geometry [equation (31.24b)] with electric and magnetic fields, as measured in the standard static orthonormal frames

$$\boldsymbol{E} = \frac{Q_e}{r^2}\boldsymbol{e}_{\hat{r}}, \quad \boldsymbol{B} = \frac{Q_m}{r^2}\boldsymbol{e}_{\hat{r}}, \quad Q = (Q_m^2 + Q_e^2)^{1/2}. \tag{32.1.1}$$

Solution:

(1) First consider regions of spacetime in which $(\boldsymbol{\nabla} r)^2 \neq 0$. In this case, one can introduce Schwarzschild coordinates

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \Phi = \Phi(t,r), \quad \Lambda = \Lambda(t,r), \tag{32.1.2}$$

according to Box 23.3 of the textbook. Using the orthonormal components of the Einstein tensor as derived in exercise 14.16, we have

$$\begin{aligned}
&G_{\hat{t}\hat{t}} = r^{-2}(1 - e^{-2\Lambda}) + 2(\Lambda_{,r}/r)e^{-2\Lambda},\\
&G_{\hat{t}\hat{r}} = G_{\hat{r}\hat{t}} = 2(\Lambda_{,t}/r)e^{-(\Lambda+\Phi)},\\
&G_{\hat{r}\hat{r}} = 2(\Phi_{,r}/r)e^{-2\Lambda} - r^{-2}(1 - e^{-2\Lambda}),\\
&G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = (\Phi_{,rr} + \Phi_{,r}^2 - \Phi_{,r}\Lambda_{,r} + \Phi_{,r}/r - \Lambda_{,r}/r)e^{-2\Lambda} - (\Lambda_{,tt} + \Lambda_{,t}^2 - \Lambda_{,t}\Phi_{,t})e^{-2\Phi},\\
&G_{\hat{t}\hat{\theta}} = G_{\hat{\theta}\hat{t}} = G_{\hat{t}\hat{\phi}} = G_{\hat{\phi}\hat{t}} = G_{\hat{r}\hat{\theta}} = G_{\hat{\theta}\hat{r}} = G_{\hat{r}\hat{\phi}} = G_{\hat{\phi}\hat{r}} = G_{\hat{\theta}\hat{\phi}} = G_{\hat{\phi}\hat{\theta}} = 0.
\end{aligned} \tag{32.1.3}$$

The Einstein equation states that $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$, where $T_{\hat{\mu}\hat{\nu}}$ is the stress energy of an an electromagnetic field. According to exercise 20.6 of the textbook, the Maxwell stress-energy tensor has zero trace, and that its square is a multiple of the unit tensor. Thus we have

$$G^{\hat{\mu}}{}_{\hat{\nu}} = 0, \quad G^{\hat{\mu}}{}_{\hat{\alpha}}G^{\hat{\alpha}}{}_{\hat{\nu}} \propto \delta^{\hat{\mu}}_{\hat{\nu}}. \tag{32.1.4}$$

Or explicitly,

$$-G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} + 2G_{\hat{\theta}\hat{\theta}} = 0, \quad G_{\hat{t}\hat{r}}(G_{\hat{t}\hat{t}} - G_{\hat{r}\hat{r}}) = 0, \quad G_{\hat{t}\hat{t}}^2 - G_{\hat{t}\hat{r}}^2 = G_{\hat{r}\hat{r}}^2 - G_{\hat{t}\hat{r}}^2 = G_{\hat{\theta}\hat{\theta}}^2. \tag{32.1.5}$$

If $G_{\hat{t}\hat{r}} = 0$, then it is easy to derive that $G_{\hat{t}\hat{t}} = -G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$.

Firstly, we have $\Lambda_{,t}=0$, according to $G_{\hat{t}\hat{r}}=0$. Next, $G_{\hat{t}\hat{t}}+G_{\hat{r}\hat{r}}=0$ leads to $\Lambda_{,r}+\Phi_{,r}=0$. Furthermore, using $G_{\hat{t}\hat{t}}=G_{\hat{\theta}\hat{\theta}}$, we find that $(r^2e^{-2\Lambda})_{,rr}=2$. The general solution is

$$e^{-2\Lambda}=1-\frac{2M}{r}+\frac{Q^2}{r^2}. \tag{32.1.6}$$

Then applying $\Lambda_{,r}+\Phi_{,r}=0$, we have

$$e^{2\Phi}=C^2(t)\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right) \tag{32.1.7}$$

By redefining the time coordinate $t_{\rm new}=\int^t C(t)dt$, we will obtain the R-N metric

$$ds^2=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{32.1.8}$$

The stress energy tensor of EM field in the local Lorentz coordinate frame is

$$T_{\hat{t}\hat{t}}=\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi},\ T_{\hat{t}\hat{i}}=T_{\hat{i}\hat{t}}=-\frac{(\boldsymbol{E}\times\boldsymbol{B})_{\hat{i}}}{4\pi},\ T_{\hat{j}\hat{k}}=\frac{1}{4\pi}\left[-(E_{\hat{j}}E_{\hat{k}}+B_{\hat{j}}B_{\hat{k}})+\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)\delta_{\hat{j}\hat{k}}\right]. \tag{32.1.9}$$

It is easy to verify that $\boldsymbol{E}$ and $\boldsymbol{B}$ must be radial to produce the required from of stress energy tensor in this case. Suppose the electromagnetic field is

$$\boldsymbol{E}=f(t,r)\boldsymbol{e}_{\hat{r}},\quad \boldsymbol{B}=g(t,r)\boldsymbol{e}_{\hat{r}}. \tag{32.1.10}$$

Then we have

$$T_{\hat{t}\hat{t}}=-T_{\hat{r}\hat{r}}=T_{\hat{\theta}\hat{\theta}}=T_{\hat{\phi}\hat{\phi}}=\frac{f^2+g^2}{8\pi}. \tag{32.1.11}$$

The field equation gives that $f^2+g^2=Q^2/r^4$. So the EM field is

$$\boldsymbol{E}=\frac{Q_e}{r^2}\boldsymbol{e}_{\hat{r}},\quad \boldsymbol{B}=\frac{Q_m}{r^2}\boldsymbol{e}_{\hat{r}},\quad Q=(Q_m^2+Q_e^2)^{1/2}. \tag{32.1.12}$$

If $G_{\hat{r}\hat{r}}\neq 0$, then it is easy to derive that $G_{\hat{t}\hat{t}}=G_{\hat{r}\hat{r}}=|G_{\hat{t}\hat{r}}|$ and $G_{\hat{\theta}\hat{\theta}}=G_{\hat{\phi}\hat{\phi}}=0$. The spacetime will be described by **Vaidya metric**,

$$\begin{aligned} ds^2&=-\left[1-\frac{2M(u)}{r}\right]du^2-2dudr+r^2d\Omega^2\quad\text{(outgoing)},\\ ds^2&=-\left[1-\frac{2M(v)}{r}\right]dv^2+2dvdr+r^2d\Omega^2\quad\text{(ingoing)}\end{aligned} \tag{32.1.13}$$

in null-radial coordinates. For outgoing Vaidya metric, the Einstein tensor is

$$G_{uu}=-2\frac{M_{,u}}{r^2}. \tag{32.1.14}$$

All other components vanish. Suppose the coordinate transformation $t(u,r)$ can bring the metric into diagonal coordinates. Then we have

$$\frac{\partial u}{\partial r}=-\left(1-\frac{2M}{r}\right)^{-1},\quad g_{tt}=\left(\frac{\partial u}{\partial t}\right)^2\left(\frac{\partial u}{\partial r}\right)^{-1},\quad g_{rr}=-\frac{\partial u}{\partial r}. \tag{32.1.15}$$

The Einstein tensor in $t-r$ coordinate is

$$G_{tt}=\left(\frac{\partial u}{\partial t}\right)^2R_{uu},\quad G_{tr}=\frac{\partial u}{\partial t}\frac{\partial u}{\partial r}R_{uu},\quad G_{rr}=\left(\frac{\partial u}{\partial r}\right)^2R_{uu}. \tag{32.1.16}$$

Written in orthonormal frame, we have

$$G_{\hat{t}\hat{t}}=-G_{\hat{t}\hat{r}}=G_{\hat{r}\hat{r}}=-\frac{\partial u}{\partial r}G_{uu}=\frac{-2M_{,u}}{r^2(1-2M/r)}. \tag{32.1.17}$$

In the case of ingoing Vaidya metric, the analysis can be carried in the same way. Please refer to the paper *Vaidya spacetime in the diagonal coordinates (Berezin et al. 2017)* for more discussion on coordinate transformation.

If the source of stress energy tensor is EM field, we can easily deduce that $E_{\hat{r}} = B_{\hat{r}} = 0$, $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ and $|\boldsymbol{E}| = |\boldsymbol{B}|$. It seems that a EM wave propagating radially can generate the Vaidya spacetime. However, according to **Vaidya metric**, the Vaidya field is a pure radiation field rather than electromagnetic fields. The emitted particles or energy-matter flows have zero rest mass and thus are generally called "null dusts", typically such as photons and neutrinos, but cannot be electromagnetic waves because the Maxwell-Newman-Penrose equations are not satisfied. I know little about Maxwell-Newman-Penrose equations and the detailed proof is beyond my ability. From know on, I will take it for granted that EM field is non-null.

The following discussion is based on the "Hint" of the exercise in the textbook.

(2) Any region of dimensionality less than four, In which $(\boldsymbol{\nabla} r)^2 = 0$ (e.g., the Schwarzschild radius), can be treated as the join between four-dimensional regions with $(\boldsymbol{\nabla} r)^2 \neq 0$. Moreover, the geometry of such a region is determined uniquely by the geometry of the adjoining four-dimensional regions ("junction conditions"; section 21.13). Since the adjoining regions are necessarily Reissner-Nordstrom, then so are such "sandwiched" regions.

(3) Next consider four-dimensional regions in which $(\boldsymbol{\nabla} r)^2 = 0$ is null and nonzero. According to Box 23.3, the metric of spherically symmetric spacetime has the form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + R^2(r,t)d\Omega^2. \tag{32.1.18}$$

Now we build a new coordinate system (T, R). If $(\boldsymbol{\nabla} R)^2 = 0$, then

$$g^{RR} = g^{tt}(\frac{\partial R}{\partial t})^2 + g^{rr}(\frac{\partial R}{\partial r})^2 = (\boldsymbol{\nabla} R)^2 = 0. \tag{32.1.19}$$

We can choose $T(r,t)$ satisfying

$$g^{tt}(\frac{\partial T}{\partial t})^2 + g^{rr}(\frac{\partial T}{\partial r})^2 = 0, \tag{32.1.20}$$

i.e. choose another null surface of the spacetime to be the T coordinate plane. We now have $g^{TT} = g^{RR} = 0$, and so $g_{TT} = g_{RR} = 0$. By a change of notation, the metric has the form

$$ds^2 = -2\Psi dr dt + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{32.1.21}$$

where $\Psi = \Psi(r,t)$. It is straight to get $R_{\theta\theta} = 1$ and $R_{\phi\phi} = \sin^2\theta$ from the metric above. In orthonormal frame, we have $R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = 1/r^2$. Whereas the stress-energy tensor for a spherically symmetric electromagnetic field has

$$8\pi\tilde{T}_{\hat{\theta}\hat{\theta}} \equiv 8\pi\left(T_{\hat{\theta}\hat{\theta}} - \frac{1}{2}g_{\hat{\theta}\hat{\theta}}T\right) = Q^2/r^4, \quad Q = \text{ constant}. \tag{32.1.22}$$

These quantities, $R_{\hat{\theta}\hat{\theta}}$ and $8\pi\tilde{T}_{\hat{\theta}\hat{\theta}}$ must be equal (Einstein's field equation) but cannot be because of their different r-dependence. Thus, an electromagnetic field cannot generate regions with $\boldsymbol{d}r = 0$, $\boldsymbol{d}r \cdot \boldsymbol{d}r = 0$.

(4) Finally, consider four-dimensional regions in which $\boldsymbol{d}r = 0$. Denote the constant value of r by a. Starting from equation 32.1.18, we have

$$ds^2 = g_{t't'}dt'^2 + g_{r'r'}dr'^2 + a^2d\Omega^2. \tag{32.1.23}$$

Clearly, (t', r') and (θ, ϕ) are decoupled with each other. So we can neglect (θ, ϕ) and only focus on the (t', r') coordinates. In two dimensional manifold, there are only 3 independent components in metric. The coordinate transformation can further eliminate two degrees of freedom. So the metric can be brought into the form

$$ds^2 = a^2(-d\tilde{\tau}^2 + e^{2\lambda}dz^2 + d\Omega^2). \tag{32.1.24}$$

We choose the initial point of $\tilde{\tau}$ coordinate for every $z=$ constant in order to make $\dot{\lambda}(z)=0$ at that point, and rescale the line $\tilde{\tau}=0$ to make $\lambda(z)=0$ at that point. Then we have

$$\lambda(\tilde{\tau},z)=0,\quad \dot{\lambda}(\tilde{\tau},z)=0. \tag{32.1.25}$$

In the associated orthonormal frame, spherical symmetry demands

$$\boldsymbol{E}=\frac{Q_e}{a^2}\boldsymbol{e}_{\hat{z}},\quad \boldsymbol{B}=\frac{Q_m}{a^2}\boldsymbol{e}_{\hat{z}},\quad Q=(Q_m^2+Q_e^2)^{1/2}. \tag{32.1.26}$$

The Ricci tensor in orthonormal coordinates are

$$R_{\hat{\tau}\hat{\tau}}=-a^{-2}(\dot{\lambda}^2+\ddot{\lambda}),\ R_{\hat{z}\hat{z}}=a^{-2}(\dot{\lambda}^2+\ddot{\lambda}),\ R_{\hat{\theta}\hat{\theta}}=R_{\hat{\phi}\hat{\phi}}=a^{-2}. \tag{32.1.27}$$

For gravitation field with E-M field as source, the trace of Einstein tensor vanish, leading to $\dot{\lambda}^2+\ddot{\lambda}=-1$. Under the constraint 32.1.25, we have the unique solution $\lambda=\ln\cos\tilde{\tau}$. Also note that $8\pi T_{\hat{\theta}\hat{\theta}}=Q^2/a^4$. We must have $Q=a$. Now the metric becomes

$$ds^2=Q^2(-d\tilde{\tau}^2+\cos^2\tilde{\tau}dz^2+d\Omega^2). \tag{32.1.28}$$

(5) The Reissner-Nordstrom solution for the special case $Q=M$ is

$$ds^2=-\left(1-\frac{Q}{r}\right)^2dt^2+\left(1-\frac{Q}{r}\right)^{-2}dr^2+r^2d\Omega^2. \tag{32.1.29}$$

Perform the following coordinate transformation on the Reissner-Nordstrom throat region ($|r-Q|\ll Q$):

$$r-Q=Qe^{-z}\cos\tilde{\tau},\quad t=Qe^z\tan\tilde{\tau}. \tag{32.1.30}$$

Firstly, we have

$$g_{\tilde{\tau}\tilde{\tau}}=\left(\frac{\partial t}{\partial\tilde{\tau}}\right)^2g_{tt}+\left(\frac{\partial r}{\partial\tilde{\tau}}\right)^2g_{rr}=-Q^2\left[\frac{1}{\cos^2\tilde{\tau}(1+e^{-z}\cos\tilde{\tau})^2}-\frac{\sin^2\tilde{\tau}(1+e^{-z}\cos\tilde{\tau})^2}{\cos^2\tilde{\tau}}\right] \tag{32.1.31}$$

Note that $z\gg 0$ in the throat region. If we keep the zero order, we have $g_{\tilde{\tau}\tilde{\tau}}=-Q^2$. Similarly, we have

$$g_{\tilde{\tau}z}=\frac{\partial t}{\partial\tilde{\tau}}\frac{\partial t}{\partial\tilde{z}}g_{tt}+\frac{\partial r}{\partial\tilde{\tau}}\frac{\partial r}{\partial\tilde{z}}g_{rr}=Q^2\tan\tilde{\tau}\left[-\frac{1}{(1+e^{-z}\cos\tilde{\tau})^2}+(1+e^{-z}\cos\tilde{\tau})^2\right]\approx 0 \tag{32.1.32}$$

and

$$g_{zz}=\left(\frac{\partial t}{\partial z}\right)^2g_{tt}+\left(\frac{\partial r}{\partial z}\right)^2g_{rr}=Q^2\left[(1+e^{-z}\cos\tilde{\tau})^2-\frac{\sin^2\tilde{\tau}}{(1+e^{-z}\cos\tilde{\tau})^2}\right]\approx Q^2\cos^2\tilde{\tau}. \tag{32.1.33}$$

So the solution of the field equations in (4) is actually the throat of the Reissner-Nordstrom solution for the special case $Q=M$.

(6) Thus, each possible case leads either to no solution at all, or to a segment of the Reissner-Nordstrom geometry.

32.2 REDSHIFTS DURING COLLAPSE

(a) Let a radio transmitter on the surface of a collapsing spherical star emit monochromatic waves of wavelength λ_e; and let a distant observer, at the same θ, ϕ, as the transmitter, receive the waves. Show that at late times the wavelength received varies as

$$\lambda_{\rm rec}/\lambda_{\rm em}\propto e^{t/4M}, \tag{32.2.1}$$

where t is proper time as measured by the distant observer.

(b) Use kinetic theory for the outgoing photons (conservation of density in phase space: Liouville's theorem; 22.6) to show that the energy flux of the radiation received ($\mathrm{ergs/cm^2\ sec}$) varies as

$$F \propto e^{-t/M}. \tag{32.2.2}$$

(c) Suppose that nuclear reactions at the center of the collapsing star generate neutrinos of energy E_e, and that these neutrinos flow freely outward (negligible absorption in star). Show that the energy of the neutrinos received by a distant observer decreases at late times as

$$E_{\mathrm{rec}}/E_{\mathrm{em}} \propto e^{-t/4M}. \tag{32.2.3}$$

(d) Show that the flux of neutrino energy dies out at late times as

$$F \propto e^{-t/2M}. \tag{32.2.4}$$

(e) Explain in elementary terms why the decay laws 32.2.1 and 32.2.3 for energy are the same, but the decay laws 32.2.2 and 32.2.4 for energy flux are different.

(f) Let a collapsing star emit photons from its surface at the black-body rate

$$\frac{dN}{d\tau} = \left(1.5 \times 10^{11} \frac{\text{photons}}{\mathrm{cm^2\ sec\ K^3}}\right) \times (\text{surface area of star}) \times (\text{temperature of surface})^3. \tag{32.2.5}$$

Let a distant observer count the photons as they pass through his sphere of radius $r \gg M$. Let him begin his count (time $t = 0$) when he sees (via photons traveling radially outward) the center of the star's surface pass through the radius $r = 3M$. Show that, in order of magnitude, the time he and his associates must wait, until the last photon that will ever get out has reached them, is

$$t = (M/M_\odot[8 \times 10^{-4} + 5 \times 10^{-5} \log_{10}(T_{11} M/M_\odot)]\text{seconds}, \tag{32.2.6}$$

where T_{11} is the star's surface temperature in units of 10^{11} K.

Solution:

Solutions to exercise (a) and (b) are based on section 15 of the paper *Relativistic astrophysics. I (Zel'dovich et al. 1964)*.

(a) In the crudest approximation, we can neglect the effects of pressure and put $P = 0$. Let us consider the surface of a collapsing star. During the course of contraction the mass M dose not change, and therefore the particle on the surface simply falls under the influence of the gravitation of the mass M. Consequently, in order to explain the character of the collapse, it is sufficient to consider the free fall of a trial particle in the field of the M.

For the rate of free fall in a Schwarzschild field, we have

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(1 - \frac{1 - 2M/r}{1 - 2M/r_0}\right)^{1/2}, \tag{32.2.7}$$

where r_0 is the distance from which the fall begins and at which $dr/dt = 0$. The local stationary observer, situated side the falling body, will measure its velocity as follows

$$v \equiv \frac{g_{rr}^{1/2} dr}{g_{00}^{1/2} dt} = -\left(1 - \frac{1 - 2M/r}{1 - 2M/r_0}\right)^{1/2}. \tag{32.2.8}$$

The orbit for a radially propagating photon satisfy that

$$\frac{dr}{dt} = 1 - \frac{2M}{r}. \tag{32.2.9}$$

The integration gives

$$\Delta t = 2M \ln \frac{R - 2M}{r - 2M} + R - r, \tag{32.2.10}$$

where R is the position of the observer.

With the aid of expression 32.2.7 we find $r = r(t')$, i.e., the position of the trial particle at the instant t' as measured by the clock of the remote observer. But this is not the same place where this observer sees the particle at the instant t', since it takes the light some time Δt to cover the path from the particle to the observer. This time can be readily calculated by formula 32.2.10. Denoting the time of arrival of the light at the observer by t:

$$t = t' + \Delta t \tag{32.2.11}$$

With the aid of the expressions given above we can easily obtain the formula $r = r(t)$ for the falling particle. At late time of collapse, $r \to 2M$, the asymptotic form of the formula is:

$$r = 2M + (r_1 - 2M)e^{-\frac{t-t_1}{4M}}, \tag{32.2.12}$$

where r_1 is the position of the particle at the instant t_1.

During the propagation of photons in Schwarzschild metric, p_0 is constant. The four momentum of photon is $p_\alpha = [p_0, p_0(1-2M/r)^{-1}, 0, 0]$. The four velocity of surface particle at r is $u^\alpha = [(1-2M/r)^{-1/2}\gamma, (1-2M/r)^{1/2}\gamma v, 0, 0]$, where $\gamma \equiv (1-v^2)^{-1/2}$. So the energy of photon according to the emitter is

$$\epsilon_{\rm em} = -p_\alpha u^\alpha = \frac{\gamma(1-v)}{(1-2M/r)^{1/2}} p_0. \tag{32.2.13}$$

The four velocity of observer at R is $u^\alpha = [(1-2M/R)^{-1/2}, 0, 0, 0]$. So the energy of photon measured by the observer is

$$\epsilon_{\rm rec} = (1-2M/R)^{-1/2} p_0. \tag{32.2.14}$$

So the redshift of the photon is

$$1 + z = \frac{\gamma(1-v)(1-2M/R)^{1/2}}{(1-2M/r)^{1/2}} \tag{32.2.15}$$

From 32.2.8 it follows that $v \to -1$ and $\gamma \to (1-2M/r)^{-1/2}$ as $r \to 2M$. Now we have

$$\frac{\lambda_{\rm rec}}{\lambda_{\rm em}} \propto (1-2M/r)^{-1} \propto e^{t/4M}. \tag{32.2.16}$$

(b) The kinetic theory for the outgoing photons ensure the conservation of density in phase space. So I_ν/ν^3 is a constant during propagation. The flux from the central point of the visible disc of a contracting star received by observer is

$$F = \Delta\Omega \int I_{\rm rec} d\nu_{\rm rec} = \Delta\Omega \int \frac{I_{\rm em}}{\nu_{\rm em}^3} \nu_{\rm rec}^3 \frac{\nu_{\rm rec}}{\nu_{\rm em}} d\nu_{\rm em} = z^{-4} \Delta\Omega \int I_{\rm em} d\nu_{\rm em}, \tag{32.2.17}$$

where $\Delta\Omega$ is kept as a constant during observation. So we have

$$F \propto e^{-t/M}. \tag{32.2.18}$$

For the entire disc, the deductions are much more complicated, since it is necessary to consider rays moving at a large angle to the radius, and the paths of such rays near the star are quite complicated. An analysis (Podurets, 1964) of this question shows that the formula for the luminosity of the entire star F is

$$F \propto \exp\left(-\frac{t}{3\sqrt{3}M}\right) \tag{32.2.19}$$

(c) We will assume that collapsing star is composed of dust with uniform density and negligible pressure. The interior of the star can be described by closed Friedmann spacetime

$$ds^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + \sin^2\chi\, d\Omega^2], \tag{32.2.20}$$

where $\chi = \chi_0$ is the surface star. And we have $r = a\sin\chi_0$, where r is the surface of star in Schwarzschild metric. Suppose the neutrino is produced at $\chi = 0$, $\eta = \eta_0$ with energy $E_{\rm em}$. It will propagate to the surface of star at $\eta = \eta_0 + \chi_0$ with energy $E_{\rm em}a\eta_0/a(\eta_0 + \chi_0)$ measured by observer fixed at $\chi = \chi_0$, i.e. observer comoving with star surface. When the neutrino propagates to the observer far way observer at R, the energy measured measured by the observer will be

$$E_{\rm rec} = \frac{r(\eta_0)\gamma(1-v)(1-2M/R)^{1/2}}{r(1-2M/r)^{1/2}}E_{\rm em}, \tag{32.2.21}$$

caused by gravitational redshift and Doppler effect. When $r \to 2M$, $r(\eta_0)/r$ can be approximated as a constant. So similarly, we will have

$$\frac{E_{\rm rec}}{E_{\rm em}} =\propto e^{-t/4M}. \tag{32.2.22}$$

(d) The energy of every neutrino measured by observer will be $\epsilon_{\rm em}(1+z)^{-1}$ due to gravitational redshift. The number of neutrinos received by observer per unit time will be $(1+z)^{-1}$ of that produced by nuclear reaction per unit time due to gravitational time dilation. So the flux of neutrinos will scale as $(1+z)^{-2}$, i.e.

$$F \propto (1+z)^{-2} \propto e^{-t/2M}. \tag{32.2.23}$$

(e) When calculating the flux from the central point of the visible disc of a contracting star. Only the photon from a fixed solid angle is taken into account. The bending of the ray trajectories in the gravitational field and the beaming effect caused by Doppler effect must be considered. Neutrino sources are situated in the center of the contracting star. There are no light ray bending or beaming effect. Only redshift and time dilation should be taken into account.

(f) I do not know how to tackle this problem yet.

32.3 EMBEDDING DIAGRAMS AND PHOTON PROPAGATION FOR COLLAPSING STAR

Verify in detail the features of homogeneous collapse described in Box 32.1.

The features of homogeneous collapse described in Box 32.1 are intuitive from diagrams in the textbook. I think a detailed demonstration is unnecessarily since there is little mathematics involved in it.

32.4 MATCH OF FRIEDMANN INTERIOR TO SCHWARZSCHILD EXTERIOR

The Einstein field equations are satisfied on a star's surface if and only if the intrinsic and extrinsic geometries of the surface's three-dimensional world tube are the same, whether measured on its interior or on its exterior. Verify that for the collapsing star discussed above, the intrinsic and extrinsic geometries match at the join between the Friedmann interior and the Schwarzschild exterior.

Solution:

The original discussion can be found in *Terminal Configurations of Stellar Evolution (Beckedorff et al. 1962)*.

(a) The intrinsic geometries of the surface's three-dimensional world tube measured on its exterior are

$$ds^2 = -\left(1-\frac{2M}{R(t)}\right)dt^2 + \left(1-\frac{2M}{R(t)}\right)^{-1}dR^2 + R^2(t)d\Omega^2, \tag{32.4.1}$$

with

$$\frac{dR}{dt} = -\left(1-\frac{2M}{R}\right)\left(1-\frac{1-2M/R}{1-2M/R_i}\right)^{1/2}. \tag{32.4.2}$$

Eliminate t and we can get

$$ds^2 = -\frac{dR^2}{2M(1/R-1/R_i)} + R^2d\Omega^2. \tag{32.4.3}$$

The intrinsic geometries measured on its interior are

$$ds^2 = a^2(\eta)(-d\eta^2 + \sin^2\chi_0 d\Omega^2). \tag{32.4.4}$$

And we have

$$R = a\sin\chi_0 = R_i(1+\cos\eta)/2, \quad 2M/R_i = \sin^2\chi_0. \tag{32.4.5}$$

As

$$\begin{aligned}\left(\frac{dR}{d\eta}\right)^2 &= \frac{R_i^2}{4}\sin^2\eta = \frac{R_i^2}{4}\left[1-\left(\frac{2R}{R_i}-1\right)^2\right] = R^2R_i\left(\frac{1}{R}-\frac{1}{R_i}\right)\\ &= a^2\sin^2\chi_0 R_i\left(\frac{1}{R}-\frac{1}{R_i}\right) = 2M\left(\frac{1}{R}-\frac{1}{R_i}\right)a^2,\end{aligned} \tag{32.4.6}$$

the intrinsic geometries are the same, whether measured on its interior or on its exterior.

(b) The inner metric is

$$ds^2 = a(\eta)^2(-d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2). \tag{32.4.7}$$

The world tube of star surface is $\chi = \chi_0$. The normal vector of the surface is $n_\alpha = (0, a, 0, 0)$. The extrinsic curvature is

$$K_{ab} = n_{\alpha;\beta}e^\alpha_a e^\beta_b = n_{\alpha,b}e^\alpha_a - \Gamma^\mu{}_{\alpha\beta}n_\mu e^\alpha_a e^\beta_b = -a\Gamma^\chi{}_{ab}. \tag{32.4.8}$$

Given the $\Gamma^\chi{}_{ab}$, we can get

$$K_{\eta\eta} = K_{\eta\theta} = K_{\eta\phi} = K_{\theta\phi} = 0, \quad K_{\theta\theta} = K_{\phi\phi}/\sin^2\theta = a\sin\chi_0\cos\chi_0. \tag{32.4.9}$$

The outer metric is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{32.4.10}$$

The world tube of star surface can be parameterize as

$$\begin{aligned}t &= 2M\ln\left|\frac{\cot\chi_0+\tan(\eta/2)}{\cot\chi_0-\tan(\eta/2)}\right| + 2M\cot\chi_0\left[\eta+\frac{1}{2}\csc^2\chi_0(\eta+\sin\eta)\right],\\ r &= 2M\csc^2\chi_0\cos^2(\eta/2).\end{aligned} \tag{32.4.11}$$

The tangent vector of the surface is

$$e^\alpha_\eta = r_{,\eta}(t_{,R},1,0,0), \quad e^\alpha_\theta = (0,0,1,0), \quad e^\alpha_\phi = (0,0,0,1). \tag{32.4.12}$$

The normal vector of the surface is

$$n_\alpha = \frac{\sqrt{1-2M/R}(1,-t_{,R},0,0)}{\sqrt{-1+(1-2M/R)^2t^2_{,R}}}. \tag{32.4.13}$$

Note that

$$t_{,R} = \frac{1}{dR/dt} = \frac{-(1-2M/R_i)^{1/2}}{(1-2M/R)(2M/R-2M/R_i)^{1/2}}. \tag{32.4.14}$$

We can deduce that

$$n_\alpha = \left[\left(\frac{2M}{R}-\frac{2M}{R_i}\right)^{1/2}, \left(1-\frac{2M}{R_i}\right)^{1/2}\left(1-\frac{2M}{R}\right)^{-1}, 0, 0\right] \tag{32.4.15}$$

The extrinsic curvature is

$$K_{ab} = n_{\alpha,b}e^\alpha_a - \Gamma^\mu{}_{\alpha\beta}n_\mu e^\alpha_a e^\beta_b. \tag{32.4.16}$$

If one of a and b are θ or ϕ, we have

$$K_{ab} = -\Gamma^t{}_{\alpha\beta}n_t - \Gamma^r{}_{\alpha\beta}n_r. \tag{32.4.17}$$

For example, $\Gamma^r{}_{\theta\theta} = 2M - r$, $\Gamma^t{}_{\theta\theta} = 0$. Thus,

$$K_{\theta\theta} = R\left(1-\frac{2M}{R_i}\right)^{1/2} = a\sin\chi_0\cos\chi_0. \tag{32.4.18}$$

Similar, we can derive that $K_{\phi\phi} = a\sin\chi_0\cos\chi_0\sin^2\theta$, $K_{\theta\phi} = K_{\eta\theta} = K_{\eta\phi} = 0$.

Next, we calculate $K_{\eta\eta}$. We have

$$\begin{aligned}
K_{\eta\eta} &= n_{t,\eta}t_{,\eta} + n_{r,\eta}r_{,\eta} - \Gamma^r{}_{tt}n_r t^2_{,\eta} - 2\Gamma^t{}_{tr}n_t t_{,\eta}r_{,\eta} - \Gamma^r{}_{rr}n_r r^2_{,\eta} \\
&= n_{t,R}r_{,\eta}t_{,\eta} + n_{r,R}r^2_{,\eta} - \Gamma^r{}_{tt}n_r t^2_{,\eta} - 2\Gamma^t{}_{tr}n_t t_{,\eta}r_{,\eta} - \Gamma^r{}_{rr}n_r r^2_{,\eta} \\
&= (n_{t,R} + n_{r,R}t^{-1}_{,R} - \Gamma^r{}_{tt}n_r t_{,R} - 2\Gamma^t{}_{tr}n_t - \Gamma^r{}_{rr}n_r t^{-1}_{,R})r_{,\eta}t_{,\eta} \\
&= 0.
\end{aligned} \tag{32.4.19}$$

So we demonstrate that extrinsic geometries are the same, whether measured on its interior or on its exterior.

32.5 STARS THAT COLLAPSE FROM INFINITY

(a) Patch together a truncated Schwarzschild geometry and the geometry of a truncated "flat" ($k = 0$), dust-filled Friedmann universe to obtain a model of a star that collapses from rest at an infinite initial radius.

(b) Similarly patch together a truncated Schwarzschild geometry and the geometry of a truncated "open" ($k = -1$), dust-filled Friedmann universe to obtain a star which collapses from infinity with finite initial inward velocity.

Solution:

(a) For a particle which free falls from infinity with zero initial zero velocity, we have

$$\frac{dt}{d\tau} = \frac{1}{1-2M/r}, \quad \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}. \tag{32.5.1}$$

The world tube of the star surface is $r = R(t)$ given by equations above. If we parameters the 3-surface with τ, θ, ϕ, we can get

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega^2. \tag{32.5.2}$$

If $R = R_i$ at $\tau = 0$, we can deduce that

$$\frac{2}{3}\left[\left(\frac{R_i}{2M}\right)^{3/2} - \left(\frac{R}{2M}\right)^{3/2}\right] = \frac{\tau}{2M}. \tag{32.5.3}$$

The metric in the star is

$$ds^2 = -d\tau^2 + a^2(\tau)(d\chi^2 + \chi^2 d\Omega^2). \tag{32.5.4}$$

The star surface is given by $\chi = \chi_0$. The evolution of a is determined by friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho_0}{3}\left(\frac{a}{a_0}\right)^{-3}, \tag{32.5.5}$$

where ρ_0 and a_0 is the density of the dust and scale factor at $\tau = 0$. The solution is

$$\frac{2}{3}\left[\left(\frac{a_0\chi_0}{2m}\right)^{3/2} - \left(\frac{a\chi_0}{2m}\right)^{3/2}\right] = \frac{\tau}{2m}, \tag{32.5.6}$$

where $m = 4\pi\rho_0\chi_0^3 a_0^3/3$. Two surfaces can be connected if $M = m$ and $a_0\chi_0 = R_i$.

(b) For a particle which free falls from infinity with zero initial zero velocity, we have

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r}, \quad \frac{dr}{d\tau} = -\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}. \tag{32.5.7}$$

The world tube of the star surface is $r = R(t)$ given by equations above. If we parameters the 3-surface with τ, θ, ϕ, we can get

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega^2. \tag{32.5.8}$$

The metric in the star is

$$ds^2 = -d\tau^2 + a^2(\tau)(d\chi^2 + \sinh^2\chi d\Omega^2). \tag{32.5.9}$$

The star surface is given by $\chi = \chi_0$. The evolution of a is determined by friedmann equation

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} = \frac{8\pi\rho_0}{3}\left(\frac{a}{a_0}\right)^{-3}, \tag{32.5.10}$$

where ρ_0 and a_0 is the density of the dust and scale factor at $\tau = 0$. The equation above can be simplified to

$$\sinh\chi_0 \frac{da}{d\tau} = -\sqrt{\frac{2m}{a\sinh\chi_0} + \sinh^2\chi_0}, \tag{32.5.11}$$

where $m = 4\pi\rho_0\sinh^3\chi_0 a_0^3/3$. Two surfaces can be connected if $M = m$, $\tilde{E} = \cosh\chi_0$ and $r = a\sinh\chi_0$.

32.6 GENERAL SPHERICAL COLLAPSE: METRIC IN COMOVING COORDINATES

Consider an inhomogeneous star with pressure, undergoing spherical collapse. Spherical symmetry alone is enough to guarantee the existence of a Schwarzschild coordinate system t, r, θ, ϕ throughout the interior and exterior of the star [see equation (32.2) and preceding discussion]. Label each spherical shell of the star by a parameter a, which tells how many baryons are contained interior to that shell. Then $r(a, t)$ is the world line of the shell with label a. The expression for these world lines can be inverted to obtain $a(r, t)$, the number of baryons interior to radius r at time t. Show that there exists a new time coordinate $\tilde{t}(t, r)$, such that the line element (32.2), rewritten in the coordinates $(\tilde{t}, a, \theta, \phi)$, has the form

$$ds^2 = -e^{2\tilde{\Phi}}d\tilde{t}^2 + \left[\frac{(\partial r/\partial a)_{\tilde{t}}da}{\Gamma}\right]^2 + r^2 d\Omega^2, \quad \tilde{\Phi} = \tilde{\Phi}(\tilde{t}, a),\ r = r(\tilde{t}, a),\ \Gamma = \Gamma(\tilde{t}, a). \tag{32.6.1}$$

These are "comoving, synchronous coordinates" for the stellar interior.

Solution:

Suppose

$$d\tilde{t} = e^{\psi}\left[e^{-2\Lambda}a_{,r}dt + e^{-2\Phi}a_{,t}dr\right], \tag{32.6.2}$$

e^{ψ} is the integrating factor to ensure the existence of $\tilde{t}$. The coordinate transformation $(t, r) \to (\tilde{t}, a)$ bring equation (32.2) into

$$ds^2 = -e^{-2\psi+2\Phi+2\Lambda}(e^{-2\Lambda}a_{,r}^2 - e^{-2\Phi}a_{,t}^2)^{-1}d\tilde{t}^2 + (e^{-2\Lambda}a_{,r}^2 - e^{-2\Phi}a_{,t}^2)^{-1}da^2 \tag{32.6.3}$$

If $e^{-2\Lambda}a_{,r}^2 - e^{-2\Phi}a_{,t}^2 > 0$, then we can bring 32.6.3 into 32.6.1 by choosing proper $\tilde{\Phi}$ and Γ. In local normalized Lorentz frame, the condition can be rewritten as

$$a_{,\hat{r}}^2 > a_{,\hat{t}}^2. \tag{32.6.4}$$

Suppose the radial velocity of the baryon is v. After time $\Delta\hat{t}$, the shell will move to $\hat{r} - v\Delta\hat{t}$. Then we have

$$a_{,\hat{r}}v\Delta\hat{t} = a_{,\hat{r}}v\Delta\hat{t}. \tag{32.6.5}$$

Since $v < 1$, we must have $a_{,\hat{r}}^2 > a_{,\hat{t}}^2$. Thus we prove the existence of comoving coordinates.

32.7 ADIABATIC SPHERICAL COLLAPSE: EQUATIONS OF EVOLUTION

Describe the interior of a collapsing star by the comoving, synchronous metric 32.6.1, by the number density of baryons n, by the total density of mass-energy ρ, and by the pressure p. The 4-velocity of the star's fluid is

$$\boldsymbol{u} = e^{-\tilde{\Phi}}\partial/\partial\tilde{t}, \tag{32.7.1}$$

since the fluid is at rest in the coordinate system. Let a dot denote a proper time derivative as seen by the fluid-e.g.,

$$\dot{n} \equiv \boldsymbol{u}[n] = e^{-\tilde{\Phi}}(\partial n/\partial\tilde{t})_a; \tag{32.7.2}$$

and let a prime denote a partial derivative with respect to baryon number,-e.g.

$$n' \equiv (\partial n/\partial a)_{\tilde{t}}. \tag{32.7.3}$$

Denote by V the rate of change of $(1/2\pi)\times$ (circumference of shell), as measured by a man riding in a given shell:

$$U \equiv \dot{r}; \tag{32.7.4}$$

and denote by $m(\tilde{t}, a)$ the "total mass-energy interior to shell a at time $\tilde{t}$":

$$m(\tilde{t}, a) \equiv \int_0^a 4\pi r^2\rho(\tilde{t}, a)r'da. \tag{32.7.5}$$

(See Box 23.1 for discussion of this method of localizing mass-energy.) Assume that the collapse is adiabatic (no energy flow between adjacent shells; stress-energy tensor entirely that of a perfect fluid).

(a) Show that the equations of collapse [baryon conservation, (22.3); local energy conservation, (22.11a); Euler equation, (22.13); and Einstein field equations (ex. 14.16)] can be reduced to the following eight

equations for the eight functions $\tilde{\Phi}$, Γ, r, n, ρ, p, V, m:

$$\dot{r} = U, \tag{32.7.6a}$$

$$\frac{(nr^2)^{\cdot}}{nr^2} = -\frac{U'}{r'}, \tag{32.7.6b}$$

$$\frac{\dot{\rho}}{\rho + p} = \frac{\dot{n}}{n}, \tag{32.7.6c}$$

$$\dot{U} = -\frac{\Gamma^2}{\rho + p}\frac{p'}{r'} - \frac{m + 4\pi r^3 p}{r^2}, \tag{32.7.6d}$$

$$p = p(n, \rho), \tag{32.7.6e}$$

$$\tilde{\Phi}' = -\frac{p'}{\rho + p}, \tilde{\Phi} = 0 \text{ at star's surface}, \tag{32.7.6f}$$

$$m' = 4\pi r^2 \rho r', \tag{32.7.6g}$$

$$\Gamma = \text{sign } (r')(1 + U^2 - 2m/r)^{1/2}. \tag{32.7.6h}$$

(b) The preceding equations are in a form useful for numerical calculations. For analytic work it is often useful to replace 32.7.6b by

$$n = \frac{\Gamma}{4\pi r^2 r'}, \tag{32.7.7}$$

and 32.7.6d by

$$\dot{m} = -4\pi r^2 p U. \tag{32.7.8}$$

Derive these equations.

(c) Explain why equations 32.7.6g and 32.7.8 justify the remarks made in Box 23.1 about localizability of energy.

Solution:

(a) Equation 32.7.6a, 32.7.6e and 32.7.6g holds automatically by the definition in the exercise.

The baryon conservation law gives

$$0 = \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = \dot{n} + n\boldsymbol{\nabla} \cdot \boldsymbol{u} = \dot{n} + n(-g)^{-1/2}[(-g)^{1/2}u^{\alpha}]_{,\alpha} = \dot{n} + \frac{\Gamma n}{r^2 r'}\left(\frac{r' r^2}{\Gamma}\right)^{\cdot}. \tag{32.7.9}$$

It can be rewritten as

$$\frac{(nr^2)^{\cdot}}{nr^2} + \frac{(r')^{\cdot}}{r'} - \frac{\dot{\Gamma}}{\Gamma} = 0. \tag{32.7.10}$$

For adiabatic flow, we have $ds/d\tau = 0$ except in shock waves. The first law of thermodynamics gives

$$\dot{\rho} = \frac{\rho + p}{n}\dot{n}. \tag{32.7.11}$$

Note that

$$u^{\alpha}{}_{;\beta}u^{\beta} = \dot{u^{\alpha}} + \Gamma^{\alpha}{}_{00}e^{-2\tilde{\Phi}} \tag{32.7.12}$$

and

$$\Gamma^{0}{}_{00} = e^{\tilde{\Phi}}\dot{\tilde{\Phi}}, \quad \Gamma^{1}{}_{00} = \frac{\Gamma^2}{r'^2}e^{2\tilde{\Phi}}\tilde{\Phi}', \quad \Gamma^{2}{}_{00} = \Gamma^{3}{}_{00} = 0. \tag{32.7.13}$$

The only none zero component of $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$ is $(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u})^1 = \Gamma^2\tilde{\Phi}'/r'^2$. The Euler equation then gives

$$(\rho + p)\tilde{\Phi}' + p' = 0. \tag{32.7.14}$$

And so $\Phi' = 0$ outside the star. If we demand that $\tilde{\Phi} = 0$ at infinity, we must have $\tilde{\Phi} = 0$ at the star's surface.

The stress energy of the fluid is $T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta}$. Explicitly, we have

$$T_{00} = e^{2\tilde{\Phi}}\rho, \quad T_{11} = \frac{pr'^2}{\Gamma^2}, \quad T_{22} = pr^2, \quad T_{33} = pr^2\sin^2\theta. \tag{32.7.15}$$

All other components vanish.

Einstein metric can be calculated from metric. Firstly, from $G_{01} = 0$ we have

$$\dot{\Gamma}r' - \Gamma\tilde{\Phi}'U = 0. \tag{32.7.16}$$

Note that $U' = (r')^{\cdot} - U\Phi'$. We have

$$\frac{U'}{r'} = \frac{(r')^{\cdot}}{r'} - \frac{\dot{\Gamma}}{\Gamma}. \tag{32.7.17}$$

Substituting 32.7.17 into 32.7.10, we can get 32.7.6b. From $G_{00} = 8\pi T_{00}$ and 32.7.17, we can get

$$-2\Gamma\Gamma' r + U^2 r' + 2UU' r + (1-\Gamma^2)r' = 8\pi\rho r^2 r', \tag{32.7.18}$$

i.e.

$$[r(1-\Gamma^2) + U^2 r)]' = 2m'. \tag{32.7.19}$$

Noting that $m(0) = 0$, the integration of equation above will give 32.7.6h. From $G_{11} = 8\pi T_{11}$, we can get

$$\frac{2\Gamma^2 r\tilde{\Phi}'}{r'} - U^2 - 2r\dot{U} + \Gamma^2 - 1 = 8\pi p r^2. \tag{32.7.20}$$

Using equation 32.7.6f and 32.7.6h, we can obtain 32.7.6d.

(b)Equation 32.7.10 can be rewritten as

$$\frac{d}{d\tilde{t}}\left(\frac{nr^2 r'}{\Gamma}\right) = 0. \tag{32.7.21}$$

So we have

$$n = f(a)\frac{\Gamma}{r^2 r'}, \tag{32.7.22}$$

where $f(a)$ is an arbitrary function of baryon number a. On the other hand, from the definition of baryon number and number density we have

$$a = \int_0^a 4\pi n r^2 \frac{r'}{\Gamma} da. \tag{32.7.23}$$

It is easy to get $f = 1/4\pi$ and so we have equation 32.7.7.

Equation 32.7.6h can be rewritten as

$$2m = r + rU^2 - r\Gamma^2. \tag{32.7.24}$$

Take the time derivatives and we can get

$$2\dot{m} = U + U^3 + 2rU\dot{U} - U\Gamma^2 - 2r\Gamma\dot{\Gamma}. \tag{32.7.25}$$

Using 32.7.16 and 32.7.20, we can obtain equation 32.7.8.

(c) From equation 32.7.6g and 32.7.8, we find that the mass-energy m associated with a given ball of matter (fixed baryon number) can change in time only to the extent that locally measurable energy fluxes can be detected at the boundary of the ball. And energy fluxes in this case it the power expended by pressure forces against the moving boundary surface. Thus the energy m is localized by the circumstance that transfer of energy is detectable by local measurements.

32.8 ANALYTIC SOLUTIONS FOR PRESSURE-FREE COLLAPSE

Show how the general solution to the equations of collapse in the case of zero pressure can be generated.

Solution:

In the case of zero pressure, the equations of collapse can be simplified to the following equations.

$$\dot{r} = U, \tag{32.8.1a}$$

$$n = \frac{\Gamma}{4\pi r^2 r'}, \tag{32.8.1b}$$

$$\frac{\dot{\rho}}{\rho} = \frac{\dot{n}}{n}, \tag{32.8.1c}$$

$$\dot{m} = 0, \tag{32.8.1d}$$

$$\tilde{\Phi} = 0, \tag{32.8.1e}$$

$$m' = 4\pi r^2 \rho r', \tag{32.8.1f}$$

$$\Gamma = \text{sign}\ (r')(1 + U^2 - 2m/r)^{1/2}. \tag{32.8.1g}$$

(1) At first, we specify mass inside shell a, $m(a)$ at the beginning. It will not change with time according to 32.8.1d.

(2) From 32.8.1c, we have $(\rho/n)^{\cdot} = 0$. ρ and n can be related by

$$\rho = \mu(a) n. \tag{32.8.2}$$

Here $\mu(a)$ can be seen as rest masses of the particles at shell a.

(3) Using 32.8.1b, 32.8.1f and 32.8.2, we can obtain

$$\Gamma = \frac{m'}{\mu}. \tag{32.8.3}$$

And so Γ is also independent of $\tilde{t}$.

(4) From 32.8.1e, we know $\dot{f} = \partial f / \partial \tilde{t}$. Now using 32.8.1a, 32.8.1g, we can obtain

$$\left(\frac{\partial r}{\partial \tilde{t}}\right)^2 - \frac{2m(a)}{r} = \Gamma^2(a) - 1. \tag{32.8.4}$$

If we specify the $r(a)$ at $\tilde{t} = 0$, $r(\tilde{t}, a)$ can be solved by the equation 32.8.4.

The metric in this case therefore

$$ds^2 = -d\tilde{t}^2 + \left(\frac{r' da}{\Gamma}\right)^2 + r^2 d\Omega^2. \tag{32.8.5}$$

The remaining quantities of interest can be calculated easily.

32.9 COLLAPSE WITH UNIFORM DENSITY

Recover the Friedmann-Schwarzschild solution for collapse with uniform density and zero pressure by specifying appropriate forms for $m(a)$ and $r(a)$ in the prescription of exercise 32.8. In the interior of

the star, give the dust particles nonzero rest masses, $\mu =$ constant $\neq 0$; in the exterior give them zero rest masses, $\mu = 0$ ("imaginary dust particles" in vacuum). Reduce the resulting metric 32.8.5 to that of Friedmann inside the star, and to that of Novikov for the Schwarzschild geometry outside the star.

Solution:

The surface of the star is $a = a_{\rm s}$. When $a <$ s, we assume $r(\tilde{t}, a) = f(\tilde{t})R(a)$, $\Gamma^2 = 1 - R^2$ and $m/R^3 = 2M/R_{\rm s}^3$, where M is a constant and $R_{\rm s} = R(a_{\rm s})$. The form of $R(a)$ is determined by 32.8.3, i.e.

$$\mu(1 - R^2) = \frac{6MR^2}{R_{\rm s}^3}\frac{dR}{da}. \tag{32.9.1}$$

The solution is

$$\operatorname{arctanh} R - R = \frac{\mu R_{\rm s}^3}{6M}a. \tag{32.9.2}$$

The form of f is determined by 32.8.4, i.e.

$$\left(\frac{\partial f}{\partial \tilde{t}}\right)^2 = \frac{2M}{R_0^3}\frac{1}{f} - 1, \tag{32.9.3}$$

and we suppose $\partial f/\partial \tilde{t} = 0$ when $\tilde{t} = 0$. So the initial distribution of r is

$$r_0(a) \equiv r(\tilde{t} = 0, a) = f(\tilde{t} = 0)R(a) = \frac{2M}{R_{\rm s}^3}R(a). \tag{32.9.4}$$

Especially, we have

$$r_{\rm s0} \equiv r_0(a_{\rm s}) = \frac{2M}{R_{\rm s}^2} \tag{32.9.5}$$

and so

$$\Gamma_s^2 = 1 - \frac{2M}{r_{\rm s0}}. \tag{32.9.6}$$

The metric becomes

$$ds^2 = -d\tilde{t}^2 + f^2\frac{dR^2}{1 - R^2} + f^2R^2d\Omega^2. \tag{32.9.7}$$

It is clear that the resulting metric is Friedmann inside the star.

When $a > a_0$, $m(a)$ is constant. And so we have $m(a) = M$ by continuity. As $m' = 0$ and $\mu = 0$ outside the star, the value of Γ is arbitrary. And $r(\tilde{t}, a)$ must satisfy that

$$\left(\frac{\partial r}{\partial \tilde{t}}\right)^2 = \frac{2M}{r} + \Gamma^2 - 1. \tag{32.9.8}$$

And $r(t, \tilde{a})$ satisfy the free fall equation for a particle in Schwartz metric with energy $\Gamma(a)$. Suppose $\partial r/\partial \tilde{t} = 0$ at $\tilde{t} = 0$. We then have $\Gamma^2 = 1 - 2M/r_0$. Note that Γ is also continuous across the star surface. The metric now becomes

$$ds^2 = -d\tilde{t}^2 + \frac{1}{1 - 2M/r_0}\left(\frac{\partial r}{\partial a}\right)^2 da^2 + r^2d\Omega^2. \tag{32.9.9}$$

Compare it with equation (31.11) and (31.12) in the textbook. It is clear that the resulting metric is Novikov outside the star.

32.10 PRICE'S THEOREM FOR A SCALAR FIELD

A collapsing spherical star, with an arbitrary nonspherical "scalar charge distribution," generates an external scalar field Φ. The vacuum field equation for Φ is $\Box\Phi = \Phi_{,\alpha}{}^{\alpha} = 0$. Ignore the back-reaction of the field's stress-energy on the geometry of spacetime.

(a) Resolve the external field into scalar spherical harmonics, using Schwarzschild coordinates for the external Schwarzschild geometry:

$$\Phi = \sum_\ell \frac{1}{r}\Psi_l(t,r)Y_{lm}(\theta,\phi). \tag{32.10.1}$$

Show that the vacuum field equation reduces to

$$-\Psi_{\ell,tt} + \Psi_{\ell,r^*r^*} = \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)\Psi_\ell. \tag{32.10.2}$$

where r^* is the "tortoise coordinate" of section 25.5 and Figure 25.4:

$$r^* = r + 2M\ln\left(\frac{r}{2M}-1\right). \tag{32.10.3}$$

Notice that 32.10.2 is a flat-space, one-dimensional wave equation with effective potential

$$V_{\rm eff} = \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right). \tag{32.10.4}$$

Part of this effective potential $[\ell(\ell+1)/r^2]$ is the "centrifugal barrier", and part $[2M/r]$ is due to the curvature of spacetime. Notice the similarity of this effective potential for scalar waves, to the effective potentials for particles and photons moving in the Schwarzschild geometry,

$$\begin{aligned}(\widetilde{V}^2)_{\rm particles} &= \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right),\\ (B^{-2})_{\rm photons} &= \left(1-\frac{2M}{r}\right)r^{-2}\end{aligned} \tag{32.10.5}$$

(Boxes 25.6 and 25.7). The scalar-wave potential, like the photon potential, is positive for all $r > 2M$. It rises, from 0 at $r = 2M$, to a barrier summit; then falls back to 0 at $r = \infty$.

(b) Show that there exist no physically acceptable static scalar-wav perturbations of a Schwarzschild black hole. [More precisely, show that all static solutions to equation 32.10.2 become infinite at either the horizon ($r = 2M$, $r^* = -\infty$) or at radial infinity.] This suggests that somehow the black hole formed by collapse must divest itself of the star's external scalar field before it can settle down into a quiescent state.

(c) The general solution to the wave equation 32.10.2 can be written in terms of a Fourier transform. For waves that begin near the horizon, propagate outward, and are partially transmitted and partially reflected ("rightward-propagating waves"), show that the general solution is

$$\Psi_\ell = \int_{-\infty}^{\infty} A(k)R_k^\ell(r^*)e^{-ikt}dk, \tag{32.10.6}$$

where

$$d^2R_k^\ell/dr^{*2} = [-k^2 + V_{\rm eff}(r)]R_k^\ell, \tag{32.10.7}$$

$$R_k^\ell \to e^{ikr^*} + \Gamma_k^{(R)}e^{-ikr^*} \text{ as } r^* \to -\infty,$$

$$R_k^\ell \to T_k^{(R)}e^{ikr^*} \text{ as } r^* \to \infty. \tag{32.10.8}$$

Show that the "reflection and transmission coefficients for rightward-propagating waves", $\Gamma_k^{(R)}$ and $T_k^{(R)}$, have the following asymptotic forms for $|k| \ll 1/M$ (short wave number; long wavelength):

$$\Gamma_k^{(R)} = -1 + \alpha 2Mik, \quad T_k^{(R)} = \frac{\beta}{(2\ell-1)!!}(2Mik)^{\ell+1}, \tag{32.10.9}$$

where α and β are constants of order unity. Give a similar analysis for waves that impinge on a Schwarzschild black hole from outside ("leftward-propagating waves").

(d) Show that, as the star collapses into the horizon, the world line of its surface in (t, r^*) coordinates is

$$r^* = R^*(t) = -t - R_0^* \exp(-t/2M) + \text{const.}, \tag{32.10.10}$$

where R_0^* is related to the magnitude a of the surface's 4-acceleration ($a > 0$ for outward 4-acceleration) by

$$R_0^* = \frac{8M}{e}\left\{1 + 16Ma\left[Ma + \left(M^2a^2 + \frac{1}{8}\right)^{1/2}\right]\right\}. \tag{32.10.11}$$

Thus, the world line of the surface appears to become null near the horizon ($t + r^* \equiv \widetilde{V} =$ constant); of course, this is due to pathology of the coordinate system there. Show, further, that the scalar field on the star's surface ($\widetilde{V} =$ constant) must vary as

$$\Psi_\ell = Q_{\ell 0} + Q_{\ell 1} e^{-\widetilde{U}/4M}, \quad \widetilde{U} = t - r^*, \tag{32.10.12}$$

when the star is approaching the horizon ($t \to \infty$, $r^* \to -\infty$, $\widetilde{U} \to \infty$), in order that the rate of change of Ψ_ℓ be finite as measured on the star's surface. Notice that $Q_{\ell 0}$ is the "final value" of the scalar field on the star's surface. It can be regarded as an outgoing wave with zero wave number (infinite wavelength); and, consequently, it gets completely and destructively reflected by the effective potential.

Conclusion: All multipoles of the scalar field die out at finite r^* as $t \to \infty$ (Price's theorem for a scalar field.)

Solution:

(a) Generally, we have

$$\Phi_{,\alpha}{}^{\alpha} = g^{\alpha\beta}\Phi_{,\alpha\beta} - g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta}\Phi_{,\mu}. \tag{32.10.13}$$

In Schwarzschild metric, we have

$$g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta} = \left[0, \frac{2(M-r)}{r^2}, -\frac{\cos\theta}{r^2\sin\theta}, 0\right]. \tag{32.10.14}$$

So the field equation can be written as

$$-\left(1 - \frac{2M}{r}\right)^{-1}\Phi_{,tt} + \left(1 - \frac{2M}{r}\right)\Phi_{,rr} - \frac{2(M-r)}{r^2}\Phi_{,r} + \frac{1}{r^2\sin\theta}\left(\sin\theta\Phi_{,\theta}\right)_{,\theta} + \frac{1}{r^2\sin^2\theta}\Phi_{,\phi\phi} = 0. \tag{32.10.15}$$

Substitute 32.10.1 into the field equation, we can get

$$-\left(1 - \frac{2M}{r}\right)^{-1}\Psi_{\ell,tt} + \left(1 - \frac{2M}{r}\right)\Psi_{\ell,rr} + \frac{2M}{r^2}\Psi_{\ell,r} - \frac{2M}{r^3}\Psi_\ell - \frac{\ell(\ell+1)}{r^2}\Psi_\ell = 0. \tag{32.10.16}$$

Using

$$\frac{dr^*}{dr} = \frac{1}{1 - 2M/r}, \tag{32.10.17}$$

we can get the field equation in tortoise coordinate, i.e. 32.10.2.

(b) For static solutions, we have

$$\Psi_{\ell,r^*r^*} = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)\Psi_\ell. \tag{32.10.18}$$

When $r^* \to -\infty$ ($r \to 2M$), we have $\Psi_{\ell,r^*r^*} \to 0$, and the asymptotic behavior of the Ψ is

$$\Psi \sim r^* \text{ or } \Psi \sim \text{ constant}. \tag{32.10.19}$$

If the solution is regular when $r^* \to -\infty$, the only possibility is $\Psi \sim$ constant. When $r^* \to \infty$ and $\ell > 0$, we have $\Psi_{\ell,r^*r^*} \to \ell(\ell+1)\Psi_l/r^{*2}$, and the asymptotic behavior of the Ψ is

$$\Psi \sim r^{*\ell+1} \text{ or } \Psi \sim r^{*-\ell}. \tag{32.10.20}$$

If the solution is regular when $r^* \to \infty$, the only possibility is $\Psi \sim r^{*-\ell}$. Now we want to connect two pieces of solution at infinity. We suppose Ψ is positive when $r^* \to -\infty$. Then we have $\Psi_{\ell,r^*r^*} > 0$. So Ψ_ℓ must be increasing as r increases, which is contradict to its behavior at $r^* \to \infty$. The analysis for the opposite case is similar.

If $\ell = 0$, we have

$$\left(1 - \frac{2M}{r}\right)\Psi_{0,rr} + \frac{2M}{r^2}\Psi_{0,r} - \frac{2M}{r^3}\Psi_0 = 0. \tag{32.10.21}$$

We have exact solution

$$\Psi_0 = c_1 r + c_2 r \ln\left(1 - \frac{2M}{r}\right). \tag{32.10.22}$$

Clearly, it is not regular at $r^* \to \pm\infty$.

Thus all static solutions to equation 32.10.2 become infinite at either the horizon ($r = 2M$, $r^* = -\infty$) or at radial infinity.

(c) If we substitute Fourier decomposition 32.10.36 into field equation 32.10.2, it is easy to obtain 32.10.37. When $r^* \to \pm\infty$, i.e. $r^* \gg k^{-1} \gg M$, we have $V_{\text{eff}} \to 0$ and so

$$R_k^\ell = C_l e^{ikr^*} + D_l e^{-ikr^*}. \tag{32.10.23}$$

If we assume the waves are rightward-propagating, D_l should be taken to 0 at $r^* \to \infty$. So we have 32.10.38 at infinity, up to a renormalization constant.

At $r^* \gg M$(but $r^* \ll k^{-1}$, or $r^* \sim k^{-1}$, or $r^* \gg k^{-1}$), we have

$$\frac{d^2 R_k^\ell}{dr^{*2}} = \left[-k^2 + \frac{\ell(\ell+1)}{r^{*2}}\right] R_k^\ell. \tag{32.10.24}$$

The solution is

$$R_k^\ell = C_l r^* h_l^{(1)}(kr^*) + D_l r^* h_l^{(2)}(kr^*), \tag{32.10.25}$$

where $h_l^{(1)}(kr^*)$ and $h_l^{(2)}(kr^*)$ are the first and second type spherical Hankel function. When $kr \gg 1$, we have asymptotic approximation

$$h_l^{(1)}(kr^*) = i^{-(\ell+1)}\frac{e^{ikr^*}}{kr^*}, \quad h_l^{(2)}(kr^*) = i^{\ell+1}\frac{e^{-ikr^*}}{kr^*}. \tag{32.10.26}$$

In order to match

$$R_k^\ell = T_k^{(R)} e^{ikr^*} \tag{32.10.27}$$

at $r \gg k^{-1}$. We must have

$$R_k^\ell = T_k^{(R)} i^{\ell+1} kr^* h_l^{(1)}(kr^*) \tag{32.10.28}$$

at $r^* \gg M$.

At $|r^*| \ll k^{-1}$ (but $|r^*| \sim M$ or $|r^*| \gg M$), $R_\ell^k = 2Miky$, where $y = y(r^*)$ satisfies static field equation.

$$\frac{d^2 y}{dr^{*2}} = V_{\text{eff}} y. \tag{32.10.29}$$

And according to discussion in (b), we have

$$y = \alpha + \frac{r^*}{M} \text{ as } r^* \to -\infty \tag{32.10.30}$$

and

$$y = \beta(2M/r^*)^l \text{ as } r^* \to +\infty. \tag{32.10.31}$$

where α and β depend on multipole, but are both of order unity.

Now, by matching 32.10.28 and 32.10.31, noting that

$$h_\ell^{(1)}(kr^*) \sim -i\frac{(2l-1)!!}{(kr^*)^{l+1}} \text{ at } kr^* \sim 0, \tag{32.10.32}$$

we can get

$$T_k^{(R)} = (-1)^l \frac{\beta}{(2\ell-1)!!}(2Mik)^{\ell+1}. \tag{32.10.33}$$

Thus waves have no transmission in limit $k \to 0$. I have an extra $(-1)^l$ factor for $T_k^{(R)}$. Please check it.

By matching 32.10.30 and $e^{ikr^*} + \Gamma_k^{(R)} e^{-ikr^*}$, noting that

$$e^{ikr^*} + \Gamma_k^{(R)} e^{-ikr^*} \sim 1 + \Gamma_k^{(R)} \text{ at } kr^* \sim 0, \tag{32.10.34}$$

we can get

$$\Gamma_k^{(R)} = -1 + \alpha 2Mik. \tag{32.10.35}$$

So waves have complete reflection and complete destructive interference in limit $k \to 0$.

Analysis for leftward-propagating waves can be carried similarly. The general solution is

$$\Psi_\ell = \int_{-\infty}^{\infty} A(k) L_k^\ell(r^*) e^{-ikt} dk, \tag{32.10.36}$$

where

$$d^2 L_k^\ell / dr^{*2} = [-k^2 + V_{\text{eff}}(r)] L_k^\ell, \tag{32.10.37}$$

$$L_k^\ell \to e^{-ikr^*} + \Gamma_k^{(R)} e^{ikr^*} \text{ as } r^* \to \infty,$$

$$L_k^\ell \to T_k^{(R)} e^{-ikr^*} \text{ as } r^* \to -\infty. \tag{32.10.38}$$

The "reflection and transmission coefficients for leftward-propagating waves", $\Gamma_k^{(R)}$ and $T_k^{(R)}$, have the following asymptotic forms for $|k| \ll 1/M$ (short wave number; long wavelength):

$$\Gamma_k^{(R)} = (-1)^{\ell+1} + \frac{\alpha}{(2l-1)!!}(2Mik)^{2\ell+1}, \quad T_k^{(R)} = \beta(2Mik)^{\ell+1} \tag{32.10.39}$$

where α and β are constants of order unity. Please see page 249 of the book *Magic without Magic (Klauder et al .1972)* for a more detailed discussion.

(4) I do not know how to get the relation between R_0^* and four acceleration a. Any help is appreciated.

Introduce Kruskal coordinates

$$\tilde{u} = -4Me^{-\widetilde{U}/4M}, \quad \tilde{v} = 4Me^{\widetilde{V}/4M}. \tag{32.10.40}$$

Since $\tilde{u}$ and $\tilde{v}$ are well-behaved coordinates at $r = 2M$, the partial derivatives $\partial\Psi/\partial\tilde{u}$ and $\partial\Psi/\partial\tilde{v}$ should be finite. This implies that $\partial\Psi/\partial\widetilde{U}$ must fall off sharply at the event horizon because

$$\frac{\partial\Psi}{\partial\widetilde{U}} = \frac{\partial\Psi}{\partial\tilde{u}} e^{-\widetilde{U}/4M}, \tag{32.10.41}$$

and $\widetilde{U} \to \infty$ at $r = 2M$. The advanced time $\widetilde{V}$ is finite $r = 2M$ so that $\partial\Psi/\partial\widetilde{V}$ is finite.

If we picture the path of the surface through space-time as $\widetilde{V}(\widetilde{U})$ or $\tilde{v}(\tilde{u})$, then near $r = 2M$ we have

$$\frac{d\widetilde{V}}{d\widetilde{U}} = \frac{d\tilde{v}}{d\tilde{u}} \frac{d\widetilde{V}}{d\tilde{v}} \Big/ \frac{d\widetilde{U}}{d\tilde{u}} = \frac{d\tilde{v}}{d\tilde{u}} e^{-(\widetilde{U}+\widetilde{V})/4M} \sim e^{-\widetilde{U}/4M} \tag{32.10.42}$$

Thus we have

$$\frac{d\Psi}{d\widetilde{U}} = \frac{\partial\Psi}{\partial\widetilde{V}} \frac{d\widetilde{V}}{d\widetilde{U}} + \frac{\partial\Psi}{\partial\widetilde{U}} \sim e^{-\widetilde{U}/4M}. \tag{32.10.43}$$

for large $\widetilde{U}$. Ψ varies as

$$\Psi = Q_0 + Q_1 e^{-\widetilde{U}/4M} \tag{32.10.44}$$

on star's surface as $R^* \to -\infty$.

The time-varying part of this surface field, $\Psi = Q_1 e^{-\widetilde{U}/4M}$, is of medium wavelength, so much of it gets through the potential and propagates off to $\widetilde{V} \to \infty$. But the static part, $\Psi = Q_0$, is of infinite wavelength and is completely reflected by the potential. Put in more physical terms, the curvature of spacetime prevents any information whatsoever about the final surface field $T = Q_0$ from propagating off to a distant external observer.

32.11 NEWMAN-PENROSE "CONSTANTS"

Wheeler (1955) showed that Maxwell's equations for an ℓ-pole electromagnetic field residing in the Schwarzschild geometry can be reduced to the wave equation

$$-\Psi_{\ell,tt} + \Psi_{\ell,r^*r^*} = \left(1 - \frac{2M}{r}\right) \frac{\ell(\ell+1)}{r^2} \Psi_\ell. \tag{32.11.1}$$

After this equation has been solved, the components of the electromagnetic field can be obtained by applying certain differential operators to $\Psi_\ell(t, r^*) Y_{\ell m}(\theta, \phi)$.

(a) Show that the general solution to the electromagnetic wave equation 32.11.1 for dipole ($\ell = 1$) fields, with outgoing-wave boundary conditions at $r^* \to \infty$, has the form

$$\Psi_1 = f_0(\widetilde{U}) + \frac{f_1(\widetilde{U})}{r} + \frac{f_2(\widetilde{U})}{r^2} + \cdots \tag{32.11.2}$$

where $\widetilde{U} = t - r^*$ is "retarded time", and

$$f_1' = f_0,\ f_2' = 0,\ \cdots,\ f_n' = -\frac{(n+1)(n-2)}{2n} f_{n-1} + (n-2) M f_{n-2}. \tag{32.11.3}$$

When spacetime is flat ($M = 0$), this solution becomes

$$\Psi_1 = f_1'(\widetilde{U}) + f_1(\widetilde{U})/r. \tag{32.11.4}$$

The terms $f_2(\widetilde{U})/r^2 + \cdots$ which are absent in flat spacetime, are attributable to backscatter of the outgoing waves by the curvature of spacetime. They are sometimes called the "tail" of the waves.

(b) Show that the general static dipole field has the form 32.11.2 with

$$(f_0)_{\text{static}} = 0; \quad (f_1)_{\text{static}} \equiv D = \text{ dipole moment}; \quad (f_2)_{\text{static}} = \frac{3}{2} M D; \cdots \tag{32.11.5}$$

(c) Consider a star (not a black hole!) with a dipole field that is initially static. At time $t = 0$, let the star suddenly change its dipole moment to a new static value D'. Equations 32.11.3 demand that f_2 be conserved ["Newman-Penrose (1965) constant"]. Hence, f_2 will always exhibit a value $3MD/2$ corresponding to the old dipole moment; it can never change to $3MD'/2$. This is a manifestation of the tail of the waves that are generated by the sudden change in dipole moment. To understand this tail effect more clearly, and to discover an important flaw in the above result, evaluate the solution 32.11.2 for retarded time $U > 0$, using the assumptions

- field has static form 32.11.5 for $\widetilde{U} < 0$,
- $f_1 = D'$ for $\widetilde{U} > 0$.

Put the answer in the form

$$\Psi_1 = \frac{D'}{r} + \frac{3MD}{2r^2} + \sum_{n=3}^{\infty} \frac{2M(D'-D)(-1)^{n+1}(n+1)\widetilde{U}^{n-2}}{(2r)^n} + O\left(\frac{M^2}{r^3}, \frac{M^2\widetilde{U}}{r^4}\right). \tag{32.11.6}$$

(The terms neglected are small compared to those kept for all $\widetilde{U}/r$, so long as $r \gg M$.) Evidently, so long as the series converges the Newman- Penrose "constant" (coefficient of $1/r^2$) remembers the old D value and is conserved, as claimed above. Show, however, that the series diverges for $\widetilde{U} > 2r$ -i.e., it diverges inside a sphere that moves outward with asymptotically $1/3$ the speed of light. Thus, the Newman-Penrose "constant" is well-defined and conserved only outside the "$1/3$-speed-of-light cone".

(d) Sum the series in 32.11.6 to obtain a solution valid even for $\tilde{U} > 2r$:

$$\begin{aligned}\Psi &= \frac{D'}{r} + \frac{3MD'}{2r^2} - \frac{2M(D'-D)}{r}\frac{(\widetilde{U}+3r)}{(\widetilde{U}+2r)^2} + O\left(\frac{M^2}{r^3}\right) \\ &= \text{the series 32.11.6 for } \widetilde{U} < 2r (\text{domain of convergence of that series}) \\ &= \frac{D'}{r} + \frac{3MD'}{2r^2} + O\left(\frac{M}{\widetilde{U}r}, \frac{M^2}{r^3}\right) \text{ for } \widetilde{U} \gg r \gg M. \end{aligned} \tag{32.11.7}$$

From this result conclude that at fixed r and late times the electromagnetic field becomes asymptotically static and the Newman-Penrose "constant" assumes the new value $3MD'/2$ appropriate to the new dipole moment.

Solution:

(a) The expansion for outgoing waves assumes that at sufficiently large r the radial dependence can be analyzed as a power series in r^{-1}. Thus we have the expansion 32.11.2. If we substitute 32.11.2 into 32.11.1, we then have

$$\sum_{k=1}^{\infty} \frac{2kf_k'(\widetilde{U})}{r^{k+1}} + \frac{k(k+1)f_k(\widetilde{U})}{r^{k+2}} - \frac{2Mk(k+2)f_k(\widetilde{U})}{r^{k+3}} = \sum_{k=0}^{\infty} \frac{2f_k(\widetilde{U})}{r^{k+2}}. \tag{32.11.8}$$

By equating the coefficients of r^{-k} of both side of the equation above, we can get

$$\begin{aligned} &2f_1' = 2f_0 \\ &4f_2' + 2f_1 = 2f_1 \\ &2kf_k' + k(k-1)f_{k-1} - 2Mk(k-2)f_{k-2} = 2f_{k-1}\ (k \geq 3). \end{aligned} \tag{32.11.9}$$

Some simple algebra manipulation will give 32.11.3. When space time is flat, we have

$$f_1' = f_0,\ f_2' = 0, \cdots,\ f_n' = -\frac{(n+1)(n-2)}{2n} f_{n-1}. \tag{32.11.10}$$

If $f_2 \neq 0$, then $f_3 = -2f_2\tilde{U}/3 + f_3(\tilde{U} = 0)$, which is unbound. Thus we must have $f_2 = 0$. By similar analysis, we can show that all higher terms vanish. The solution is therefore $\Psi_1 = f_1'(\widetilde{U}) + f_1(\widetilde{U})/r$.

(b) For static dipole field, we have $f_n' = 0$, i.e.

$$f_0 = 0,\ f_{n-1} = \frac{2nM}{n+1} f_{n-2}\ (n \geq 3). \tag{32.11.11}$$

If $f_1 = D$, we can get

$$f_n = \frac{3D}{n+2}(2M)^{n-1}\ (n \geq 2). \tag{32.11.12}$$

(c) When f_1 changed form D to D', we have

$$f_3' = M\Delta f_1 = M(D'-D) \tag{32.11.13}$$

and so

$$f_3 = \frac{12DM^2}{5} + M(D' - D)\widetilde{U}. \tag{32.11.14}$$

In the case of $\tilde{U} \gg M$ and $r \gg M$, we can neglect the constant term in f_3. Generally, we can only keep terms with highest order of $\widetilde{U}$. Suppose $f_n = C_n M(D' - D)\widetilde{U}^{n-2}$. We have

$$f'_n = -\frac{(n+1)(n-2)}{2n} C_{n-1} M(D' - D)\widetilde{U}^{n-3} \tag{32.11.15}$$

and so

$$C_n = -\frac{(n+1)}{2n} C_{n-1}. \tag{32.11.16}$$

As $C_3 = 1$, we can obtain

$$C_n = \frac{n+1}{2^{n-1}}(-1)^{n+1}. \tag{32.11.17}$$

Now the field can be written as 32.11.6.

If $\widetilde{U} > 2r$, the term $2M(D' - D)(-1)^{n+1}(n+1)r^{-2}(\widetilde{U}/2r)^{n-2}$ is infinite as $n \to \infty$ and so the summation dose not converge mathematically. Though the value of the field may be finite by cancellation between infinite terms (called analytic continuation mathematically or renormalization physically), the Newman-Penrose "constant" become ill defined and cannot be evaluated uniquely from data in region $\tilde{U} > 2r$.

When $r \gg M$, $\tilde{U} > 2r$ is equivalent to $r < t/3$, so the divergent sphere move outward with $1/3$ of speed of light.

(4)Equation 32.11.6 can be rewritten as

$$\Psi_1 = \frac{D'}{r} + \frac{3MD'}{2r^2} - \frac{M(D' - D)}{2r^2} \sum_{n=2}^{\infty} (n+1) \left(-\frac{\widetilde{U}}{2r} \right)^{n-2} + O\left(\frac{M^2}{r^3}, \frac{M^2\widetilde{U}}{r^4} \right). \tag{32.11.18}$$

Note that

$$\sum_{n=0}^{\infty} (n+3)a^n = \frac{3 - 2a}{(1-a)^2} \text{ for } |a| < 1. \tag{32.11.19}$$

By analytic continuation, we can obtain

$$\Psi_1 = \frac{D'}{r} + \frac{3MD'}{2r^2} - \frac{2M(D' - D)}{r} \frac{\tilde{U} + 3r}{(\tilde{U} + 2r)^2} + O\left(\frac{M^2}{r^3}, \frac{M^2\widetilde{U}}{r^4} \right). \tag{32.11.20}$$

If $\widetilde{U} \gg r \gg M$, we have

$$\Psi_1 = \frac{D'}{r} + \frac{3MD'}{2r^2} + O\left(\frac{M^2}{r^3}, \frac{M^2\widetilde{U}}{r^4} \right). \tag{32.11.21}$$

At fixed r and late times the electromagnetic field becomes asymptotically static and the Newman-Penrose "constant" assumes the new value $3MD'/2$ appropriate to the new dipole moment.

Chapter 33

BLACK HOLES

33.1 KERR DESCRIPTION OF KILLING VECTORS

(a) Use the transformation law from Boyer-Lindquist coordinates to Kerr coordinates [equation (4) of Box 33.2] to show that

$$\boldsymbol{\xi}_{(t)} \equiv \left(\frac{\partial}{\partial t}\right)_{r,\theta,\phi} = \left(\frac{\partial}{\partial \widetilde{V}}\right)_{r,\theta,\widetilde{\phi}} \tag{33.1.1a}$$

$$\boldsymbol{\xi}_{(\phi)} \equiv \left(\frac{\partial}{\partial \phi}\right)_{t,r,\theta} = \left(\frac{\partial}{\partial \widetilde{\phi}}\right)_{\widetilde{V},r,\theta}. \tag{33.1.1b}$$

Verify explicitly by examining metric components that

$$g_{\widetilde{V}\widetilde{V}} = g_{tt}, \quad g_{\widetilde{V}\widetilde{\phi}} = g_{t\phi}, \quad g_{\widetilde{\phi}\widetilde{\phi}} = g_{\phi\phi}, \tag{33.1.2}$$

In accordance with equations 33.1.1.

(b) Show that for a stationary observer (world line of constant r, θ), the angular velocity expressed in terms of Kerr coordinates is

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\widetilde{\phi}}{d\widetilde{V}} = \frac{u^{\widetilde{\phi}}}{u^{\widetilde{V}}}. \tag{33.1.3}$$

Solution:

(a) The transformation matrix is

$$\frac{\partial x^{\tilde{\alpha}}}{\partial x^{\beta}} = \begin{pmatrix} 1 & (r^2+a^2)/\Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a/\Delta & 0 & 1 \end{pmatrix}_{\tilde{\alpha}\beta}. \tag{33.1.4}$$

It is easy to see that

$$\frac{\partial}{\partial t} = \frac{\partial x^{\tilde{\alpha}}}{\partial t}\frac{\partial}{\partial x^{\tilde{\alpha}}} = \frac{\partial}{\partial x^{\tilde{V}}} \tag{33.1.5}$$

and

$$\frac{\partial}{\partial \phi} = \frac{\partial x^{\tilde{\alpha}}}{\partial \phi}\frac{\partial}{\partial x^{\tilde{\alpha}}} = \frac{\partial}{\partial x^{\tilde{\phi}}}. \tag{33.1.6}$$

In Boyer-Lindquist coordinates, we have

$$g_{tt} = -\frac{\Delta}{\rho^2} + \frac{a^2 \sin^2\theta}{\rho^2} = -\frac{r^2 - 2Mr + a^2 + Q^2 - a^2\sin^2\theta}{r^2 + a^2\cos^2\theta} = -1 - \frac{-2Mr + Q^2}{r^2 + a^2\cos^2\theta}, \tag{33.1.7}$$

$$g_{t\phi} = \frac{a\Delta\sin^2\theta}{\rho^2} - \frac{a(a^2+r^2)\sin^2\theta}{\rho^2} = -\frac{(a^2 + r^2 - \Delta)a\sin^2\theta}{\rho^2} = -\frac{(2Mr - Q^2)a\sin^2\theta}{\rho^2} \tag{33.1.8}$$

and

$$g_{\phi\phi} = -\frac{\Delta a^2\sin^4\theta}{\rho^2} + \frac{(r^2+a^2)^2\sin^2\theta}{\rho^2} = \frac{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta}{\rho^2}. \tag{33.1.9}$$

In Kerr coordinates, we have

$$g_{\widetilde{V}\widetilde{V}} = -1 + \frac{2Mr - Q^2}{\rho^2}, \tag{33.1.10}$$

$$g_{\widetilde{V}\widetilde{\phi}} = -\frac{a(2Mr - Q^2)\sin^2\theta}{\rho^2} \tag{33.1.11}$$

and

$$g_{\widetilde{\phi}\widetilde{\phi}} = \frac{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta}{\rho^2}. \tag{33.1.12}$$

Now, we have verified equation 33.1.2.

(b) The transformation matrix gives

$$\boldsymbol{d}\widetilde{V} = \boldsymbol{d}t + (r^2+a^2)(\boldsymbol{d}r/\Delta), \quad \boldsymbol{d}\widetilde{\phi} = \boldsymbol{d}\phi + a(\boldsymbol{d}r/\Delta). \tag{33.1.13}$$

For stationary object, we have $\boldsymbol{d}r = 0$ and so

$$\boldsymbol{d}\widetilde{V} = \boldsymbol{d}t, \quad \boldsymbol{d}\widetilde{\phi} = \boldsymbol{d}\phi. \tag{33.1.14}$$

Now, we have verified equation 33.1.3.

33.2 OBSERVATIONS OF ANGULAR VELOCITY

An observer, far from a black hole and at rest in the hole's asymptotic Lorentz frame, watches (with his eyes) as a particle moves along a stationary (nongeodesic) orbit near the black hole. Let $\Omega = d\phi/dt$ be the particle's angular velocity, as defined and discussed above. The distant observer uses his stopwatch to measure the time required for the particle to make one complete circuit around the black hole (one complete circuit relative to the distant observer himself; i.e., relative to the hole's asymptotic Lorentz frame).

(a) Show that the circuit time measured is $2\pi/\Omega$. Thus, Ω can be regarded as the particle's "angular velocity as measured from infinity."

(b) Let the observer moving with the particle measure its circuit time relative to the asymptotic Lorentz frame, using his eyes and a stopwatch he carries. Show that his answer for the circuit time must be

$$\Delta\tau = \frac{2\pi}{\Omega}(-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{1/2}. \tag{33.2.1}$$

Solution:

(a) Suppose the observer is at (R, θ_0, ϕ_0) and $R \gg M$. At time t, the particle pass ϕ_0 and emit a photon to observer. The observer receive the photon at $t + \Delta t$. At time $t + 2\pi/\Omega$, the particle pass ϕ_0 again and emit

a photon to observer. The observer then will receive the photon at $t + 2\pi/\Omega + \Delta t$. The time lag between two signals received by observer is therefore $2\pi/\Omega$, i.e. the circuit time measured is $2\pi/\Omega$.

(b) The four velocity of the particle is $u^t(1, 0, 0, \Omega)$. Normalization gives

$$(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)u^{t2} = -1. \tag{33.2.2}$$

We can get

$$\frac{d\phi}{d\tau} = \Omega u^t = \Omega(-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{-1/2}. \tag{33.2.3}$$

The circuit time is

$$\Delta\tau = \frac{2\pi}{d\phi/d\tau} = \frac{2\pi}{\Omega}(-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{1/2}. \tag{33.2.4}$$

33.3 LOCALLY NONROTATING OBSERVERS

(a) Place a rigid, circular mirror ("ring mirror") at fixed (r, θ) around a black hole. Let an observer at (r, θ) with angular velocity Ω emit a flash of light. Some of the photons will get caught by the mirror and will skim along its surface, circumnavigating the black hole in the positive-ϕ direction. Others will get caught and will skim along in the negative-ϕ direction. Show that the observer will receive back the photons from both directions simultaneously only if his angular velocity is

$$\Omega = \omega(r, \theta) \equiv \frac{1}{2}(\Omega_{\min} + \Omega_{\max}) = -\frac{g_{\phi t}}{g_{\phi\phi}} = \frac{(2Mr - Q^2)a}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}. \tag{33.3.1}$$

Thus in this case, and only in this case, can the observer regard the $+\phi$ and $-\phi$ directions as equivalent in terms of local geometry. Put differently, in this case and only in this case is the observer "nonrotating relative to the local spacetime geometry". Thus, it is appropriate to use the name "locally nonrotating observer" for an observer who moves with the angular velocity $\Omega = \omega(r, \theta)$.

(b) Associated with the axial symmetry of a black hole is a conserved quantity, $p_\phi \equiv \boldsymbol{p} \cdot \boldsymbol{\xi}_{(\phi)}$ for geodesic motion. This quantity for any particle-whether it is moving along a geodesic or not-is called the "component of angular momentum along the black hole's spin axis", or simply the particle's "angular momentum". Show that of all stationary observers at fixed (r, θ), only the "locally nonrotating observer" has zero angular momentum. [Note: Bardeen, Press, and Teukolsky (1972) have shown that the "locally nonrotating observer" can be a powerful tool in the analysis of physical processes near a black hole.]

Solution:

(a) Suppose the angular velocity of the photon is $\Omega_{\rm ph}$. The four velocity of the photon is $u^t(1, 0, 0, \Omega_{\rm ph})$. Using normalization condition, we have

$$g_{tt} + 2g_{t\phi}\Omega_{\rm ph} + g_{\phi\phi}\Omega_{\rm ph}^2 = 0. \tag{33.3.2}$$

There are two solutions $\Omega_{\min}$ and $\Omega_{\min}$, satisfying

$$\Omega_{\min} + \Omega_{\max} = -2\frac{g_{t\phi}}{g_{\phi\phi}}. \tag{33.3.3}$$

If the observer will receive back the photons from both directions simultaneously, we have

$$\frac{2\pi}{\Omega_{\max} - \Omega} = \frac{2\pi}{\Omega - \Omega_{\min}}, \tag{33.3.4}$$

i.e. $\Omega = \omega(r, \theta)$.

(b) The four momentum of the stationary observer is $p^t(1,0,0,\Omega)$. So we have

$$p_\phi = g_{\phi\alpha}p^\alpha = p^t(g_{t\phi} + g_{\phi\phi}\Omega). \tag{33.3.5}$$

If $p_\phi = 0$, we then have

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \omega(r,\theta). \tag{33.3.6}$$

33.4 ORTHONORMAL FRAMES OF LOCALLY NONROTATING OBSERVERS

(a) Let spacetime be filled with world lines of locally nonrotating observers, and let each such observer carry an orthonormal frame with himself. Show that the spatial orientations of these frames can be so chosen that their basis 1-forms are

$$\boldsymbol{\omega}^{\hat{t}} = |g_{tt} - \omega^2 g_{\phi\phi}|^{1/2}\boldsymbol{d}t,\ \boldsymbol{\omega}^{\hat{\phi}} = g_{\phi\phi}^{1/2}(\boldsymbol{d}\phi - \omega\boldsymbol{d}t),\ \boldsymbol{\omega}^{\hat{r}} = (\rho/\Delta^{1/2})\boldsymbol{d}r,\ \boldsymbol{\omega}^{\hat{\theta}} = \rho\boldsymbol{d}\theta. \tag{33.4.1}$$

More specifically, show that these 1-forms are orthonormal and that the dual basis has

$$\frac{\partial}{\partial\hat{t}} = \boldsymbol{u} \equiv 4-\text{velocity of locally nonrotating observer.} \tag{33.4.2}$$

Show that $-\boldsymbol{\omega}^{\hat{t}}$ is a rotation-free field of 1-forms [$\boldsymbol{d}\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{t}} = 0$; exercise 4.4].

(b) One sometimes meets the mistaken notion that a "locally nonrotating observer" is in some sense locally inertial. To destroy this false impression, verify that: (i) such an observer has nonzero 4-acceleration,

$$\boldsymbol{a} = \Gamma_{\hat{j}\hat{t}\hat{t}}\boldsymbol{e}_{\hat{j}} = \frac{1}{2}\boldsymbol{\nabla}\ln|g_{tt} - \omega^2 g_{\phi\phi}|; \tag{33.4.3}$$

(ii) if such an observer carries gyroscopes with himself, applying the necessary accelerations at the gyroscope centers of mass, he sees the gyroscopes precess relative to his orthonormal frame with angular velocity

$$\boldsymbol{\Omega}^{\text{precess}} = \Gamma_{\hat{\theta}\hat{\phi}\hat{t}}\boldsymbol{e}_{\hat{r}} + \Gamma_{\hat{\phi}\hat{r}\hat{t}}\boldsymbol{e}_{\hat{\theta}} = \frac{1}{2}\frac{g_{\phi\phi}^{1/2}}{|g_{tt} - \omega^2 g_{\phi\phi}|^{1/2}}\left[\frac{\omega_{,\theta}}{\rho}\boldsymbol{e}_{\hat{r}} - \frac{\Delta^{1/2}\omega_{,r}}{\rho}\boldsymbol{e}_{\hat{\theta}}\right]. \tag{33.4.4}$$

Solution:

(a) The metric of the spacetime is

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & -\omega g_{\phi\phi} \\ 0 & \rho^2/\Delta & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\omega g_{\phi\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix},\ g^{\mu\nu} = \begin{pmatrix} 1/(g_{tt} - \omega^2 g_{\phi\phi}) & 0 & 0 & \omega/(g_{tt} - \omega^2 g_{\phi\phi}) \\ 0 & \Delta/\rho^2 & 0 & 0 \\ 0 & 0 & 1/\rho^2 & 0 \\ \omega/(g_{tt} - \omega^2 g_{\phi\phi}) & 0 & 0 & g_{tt}/[g_{\phi\phi}(g_{tt} - \omega^2 g_{\phi\phi})] \end{pmatrix}. \tag{33.4.5}$$

Note that

$$\begin{aligned} (\boldsymbol{\omega}^{\hat{t}})_\alpha &= |g_{tt} - \omega^2 g_{\phi\phi}|^{1/2}(1,0,0,0), \\ (\boldsymbol{\omega}^{\hat{\phi}})_\alpha &= g_{\phi\phi}^{1/2}(-\omega,0,0,1), \\ (\boldsymbol{\omega}^{\hat{r}})_\alpha &= \rho/\Delta^{1/2}(0,1,0,0), \\ (\boldsymbol{\omega}^{\hat{\theta}})_\alpha &= \rho(0,0,1,0). \end{aligned} \tag{33.4.6}$$

Direct calculation can show that these 1-forms are orthonormal. Also note that $(\partial/\partial\hat{t})^\alpha(\boldsymbol{\omega}^{\hat{\beta}})_\alpha = \delta_{\hat{t}}^{\hat{\beta}}$. We can get

$$\left(\frac{\partial}{\partial\hat{t}}\right)^{\hat{\alpha}} = |g_{tt} - \omega^2 g_{\phi\phi}|^{-1/2}(1,0,0,\omega). \tag{33.4.7}$$

Using $d\phi/dt=\omega(r,\theta)$ for nonrotating observer, we know $\partial/\partial\hat{t}$ is his four velocity. Since we have

$$\boldsymbol{d\omega}^{\hat{t}}=|f|^{1/2}_{,r}\boldsymbol{d}r\wedge\boldsymbol{d}t+|f|^{1/2}_{,\theta}\boldsymbol{d}\theta\wedge\boldsymbol{d}t \tag{33.4.8}$$

where $f=g_{tt}-\omega^2 g_{\phi\phi}$, we can easily deduce that $\boldsymbol{d\omega}^{\hat{t}}\wedge\boldsymbol{\omega}^{\hat{t}}=0$, i.e. $-\boldsymbol{\omega}^{\hat{t}}$ is a rotation-free field of 1-forms.

(b) Recall section 5 of the chapter 14. Note that

$$\begin{aligned}
\boldsymbol{d\omega}^{\hat{t}}&=\frac{1}{2}(\ln|f|)_{,r}\Delta^{1/2}\rho^{-1}\boldsymbol{\omega}^{\hat{r}}\wedge\boldsymbol{\omega}^{\hat{t}}+\frac{1}{2}(\ln|f|)_{,\theta}\rho^{-1}\boldsymbol{\omega}^{\hat{\theta}}\wedge\boldsymbol{\omega}^{\hat{t}}\\
\boldsymbol{d\omega}^{\hat{\phi}}&=\frac{1}{2}(\ln g_{\phi\phi})_{,r}\Delta^{1/2}\rho^{-1}\boldsymbol{\omega}^{\hat{r}}\wedge\boldsymbol{\omega}^{\hat{\phi}}+\frac{1}{2}(\ln g_{\phi\phi})_{,\theta}\rho^{-1}\boldsymbol{\omega}^{\hat{\theta}}\wedge\boldsymbol{\omega}^{\hat{\phi}}\\
&\quad-g^{1/2}_{\phi\phi}\omega_{,r}|f|^{-1/2}\Delta^{1/2}\rho^{-1}\boldsymbol{\omega}^{\hat{r}}\wedge\boldsymbol{\omega}^{\hat{t}}-g^{1/2}_{\phi\phi}\omega_{,\theta}|f|^{-1/2}\rho^{-1}\boldsymbol{\omega}^{\hat{\theta}}\wedge\boldsymbol{\omega}^{\hat{t}}\\
\boldsymbol{d\omega}^{\hat{r}}&=\ln\Big(\rho/\Delta^{1/2}\Big)_{,\theta}\rho^{-1}\boldsymbol{\omega}^{\hat{\theta}}\wedge\boldsymbol{\omega}^{\hat{r}}\\
\boldsymbol{d\omega}^{\hat{\theta}}&=\ln(\rho)_{,r}\Delta^{1/2}\rho^{-1}\boldsymbol{\omega}^{\hat{r}}\wedge\boldsymbol{\omega}^{\hat{\theta}}.
\end{aligned}\tag{33.4.9}$$

We then find that

$$\begin{aligned}
&c_{\hat{t}\hat{r}}{}^{\hat{t}}=\frac{1}{2}(\ln|f|)_{,r}\Delta^{1/2}\rho^{-1},\ c_{\hat{t}\hat{\theta}}{}^{\hat{t}}=\frac{1}{2}(\ln|f|)_{,\theta}\rho^{-1},\ c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}}=\frac{1}{2}(\ln g_{\phi\phi})_{,r}\Delta^{1/2}\rho^{-1},\\
&c_{\hat{\phi}\hat{\theta}}{}^{\hat{\phi}}=\frac{1}{2}(\ln g_{\phi\phi})_{,\theta}\rho^{-1},\ c_{\hat{r}\hat{t}}{}^{\hat{\phi}}=g^{1/2}_{\phi\phi}\omega_{,r}|f|^{-1/2}\Delta^{1/2}\rho^{-1},\ c_{\hat{\theta}\hat{t}}{}^{\hat{\phi}}=g^{1/2}_{\phi\phi}\omega_{,\theta}|f|^{-1/2}\rho^{-1},\\
&c_{\hat{r}\hat{\theta}}{}^{\hat{r}}=\ln\Big(\rho/\Delta^{1/2}\Big)_{,\theta}\rho^{-1},\ c_{\hat{\theta}\hat{r}}{}^{\hat{\theta}}=\ln(\rho)_{,r}\Delta^{1/2}\rho^{-1}.
\end{aligned}\tag{33.4.10}$$

Since

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}=\frac{1}{2}(c_{\hat{\mu}\hat{\nu}\hat{\alpha}}+c_{\hat{\mu}\hat{\alpha}\hat{\nu}}-c_{\hat{\nu}\hat{\alpha}\hat{\mu}}),\tag{33.4.11}$$

we have

$$\begin{aligned}
\Gamma_{\hat{t}\hat{r}\hat{t}}&=-\frac{1}{2}(\ln|f|)_{,r}\Delta^{1/2}\rho^{-1},\ \Gamma_{\hat{t}\hat{r}\hat{\phi}}=-\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,r}|f|^{-1/2}\Delta^{1/2}\rho^{-1},\ \Gamma_{\hat{t}\hat{\theta}\hat{t}}=-\frac{1}{2}(\ln|f|)_{,\theta}\rho^{-1},\\
\Gamma_{\hat{t}\hat{\theta}\hat{\phi}}&=-\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,\theta}|f|^{-1/2}\rho^{-1},\ \Gamma_{\hat{t}\hat{\phi}\hat{r}}=-\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,r}|f|^{-1/2}\Delta^{1/2}\rho^{-1},\ \Gamma_{\hat{t}\hat{\phi}\hat{\theta}}=-\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,\theta}|f|^{-1/2}\rho^{-1},\\
\Gamma_{\hat{r}\hat{\theta}\hat{r}}&=\ln\Big(\rho/\Delta^{1/2}\Big)_{,\theta}\rho^{-1}.\ \Gamma_{\hat{r}\hat{\theta}\hat{\theta}}=-\ln(\rho)_{,r}\Delta^{1/2}\rho^{-1},\ \Gamma_{\hat{r}\hat{\phi}\hat{t}}=\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,r}|f|^{-1/2}\Delta^{1/2}\rho^{-1}\\
\Gamma_{\hat{r}\hat{\phi}\hat{\phi}}&=-\frac{1}{2}(\ln g_{\phi\phi})_{,r}\Delta^{1/2}\rho^{-1},\ \Gamma_{\hat{\theta}\hat{\phi}\hat{t}}=\frac{1}{2}g^{1/2}_{\phi\phi}\omega_{,\theta}|f|^{-1/2}\rho^{-1},\ \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi}}=-\frac{1}{2}(\ln g_{\phi\phi})_{,\theta}\rho^{-1}.
\end{aligned}\tag{33.4.12}$$

All other terms either can be calculated by $\Gamma_{\hat{mu}\hat{\nu}\hat{\alpha}}=-\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}}$ or vanish.

The acceleration of the observer is

$$\boldsymbol{a}=\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}=\Gamma^{\alpha}{}_{\hat{t}\hat{t}}\boldsymbol{e}_{\alpha}=\Gamma_{\hat{j}\hat{t}\hat{t}}\boldsymbol{e}_{j}=\frac{1}{2}(\ln|f|)_{,r}\partial_r+\frac{1}{2}(\ln|f|)_{,\theta}\partial_\theta=\frac{1}{2}\boldsymbol{\nabla}|g_{tt}-\omega^2 g_{\phi\phi}|.\tag{33.4.13}$$

The angular velocity of the observer's spatial frame is

$$\boldsymbol{\Omega}=\frac{1}{2}\boldsymbol{e}_{\hat{i}}\epsilon_{\hat{i}\hat{j}\hat{k}}\boldsymbol{e}_k\cdot\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{e}_{\hat{j}}=\frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}}\Gamma_{\hat{k}\hat{j}\hat{t}}\boldsymbol{e}_{\hat{i}}=\Gamma_{\hat{\phi}\hat{\theta}\hat{t}}\boldsymbol{e}_{\hat{r}}+\Gamma_{\hat{r}\hat{\phi}\hat{t}}\boldsymbol{e}_{\hat{\theta}}.\tag{33.4.14}$$

The gyroscopes precess relative to his orthonormal frame with angular velocity

$$\boldsymbol{\Omega}^{\text{precess}}=-\boldsymbol{\Omega}=\Gamma_{\hat{\theta}\hat{\phi}\hat{t}}\boldsymbol{e}_{\hat{r}}+\Gamma_{\hat{\phi}\hat{r}\hat{t}}\boldsymbol{e}_{\hat{\theta}}=\frac{1}{2}\frac{g^{1/2}_{\phi\phi}}{|g_{tt}-\omega^2 g_{\phi\phi}|^{1/2}}\left[\frac{\omega_{,\theta}}{\rho}\boldsymbol{e}_{\hat{r}}-\frac{\Delta^{1/2}\omega_{,r}}{\rho}\boldsymbol{e}_{\hat{\theta}}\right].\tag{33.4.15}$$

33.5 LOCAL LIGHT CONES

Calculate the shapes of the light cones depicted in the Kerr diagram for an uncharged Kerr black hole (part II.F of Box 33.2). In particular, introduce a new time coordinate

$$\tilde{t} = \tilde{V} - r \tag{33.5.1}$$

for which the slices of constant $\tilde{t}$ are horizontal surfaces in the Kerr diagram. Then the Kerr diagram plots $\tilde{t}$ vertically, r radially, and $\tilde{\phi}$ azimuthally, while holding $\theta = \pi/2$ ("equatorial slice through black hole").

(a) Show that the light cone emanating from given $\tilde{t}$, r and $\tilde{\phi}$ has the form

$$\frac{dr}{d\tilde{t}} = a\left(\frac{d\tilde{\phi}}{d\tilde{t}}\right) - \frac{2M/r}{1+2M/r} \pm \sqrt{\frac{1}{(1+2M/r)^2} - \frac{r^2(d\tilde{\phi}/d\tilde{t})^2}{1+2M/r}}. \tag{33.5.2}$$

(b) Show that the light cone slices through the surface of constant radius along the curves

$$dr/d\tilde{t} = 0, \quad d\tilde{\phi}/d\tilde{t} = \Omega_{\min} \text{ and } \Omega_{\max}, \tag{33.5.3}$$

where $\Omega_{\min}$ and $\Omega_{\max}$ are given by expressions (33.15a,b) (minimum and maximum allowed angular velocities for stationary observers).

(c) Show that at the static limit, $r = r_0(\pi/2)$, the light cone is tangent to a curve of constant r, θ and $\tilde{\phi}$.

(d) Show that the light cone slices the surface of constant $\tilde{\phi}$ along the curves

$$\frac{d\tilde{\phi}}{d\tilde{t}} = 0, \quad \frac{dr}{d\tilde{t}} = -1 \text{ and } \frac{1-2M/r}{1+2M/r}. \tag{33.5.4}$$

(e) Show that the light cone is tangent to the horizon.

(f) Make pictures of the shapes of the light cone as a function of radius.

(g) Describe qualitatively how the light cone must look near the horizon in Boyer-Lindquist coordinates. (Note: it will look "crazy" because the coordinates are singular at the horizon.)

Solution:

(a)Take $\theta = \pi/2$ and $Q = 0$, the metric in Kerr frame is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{V}^2 + 2dr d\tilde{V} + \left[r^2 + a^2\left(1 + \frac{2M}{r}\right)\right]d\tilde{\phi}^2 - 2a d\tilde{\phi} dr - \frac{4Ma}{r} d\tilde{\Phi} d\tilde{V}. \tag{33.5.5}$$

In new coordinates $\tilde{t} = \tilde{V} - r$, we have $d\tilde{V} = d\tilde{r} + d\tilde{t}$, and so

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\tilde{t}^2 + \left(1 + \frac{2M}{r}\right)dr^2 + \frac{4M}{r} dr d\tilde{t} + \left[r^2 + a^2\left(1 + \frac{2M}{r}\right)\right]d\tilde{\phi}^2 - 2a\left(1 + \frac{2M}{r}\right)d\tilde{\Phi} dr - \frac{4Ma}{r} d\tilde{\phi} d\tilde{t}. \tag{33.5.6}$$

For null geodesics, we have $ds^2 = 0$ and so

$$\left(1 + \frac{2M}{r}\right)\left(\frac{dr}{d\tilde{t}}\right)^2 + \left[\frac{4M}{r} - 2a\left(1 + \frac{2M}{r}\right)\frac{d\tilde{\phi}}{dt}\right]\frac{dr}{d\tilde{t}} + \left\{\left[r^2 + a^2\left(1 + \frac{2M}{r}\right)\right]\left(\frac{d\tilde{\phi}}{d\tilde{t}}\right)^2 - \frac{4Ma}{r}\frac{d\tilde{\phi}}{d\tilde{t}} - \left(1 - \frac{2M}{r}\right)\right\} = 0. \tag{33.5.7}$$

Solving the quadratic equation for $dr/d\tilde{t}$ above, we can obtain 33.5.2.

(b) Using 33.5.6 and $dr/d\tilde{t} = 0$, we can get

$$\left[r^2 + a^2\left(1 + \frac{2M}{r}\right)\right]\left(\frac{d\tilde{\phi}}{d\tilde{t}}\right)^2 - \frac{4Ma}{r}\frac{d\tilde{\phi}}{d\tilde{t}} - \left(1 - \frac{2M}{r}\right) = 0. \tag{33.5.8}$$

Note that

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{t\phi} = -\frac{2Ma}{r}, \quad g_{\phi\phi} = r^2 + a^2\left(1 + \frac{2M}{r}\right) \tag{33.5.9}$$

when $Q = 0$ and $\theta = \pi/2$ in Boyer-Lindquist coordinates. We can conclude that $\Omega_{\min}$ and $\Omega_{\max}$ are the solution of 33.5.8.

(c) At static limit, we have $\Omega_{\min} = 0$, i.e. we can have $d\tilde{\phi}/d\tilde{t} = 0$ when $dr/d\tilde{t} = 0$ for null geodesics. And so the light cone is tangent to a curve of constant r, θ and $\tilde{\phi}$.

(d) According to 33.5.2, we have

$$\frac{dr}{d\tilde{t}} = -\frac{2M/r}{1 + 2M/r} \pm \frac{1}{1 + 2M/r} = -1 \text{ and } \frac{1 - 2M/r}{1 + 2M/r}. \tag{33.5.10}$$

when $d\tilde{\phi}/d\tilde{t} = 0$.

(e) At horizon, we have

$$\frac{d\tilde{\phi}}{d\tilde{t}} = \Omega_{\min} = \Omega_{\max} = \frac{a}{a^2 + r_+^2} \tag{33.5.11}$$

when $dr/d\tilde{t} = 0$. So the light cone is tangent to the horizon.

33.6 SUPERHAMIlTONIAN FOR CHARGED-PARTICLE MOTION

Show that Hamilton's equations (33.28a) for the Hamiltonian (33.28b) reduce to equation (33.29a) for the value of the generalized momentum, and to the Lorentz force equation (33.29b).

Solution:

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}(\pi_\mu - eA_\mu)(\pi_\nu - eA_\nu). \tag{33.6.1}$$

So we have

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} = \frac{\partial\mathcal{H}}{\partial\pi_\mu} = g^{\mu\nu}(\pi_\nu - eA_\nu) = \pi^\mu - eA^\mu \tag{33.6.2}$$

and

$$\frac{d\pi_\mu}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial x^\mu} = -\frac{1}{2}g^{\alpha\beta}{}_{,\mu}(\pi_\alpha - eA_\alpha)(\pi_\beta - eA_\beta) + eA_{\alpha,\mu}(\pi^\alpha - eA^\alpha). \tag{33.6.3}$$

Note that

$$0 = (g^{\alpha\beta}g_{\beta\gamma})_{,\mu} = g_{\beta\gamma}g^{\alpha\beta}{}_{,\mu} + g^{\alpha\beta}g_{\beta\gamma,\mu}. \tag{33.6.4}$$

We have

$$\begin{aligned}\frac{dp^\mu}{d\lambda} &= \frac{dg^{\mu\nu}}{d\lambda}p_\nu + g^{\mu\nu}\frac{dp_\nu}{d\lambda} \\ &= g^{\mu\nu}{}_{,\alpha}p^\alpha p_\nu + g^{\mu\nu}\frac{d}{d\lambda}(\pi_\nu - eA_\nu) \\ &= -g^{\mu\rho}g_{\rho\beta,\alpha}p^\alpha p^\beta + g^{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}{}_{,\nu}p_\alpha p_\beta + eA_{\alpha,\nu}p^\alpha\right] - eg^{\mu\nu}A_{\nu,\alpha}p^\alpha \\ &= g^{\mu\nu}\left(\frac{1}{2}g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha}\right)p^\alpha p^\beta + eg^{\mu\nu}F_{\nu\alpha}p^\alpha \\ &= -\frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})p^\alpha p^\beta + eF^{\mu\alpha}p_\alpha \\ &= -\Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta + eF^{\mu\alpha}p_\alpha. \end{aligned} \tag{33.6.5}$$

33.7 HAMILTON-JACOBI DERIVATION OF EQUATIONS OF MOTION

Derive the first-order equations of motion (33.32) for a charged particle moving in the external fields of a Kerr-Newman black hole. Use the Hamilton-Jacobi method.

Solution:

(a) We first replace the generalized momentum π_a by the gradient $\partial S/\partial x^\alpha$ a of the Hamilton-Jacobi function hroughout the superhamiltonian $\mathcal{H}$ of equation (33.28b)

(b) Then we write down the Hamilton-Jacobi equation in the form

$$-\frac{\partial S}{\partial \lambda} = \mathcal{H}\left[x^\alpha, \frac{\partial S}{\partial x^\alpha}\right] = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} - eA_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} - eA_\beta\right). \tag{33.7.1}$$

(c) In Boyer-Lindquist coordinates, the metric restricted to (t, ϕ) for a Kerr-Newman black hole is

$$g_{AB} = \rho^{-2}\begin{pmatrix} -\Delta + a^2\sin^2\theta & a(\Delta - r^2 - a^2)\sin^2\theta \\ a(\Delta - r^2 - a^2)\sin^2\theta & [(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta \end{pmatrix} \tag{33.7.2}$$

The inverse metric is

$$g^{AB} = -\frac{1}{\Delta\rho^2\sin^2\theta}\begin{pmatrix} [(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta & a(r^2+a^2-\Delta)\sin^2\theta \\ a(r^2+a^2-\Delta)\sin^2\theta & -\Delta + a^2\sin^2\theta \end{pmatrix} \tag{33.7.3}$$

Put back r and θ coordinates and we can get

$$\begin{aligned}\boldsymbol{g} \equiv g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} &= -\frac{1}{\Delta\rho^2}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\phi}\right]^2 + \frac{1}{\rho^2\sin^2\theta}\left[\frac{\partial}{\partial\phi} + a\sin^2\theta\frac{\partial}{\partial t}\right]^2 \\ &+ \frac{\Delta}{\rho^2}\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\theta}\right)^2. \end{aligned} \tag{33.7.4}$$

(d) The vector potential of EM field is

$$A_\alpha = -\frac{Qr}{\rho^2}(1, 0, 0, -a\sin^2\theta). \tag{33.7.5}$$

So we have

$$(r^2+a^2)A_t + aA_\phi = -Qr, \quad A_\phi + a\sin^2\theta A_t = 0 \tag{33.7.6}$$

Now the Hamilton-Jacobi equation can be written as

$$-\frac{\partial S}{\partial \lambda} = -\frac{1}{2}\frac{1}{\Delta\rho^2}\left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \phi} + eQr\right]^2 + \frac{1}{2}\frac{1}{\rho^2\sin^2\theta}\left[\frac{\partial S}{\partial \phi} + a\sin^2\theta\frac{\partial S}{\partial t}\right]^2 + \frac{1}{2}\frac{\Delta}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2}\frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2. \tag{33.7.7}$$

(e) Because the equation has no explicit dependence on λ, ϕ, or t, the solution must take the form

$$S = \frac{1}{2}\mu^2\lambda - Et + L_z\phi + S_r(r) + S_\theta(\theta), \tag{33.7.8}$$

where the values of the "integration constants" follow from $\partial S/\partial\lambda = -\mathcal{H}$, $\partial S/\partial\phi = \pi_\phi$ and $\partial S/\partial t = \pi_t$. Insert this assumed form into (33.35), we have

$$-\mu^2 = -\frac{1}{\Delta\rho^2}\left[-(r^2+a^2)E + aL_z + eQr\right]^2 + \frac{1}{\rho^2\sin^2\theta}\left[L_z - aE\sin^2\theta\right]^2 + \frac{\Delta}{\rho^2}\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2. \tag{33.7.9}$$

Multiply both sides of equation above by $\rho^2 \equiv r^2 + a^2\cos^2\theta$,

$$-(r^2+a^2\cos^2\theta)\mu^2 = -\frac{1}{\Delta}\left[-(r^2+a^2)E + aL_z + eQr\right]^2 + L_z^2\left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right) - 2aL_zE + a^2E^2(1-\cos^2\theta) + \Delta\left(\frac{\partial S_r}{\partial r}\right)^2 + \left(\frac{\partial S_\theta}{\partial\theta}\right)^2. \tag{33.7.10}$$

Now we can separate the variable and get

$$-a^2\cos^2\theta\mu^2 + \mathcal{L} = \frac{L_z^2\cos^2\theta}{\sin^2\theta} - a^2E^2\cos^2\theta + \left(\frac{\partial S_\theta}{\partial\theta}\right)^2 \tag{33.7.11}$$

and

$$-r^2\mu^2 - \mathcal{L} = -\frac{1}{\Delta}\left[-(r^2+a^2)E + aL_z + eQr\right]^2 + (L_z - aE)^2 + \Delta\left(\frac{\partial S_r}{\partial r}\right)^2, \tag{33.7.12}$$

where the constant $\mathcal{L}$ arises naturally as a "separation-of-variables constant" in this procedure. Using the definition

$$\begin{aligned}\Theta &\equiv \mathcal{L} - \cos^2\theta\left[a^2(\mu^2 - E^2) + \frac{L_z^2}{\sin^2\theta}\right],\\ P &\equiv E(r^2+a^2) - L_z a - eQr,\\ R &\equiv P^2 - \Delta[\mu^2 r^2 + (L_z - aE)^2 + \mathcal{L}],\end{aligned} \tag{33.7.13}$$

we can obtain the solution for S_r and S_θ:

$$S_r = \int\frac{\sqrt{R}}{\Delta}dr, \quad S_\theta = \int\sqrt{\Theta}d\theta. \tag{33.7.14}$$

(f) We will use $\mathcal{L} + (L_z - aE)^2$ as a new constant of motion instead of $\mathcal{L}$. Therefore $X = \mathcal{L} + (L_z - aE)^2$, μ^2, E or L_z are new canonical momentum brought by generating function S. New canonical coordinates $Y = \partial S/\partial X$ are also constant of motion. The equation of motion is given by $Y = \partial S/\partial X$.

We can set all Y to 0, as they can be absorbed into the integration of r and θ or eliminated by the redefinition of the origin of t and ϕ. By successively setting $\partial S/\partial X$ to zero, we can obtain the equations describing the test-particle orbits. Note that

$$\Theta = \mathcal{L} + (L_z - aE)^2 - a^2\mu^2\cos^2\theta - \frac{1}{\sin^2\theta}\left(L_z - aE\sin^2\theta\right)^2. \tag{33.7.15}$$

We have

$$\frac{\partial R}{\partial[\mathcal{L}+(L_z-aE)^2]}=-\Delta,\quad \frac{\partial\Theta}{\partial[\mathcal{L}+(L_z-aE)^2]}=1. \tag{33.7.16a}$$

$$\frac{\partial R}{\partial\mu^2}=-\Delta r^2,\quad \frac{\partial\Theta}{\partial\mu^2}=-a^2\cos^2\theta. \tag{33.7.16b}$$

$$\frac{\partial R}{\partial E}=2P(r^2+a^2),\quad \frac{\partial\Theta}{\partial E}=2a(L_z-aE\sin^2\theta). \tag{33.7.16c}$$

$$\frac{\partial R}{\partial L_z}=-2aP,\quad \frac{\partial\Theta}{\partial L_z}=-2(L_z-aE\sin^2\theta)/\sin^2\theta. \tag{33.7.16d}$$

Using

$$\frac{\partial S_r}{\partial X}=\int\frac{dr}{2\Delta\sqrt{R}}\frac{\partial R}{\partial X},\quad \frac{\partial S_\theta}{\partial X}=\int\frac{dr}{2\sqrt{\Theta}}\frac{\partial\Theta}{\partial X}, \tag{33.7.17}$$

we have the equation of motion

$$\int^\theta\frac{d\theta}{\sqrt{\Theta}}=\int^r\frac{dr}{\sqrt{R}}, \tag{33.7.18a}$$

$$\lambda=\int^\theta\frac{a^2\cos^2\theta}{\sqrt{\Theta}}+\int^r\frac{r^2}{\sqrt{R}} \tag{33.7.18b}$$

$$t=\int^\theta\frac{a(L_z-aE\sin^2\theta)}{\sqrt{\Theta}}+\int^r\frac{P(r^2+a^2)}{\Delta\sqrt{R}} \tag{33.7.18c}$$

$$\phi=\int^\theta\frac{L_z-aE\sin^2\theta}{\sin^2\theta\sqrt{\Theta}}+\int^r\frac{aP}{\Delta\sqrt{R}}. \tag{33.7.18d}$$

(g) By differentiating these equations, we have

$$\frac{d\theta}{dr}=\sqrt{\frac{\Theta}{R}}, \tag{33.7.19a}$$

$$\frac{d\lambda}{dr}=\frac{a^2\cos^2\theta}{\sqrt{\Theta}}\frac{d\theta}{dr}+\frac{r^2}{\sqrt{R}}=\frac{\rho^2}{\sqrt{R}}, \tag{33.7.19b}$$

$$\frac{dt}{dr}=\frac{a(L_z-aE\sin^2\theta)}{\sqrt{\Theta}}\frac{d\theta}{dr}+\frac{P(r^2+a^2)}{\Delta\sqrt{R}}=\frac{a(L_z-aE\sin^2\theta)+P(r^2+a^2)\Delta^{-1}}{\sqrt{R}}, \tag{33.7.19c}$$

$$\frac{d\phi}{dr}=\frac{L_z-aE\sin^2\theta}{\sin^2\theta\sqrt{\Theta}}\frac{d\theta}{dr}+\frac{aP}{\Delta\sqrt{R}}=\frac{L_z\sin^{-2}\theta-aE+aP\Delta^{-1}}{\sqrt{R}}. \tag{33.7.19d}$$

After combining them, we have

$$\rho^2\frac{d\theta}{d\lambda}=\sqrt{\Theta}, \tag{33.7.20a}$$

$$\rho^2\frac{dr}{d\lambda}=\sqrt{R}, \tag{33.7.20b}$$

$$\rho^2\frac{dt}{d\lambda}=a(L_z-aE\sin^2\theta)+P(r^2+a^2)\Delta^{-1}, \tag{33.7.20c}$$

$$\rho^2\frac{d\phi}{d\lambda}=L_z\sin^{-2}\theta-aE+aP\Delta^{-1}. \tag{33.7.20d}$$

$$\tag{33.7.20e}$$

(h) Using $\pi_\alpha=\partial S/\partial x^\alpha$, we have

$$-\frac{1}{2}\mu^2=\frac{1}{2}g^{\alpha\beta}(\pi_\alpha-eA_\alpha)(\pi_\beta-eA_\beta),\quad \pi_t=-E,\quad \pi_\phi=L_z \tag{33.7.21}$$

and

$$\pi_\theta=\sqrt{\mathcal{L}-\cos^2\theta\left[a^2(\mu^2-E^2)+\frac{L_z^2}{\sin^2\theta}\right]}. \tag{33.7.22}$$

Using $\pi_\alpha - eA_\alpha = p_\alpha$, we have

$$g^{\alpha\beta}p_\alpha p_\beta = -\mu^2, \quad E = -(p_t + eA_t), \quad L_z = p_\phi + eA_\phi \tag{33.7.23}$$

and

$$\mathcal{L} = p_\theta^2 + \cos^2\theta\left[a^2(\mu^2 - E^2) + \frac{L_z^2}{\sin^2\theta}\right]. \tag{33.7.24}$$

33.8 KERR-SCHILD COORDINATES

(a) Show that In Kerr coordinates the ingoing null congruence (33.39) has the form (33.42a). Also show that the covariant components of the wave vector - after changing to a new affine parameter $\lambda_{\rm new} = \lambda_{\rm old}E$ - are

$$k_r^{(\rm in)} = 0, \quad k_\theta^{(\rm in)} = 0, \quad k_{\tilde\phi}^{(\rm in)} = a\sin^2\theta, \quad k_{\tilde V}^{(\rm in)} = -1. \tag{33.8.1}$$

(b) Introduce new coordinates $\tilde t$, x, y, z, defined by

$$x + iy = (r + ia)e^{i\tilde\phi}\sin\theta, \quad z = r\cos\theta, \quad \tilde t = \tilde V - r; \tag{33.8.2}$$

and show that in tills "Kerr-Schild coordinate system" the metric takes the form

$$ds^2 = (\eta_{\alpha\beta} + 2Hk_\alpha^{(\rm in)}k_\beta^{(\rm in)})dx^\alpha dx^\beta, \tag{33.8.3}$$

where

$$H = \frac{Mr - \frac{1}{2}Q^2}{r^2 + a^2(z/r)^2}, \tag{33.8.4}$$

$$k_\alpha^{(\rm in)}dx^\alpha = -\frac{r(xdy + ydx) - a(xdy - ydx)}{r^2 + a^2} - \frac{zdz}{r} - d\tilde t. \tag{33.8.5}$$

For the transformation to analogous coordinates in which

$$ds^2 = (\eta_{\alpha\beta} + 2Hk_\alpha^{(\rm out)}k_\beta^{(\rm out)})dx^\alpha dx^\beta, \tag{33.8.6}$$

see, e.g., Boyer and Lindquist (1967).

Solution:

(a) In Boyer-Lindquist coordinates, the four momentum of ingoing photons whose trajectories belong to principal null congruences are

$$k^t = \frac{(r^2 + a^2)E}{\Delta},\ k^r = -E,\ k^\theta = 0,\ k^\phi = \frac{aE}{\Delta}, \tag{33.8.7}$$

Transformation between Boyer-Lindquist coordinates Kerr coordinates are described by

$$\boldsymbol{d}\widetilde{V} = \boldsymbol{d}t + (r^2 + a^2)(\boldsymbol{d}r/\Delta), \quad \boldsymbol{d}\widetilde{\phi} = \boldsymbol{d}\phi + a(\boldsymbol{d}r/\Delta). \tag{33.8.8}$$

And so we can get

$$k^{\tilde V} = k^t + \frac{r^2 + a^2}{\Delta}k^r = 0, \quad k^{\tilde\phi} = k^\phi + \frac{a}{\Delta}k^r = 0, \tag{33.8.9}$$

i.e. $k^{\tilde\alpha} = (0, -E, 0, 0)$. The metric in Kerr coordinates is

$$g_{\tilde\alpha\tilde\beta} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ 0 & -a\sin^2\theta & 0 & (r^2 + a^2)\sin^2\theta \end{pmatrix} + \frac{2Mr - Q^2}{\rho^2}\begin{pmatrix} 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a\sin^2\theta & 0 & 0 & a^2\sin^4\theta \end{pmatrix}. \tag{33.8.10}$$

So we have $k_{\tilde{\alpha}} = g_{\tilde{\alpha}\tilde{\beta}}k^{\tilde{\beta}} = -g_{\tilde{\alpha}r}E = -E(1,0,0,-a\sin\theta)$. Changing to a new affine parameter $\lambda_{\rm new} = \lambda_{\rm old}E$, we can get equation 33.8.1.

(b)The transformation matrix between new coordinates $\alpha = \tilde{t}, x, y, z$ and Kerr coordinates $\tilde{\alpha} = \tilde{V}, r, \theta, \tilde{\phi}$ is

$$\frac{\partial x^{\alpha}}{\partial x^{\tilde{\alpha}}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \cos\tilde{\phi}\sin\theta & (r\cos\tilde{\phi} - a\sin\tilde{\phi})\cos\theta & -(r\sin\tilde{\phi} + a\cos\tilde{\phi})\sin\theta \\ 0 & \sin\tilde{\phi}\sin\theta & (r\sin\tilde{\phi} + a\cos\tilde{\phi})\cos\theta & (r\cos\tilde{\phi} - a\sin\tilde{\phi})\sin\theta \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix}. \tag{33.8.11}$$

Note that in Kerr coordinates, $\eta_{\alpha\beta} + 2Hk^{(\rm in)}_{\alpha}k^{(\rm in)}_{\beta}$ will be transformed to

$$\frac{\partial x^{\alpha}}{\partial x^{\tilde{\alpha}}}\eta_{\alpha\beta}\frac{\partial x^{\beta}}{\partial x^{\tilde{\beta}}} + 2Hk^{(\rm in)}_{\tilde{\alpha}}k^{(\rm in)}_{\tilde{\beta}}. \tag{33.8.12}$$

It is easy to verify that

$$\frac{\partial x^{\alpha}}{\partial x^{\tilde{\alpha}}}\eta_{\alpha\beta}\frac{\partial x^{\beta}}{\partial x^{\tilde{\beta}}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ 0 & -a\sin^2\theta & 0 & (r^2+a^2)\sin^2\theta \end{pmatrix} \tag{33.8.13}$$

and

$$2Hk^{(\rm in)}_{\tilde{\alpha}}k^{(\rm in)}_{\tilde{\beta}} = \frac{2Mr - Q^2}{\rho^2}\begin{pmatrix} 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a\sin^2\theta & 0 & 0 & a^2\sin^4\theta \end{pmatrix}. \tag{33.8.14}$$

So in tills Kerr-Schild coordinate system the metric takes the form of 33.8.3.

33.9 NULL GENERATORS OF HORIZON

(a) Show that in Kerr coordinates the outgoing principle null congruence is described by the tangent vector

$$\frac{d\theta}{d\lambda} = 0,\ \frac{dr}{d\lambda} = E,\ \frac{d\tilde{\phi}}{d\lambda} = 2a\frac{E}{\Delta},\ \frac{d\tilde{V}}{d\lambda} = 2(r^2+a^2)\frac{E}{\Delta}. \tag{33.9.1}$$

(b) These components of the wave vector become singular at the horizon ($\Delta = 0$), not because of a singularity in the coordinate system-the coordinates are well-behaved! - but because of poor normalization of the affine parameter. For each outgoing geodesic, let Δ_0 be a constant, defined as the value of Δ at the event where the geodesic slices the hypersurface $\tilde{V} = 0$. Then renormalize the affine parameter for each geodesic

$$\lambda_{\rm new} = \frac{E}{\Delta_0}\lambda_{\rm old}. \tag{33.9.2}$$

Show that the resulting wave vectors

$$\frac{d\theta}{d\lambda} = 0,\ \frac{dr}{d\lambda} = \Delta_0,\ \frac{d\tilde{\phi}}{d\lambda} = 2a\frac{\Delta_0}{\Delta},\ \frac{d\tilde{V}}{d\lambda} = 2(r^2+a^2)\frac{\Delta_0}{\Delta} \tag{33.9.3}$$

are well-behaved as one approaches the horizon; and show that the geodesics on the horizon have the form

$$\theta = \text{ const.},\ r = r_+ = \text{ const.},\ \tilde{\phi} = 2a\lambda,\ \tilde{V} = 2(r_+^2 + a^2)\lambda. \tag{33.9.4}$$

(c) Show that these are the only test-particle trajectories that remain forever on the horizon.

Solution:

(a) In Boyer-Lindquist coordinates, the outgoing principle null congruence is described by the tangent vector

$$\frac{d\theta}{d\lambda}=0,\ \frac{dr}{d\lambda}=E,\ \frac{d\phi}{d\lambda}=\frac{aE}{\Delta}.\ \frac{dt}{d\lambda}=\frac{(r^2+a^2)E}{\Delta}. \tag{33.9.5}$$

Using

$$\boldsymbol{d}\widetilde{V}=\boldsymbol{d}t+(r^2+a^2)(\boldsymbol{d}r/\Delta),\quad \boldsymbol{d}\widetilde{\phi}=\boldsymbol{d}\phi+a(\boldsymbol{d}r/\Delta), \tag{33.9.6}$$

we can get

$$\frac{d\tilde{\phi}}{d\lambda}=2a\frac{E}{\Delta},\ \frac{d\tilde{V}}{d\lambda}=2(r^2+a^2)\frac{E}{\Delta}. \tag{33.9.7}$$

(b) When the photon approaches the horizon, we have $\Delta_0\to 0$ and so $dr/d\lambda\to 0$. $\Delta(r)$ will be a constant equal to Δ_0. So we have

$$\frac{d\theta}{d\lambda}=0,\ \frac{dr}{d\lambda}=0,\ \frac{d\tilde{\phi}}{d\lambda}=2a,\ \frac{d\tilde{V}}{d\lambda}=2(r^2+a^2). \tag{33.9.8}$$

The geodesic is well behaved and can be parameterized by 33.9.4.

(c) The line element at horizon with $dr=0$ is

$$ds^2=-[1-\rho^{-2}(2Mr_+-Q^2)]d\tilde{V}^2+\rho^2d\theta^2+\rho^{-2}(r^2+a^2)^2\sin^2\theta d\tilde{\phi}^2-2a\rho^{-2}(2Mr_+-Q^2)\sin^2\theta d\tilde{\phi}d\tilde{V}. \tag{33.9.9}$$

Using $\Delta=r_+^2+a^2+Q^2-2Mr_+=0$ and $\rho^2=r_+^2+a^2\cos^2\theta$, we can obtain

$$ds^2=\rho^{-2}\sin^2\theta[ad\tilde{V}-(r_+^2+a^2)d\tilde{\phi}]^2+\rho^2d\theta^2. \tag{33.9.10}$$

As $ds^2\le 0$, we must have

$$d\theta=0,\ ad\tilde{V}=(r_+^2+a^2)d\tilde{\phi}. \tag{33.9.11}$$

So 33.9.4 are the only test-particle trajectories that remain forever on the horizon.

33.10 ANGULAR MOMENTUM VECTOR FOR IN FALLING PARTICLE

Derive equations (33.49d,e,f) for the components L_x and L_y of the orbital angular momentum of a particle falling into a black hole. Assume negligible initial speed, $E^2-\mu^2\approx 0$.

Solution:

At infinity, we have

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta. \tag{33.10.1}$$

Assume negligible initial speed, we can get

$$L_x/\mu=y\dot{z}-z\dot{y}=-r^2\sin\phi\dot{\theta}-r^2\cos\theta\sin\theta\cos\phi\dot{\phi}, \tag{33.10.2a}$$

$$L_y/\mu=z\dot{x}-x\dot{z}=r^2\cos\phi\dot{\theta}-r^2\cos\theta\sin\theta\sin\phi\dot{\phi}, \tag{33.10.2b}$$

$$L_z/\mu=x\dot{y}-y\dot{x}=r^2\sin^2\theta\dot{\phi}. \tag{33.10.2c}$$

Using equations 33.7.13 and 33.7.18 with conditions $r\gg M,a,Q$ and $E^2-\mu^2\approx 0$, we have

$$\Theta=\mathcal{L}-L_z^2\cot^2\theta,\ P=Er^2,\ \Delta=r^2,\ \rho^2=r^2 \tag{33.10.3}$$

and so

$$r^2\frac{d\theta}{d\lambda}=\sqrt{\Theta},\quad r^2\frac{d\phi}{d\lambda}=\frac{L_z}{\sin^2\theta},\quad r^2\frac{dt}{d\lambda}=r^2E. \tag{33.10.4}$$

Now we have

$$L_x = -\sin\phi_\infty\sqrt{\Theta_\infty} - \cot\theta_\infty\cos\phi_\infty L_z, \quad L_y = \cos\phi_\infty\sqrt{\Theta_\infty} - \cot\theta_\infty\sin\phi_\infty L_z \tag{33.10.5}$$

and

$$\sqrt{L_x^2 + L_y^2} = \sqrt{\Theta_\infty + \cot^2\theta_\infty L_z^2} = \sqrt{\mathcal{L}}. \tag{33.10.6}$$

33.11 IRREDUCIBLE MASS IS IRREDUCIBLE

Show that condition (33.56) is equivalent to $\delta M_{\rm ir} \geq 0$.

Solution:

The irreducible mass is defined as

$$M_{\rm ir} \equiv \left(\frac{A}{16\pi}\right)^{1/2}, \tag{33.11.1}$$

where A is the surface area of horizon,

$$A = 4\pi(r_+^2 + a^2). \tag{33.11.2}$$

So $\delta M_{\rm ir} \geq 0$ is equivalent to $r_+\delta r_+ + a\delta a \geq 0$. Using $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$, we have

$$\begin{aligned} r_+\delta r_+ + a\delta a &= r_+\left(\delta M + \frac{M\delta M - Q\delta Q - a\delta a}{r_+ - M}\right) + a\delta a \\ &= \frac{r_+^2\delta M - r_+ Q\delta Q - a r_+\delta a}{r_+ - M} + \frac{a(r_+ - M)\delta a}{r_+ - M} \\ &= \frac{r_+^2\delta M - r_+ Q\delta Q - aM\delta(S/M)}{r_+ - M} \\ &= \frac{(r_+^2 + a^2)\delta M - r_+ Q\delta Q - a\delta S}{r_+ - M}. \end{aligned} \tag{33.11.3}$$

So $\delta M_{\rm ir} \geq 0$ is equivalent to

$$\delta M \geq \frac{r_+ Q\delta Q + a\delta S}{r_+^2 + a^2}. \tag{33.11.4}$$

33.12 SURFACE AREA OF A BLACK HOLE

Show that the surface area of the horizon of the Kerr-Newman geometry [area of surface $r = r_+$ and $t =$ const (Boyer-Lindquist coordinates) or $\tilde{V} =$ const (Kerr coordinates)] is $16\pi M_{\rm ir}^2$.

Solution:

In Boyer-Lindquist or Kerr coordinates, the line element on the horizon is

$$ds^2 = \rho^2 d\theta^2 + \rho^{-2}(r_+^2 + a^2)^2\sin^2\theta d\phi^2. \tag{33.12.1}$$

The surface element is therefore

$$dS = (r_+^2 + a^2)\sin\theta d\theta d\phi. \tag{33.12.2}$$

The surface area is

$$\begin{aligned} A &= \int (r_+^2 + a^2) \sin\theta d\theta d\phi \\ &= 4\pi(r_+^2 + a^2) \\ &= 4\pi[(M + \sqrt{M^2 - Q^2 - a^2})^2 + a^2] \\ &= 4\pi(2M^2 - Q^2 + 2\sqrt{M^4 - Q^2M^2 - S^2}) \\ &= 4\pi(2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - Q^4 - 4S^2}) \end{aligned} \tag{33.12.3}$$

Note that

$$M^2 = \left(M_{\rm ir} + \frac{Q^2}{4M_{\rm ir}}\right)^2 + \frac{S^2}{4M_{\rm ir}^2}. \tag{33.12.4}$$

We can obtain

$$A = 16\pi M_{\rm ir}^2. \tag{33.12.5}$$

33.13 ANGULAR VELOCITY OF A BLACK HOLE

A general theorem [Hartle (1970) for relativistic case; Ostriker and Gunn (1969) for nonrelativistic case] says that, if one injects angular momentum into a rotating star while holding fixed all other contributions to its total mass-energy (contributions from entropy and from baryonic rest mass), then the injection produces a change in total mass-energy given by

$$\delta(\text{mass-energy}) = (\text{angular velocity of star at point of injection})\, \delta(\text{angular momentum}). \tag{33.13.1}$$

By analogy, if one injects an angular momentum δS into a rotating black hole while holding fixed all other contributions to its total mass-energy (contributions from irreducible mass and from charge), one identifies the coefficient Ω_h in the equation

$$\delta M = \Omega_h \delta S \tag{33.13.2}$$

as the angular velocity of the hole:

$$\Omega_h = \left(\frac{\partial M}{\partial S}\right)_{Q, M_{\rm ir}}. \tag{33.13.3}$$

(a) Show that the angular velocity of a black hole is equal to

$$\Omega_h = \frac{a}{r_+^2 + a^2}. \tag{33.13.4}$$

Notice that this is precisely the angular velocity of photons that live forever on the horizon [equation (33.42b); "barber-pole twist" of null generators of horizon].

(b) Show that any object falling into a black hole acquires an angular velocity (relative to Boyer-Lindquist coordinates) of $\Omega = \frac{d\phi}{dt}$ in the late stages, as it approaches the horizon. (Recall that the horizon is a singularity of the Boyer-Lindquist coordinates. This is the reason that every object, regardless of its L_z, E, e, μ, $\mathcal{L}$ can approach and does approach $\Omega = \Omega_h$.)

Solution:

(a) Generally, we have equation 33.11.4. If we demand that $\delta M_{\rm ir} = 0$ and $\delta Q = 0$, we will have

$$\delta M = \frac{a}{r_+^2 + a^2} \delta S. \tag{33.13.5}$$

And so we have 33.13.4.

(b) From equation 33.7.20, we have

$$\frac{d\phi}{dt} = \frac{(L_z/\sin^2\theta - aE)\Delta + aP}{(L_z - aE\sin^2\theta)a\Delta + (r^2+a^2)P}. \tag{33.13.6}$$

As the particle approaches the horizon, $\Delta \to 0$, and so we have

$$\frac{d\phi}{dt} \to \frac{a}{r_+^2 + a^2}. \tag{33.13.7}$$

33.14 SEPARATION OF VARIABLES FOR WAVE EQUATIONS

This chapter has studied extensively the motion of small objects in the external fields of black holes. Of almost equal importance, but not so well-understood yet because of its complexity, is the evolution of weak electromagnetic and gravitational perturbations ("waves") in the Kerr-Newman geometry. Just as one had no a priori reason to expect a "fourth constant" for test particle motion in the Kerr-Newman geometry, so one had no reason to expect separability for Maxwell's equations, or for the wave equations describing gravitational perturbations-or even for the scalar wave equation $\Box\psi \equiv -\psi_{,\alpha}{}^{\alpha}$. Thus it came as a great surprise when Carter (1968c) proved separability for the scalar wave equation, and later when Teukolsky (1972, 1973) separated both Maxwell's equations and the wave equations for gravitational perturbations.

Show that separation of variables for the scalar-wave equation in the (uncharged) Kerr geometry yields solutions of the form

$$\psi = (r^2+a^2)^{-1/2} u_{\ell m}(r) S_{m\ell}(-i\omega a, \cos\theta) e^{i(m\phi - \omega t)}, \tag{33.14.1}$$

where m and ℓ are integers with $0 < |m| < \ell$; $S_{m\ell}$ is a spheroidal harmonic [see Meixner and Schärfke (1954)]; and $u_{\ell m}$ satisfies the differential equation

$$-\frac{d^2u}{dr^{*2}} + Vu = 0. \tag{33.14.2}$$

In order to put the equation in this form, define a Regge-Wheeler (1957) "tortoise" -type radial coordinate r^* by

$$dr^* = \Delta^{-1}(r^2+a^2)dr, \tag{33.14.3}$$

and find an effective potential $V(r^*)$ given by

$$\begin{aligned} V = &-\left(\omega - \frac{ma}{r^2+a^2}\right)^2 + [(m-\omega a)^2 + \mathcal{L}](r^2+a^2)^{-2}\Delta \\ &+ 2(Mr - a^2)(r^2+a^2)^{-3}\Delta + 3a^2(r^2+a^2)^{-4}\Delta^2. \end{aligned} \tag{33.14.4}$$

In this radial equation $\mathcal{L}$ is a constant (analog of Carter's constant for particle motion), given in terms of m and ℓ by

$$\mathcal{L} \equiv \lambda_{m\ell} - m^2; \tag{33.14.5}$$

$\lambda_{m\ell}$ is eigenfunction of spheroidal harmonic.

Solution:

The field equation $\Box\psi = 0$ is equivalent to

$$g^{\alpha\beta}\psi_{,\alpha\beta} - g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta} = 0. \tag{33.14.6}$$

The components of metric and inverse metric in Boyer-Lindquist coordinates are given by 33.7.2, 33.7.3 and 33.7.4. Detailed calculations show that

$$g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{\rho^2}(0, 2M - 2r, \cot\theta, 0), \tag{33.14.7}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. The field equations can be written explicitly:

$$-\frac{(r^2+a^2)^2}{\Delta}\psi_{,tt}+a^2\sin^2\theta\psi_{,tt}-\frac{4Mar}{\Delta}\psi_{,t\phi}-\frac{a^2}{\Delta}\psi_{,\phi\phi}+\frac{1}{\sin^2\theta}\psi_{,\phi\phi}+\Delta\psi_{,rr}+2(r-M)\psi_{,r}+\psi_{,\theta\theta}+\cot\theta\psi_{,\theta}=0, \tag{33.14.8}$$

where $\Delta = r^2 - 2Mr + a^2$.

Suppose the field equation can be solved by separating variables and note that the field equation does not contain t and ϕ explicitly. We would like to assume

$$\psi = R(r)\Theta(\theta)e^{im\phi}e^{-i\omega t}. \tag{33.14.9}$$

Substitute 33.14.9 into 33.14.8. We can obtain

$$\frac{(r^2+a^2)^2\omega^2 - 4Marm\omega + a^2m^2}{\Delta} + \frac{\Delta R_{,rr} + 2(r-M)R_{,r}}{R} - a^2\omega^2\sin^2\theta - \frac{m^2}{\sin^2\theta} + \frac{\Theta_{,\theta\theta}+\cot\theta\Theta_{,\theta}}{\Theta}. \tag{33.14.10}$$

The first two terms of the equation functions of r only while the last three terms are functions of θ only. So we can introduce a constant $\mathcal{L}$ such that

$$\frac{\Theta_{,\theta\theta}+\cot\theta\Theta_{,\theta}}{\Theta} - a^2\omega^2\sin^2\theta - \frac{m^2}{\sin^2\theta} = -a^2\omega^2 - m^2 - \mathcal{L}. \tag{33.14.11}$$

The equation above can be rewritten as

$$\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right)\Theta = -(\mathcal{L}+m^2)\Theta. \tag{33.14.12}$$

The general solution is spheroidal harmonic $\Theta = S_{m\ell}(-i\omega a, \cos\theta)$. And $\lambda_{m\ell} = \mathcal{L} + m^2$ is the eigenvalue of this Sturm-Liouville differential equation and fixed by the requirement that $S_{m\ell}(-i\omega a, \cos\theta)$ must be finite for $|\cos\theta| = 1$.

The differential equation for $R(r)$ is

$$\Delta[\Delta R_{,rr} + 2(r-M)R_{,r}] + [(r^2+a^2)^2\omega^2 - 4Marm\omega + a^2m^2]R = (\mathcal{L} + a^2\omega^2 + m^2)\Delta R \tag{33.14.13}$$

Now we introduce $u = R(r^2+a^2)^{1/2}$ and note that $d/dr = (r^2+a^2)\Delta^{-1}d/dr^*$. We have

$$R_{,r} = u_{,r^*}\frac{(r^2+a^2)^{1/2}}{\Delta} - u\frac{r}{(r^2+a^2)^{3/2}} \tag{33.14.14}$$

and

$$\begin{aligned} R_{,rr} = {} & u_{,r^*r^*}\frac{(r^2+a^2)^{3/2}}{\Delta^2} - u_{,r*}\frac{2(r-M)(r^2+a^2)^{1/2}}{\Delta^2} + \frac{u_{,r^*}}{\Delta}\frac{r}{(r^2+a^2)^{1/2}} \\ & - u_{,r^*}\frac{r}{\Delta(r^2+a^2)^{1/2}} - \frac{u}{(r^2+a^2)^{3/2}} + \frac{3ur^2}{(r^2+a^2)^{5/2}} \end{aligned} \tag{33.14.15}$$

It is easy to get

$$\Delta[\Delta R_{,rr} + 2(r-M)R_{,r}] = (r^2+a^2)^{3/2}u_{,r^*r^*} - \frac{u\Delta^2 + 2u\Delta(r-M)r}{(r^2+a^2)^{3/2}} + \frac{3ur^2\Delta^2}{(r^2+a^2)^{5/2}}. \tag{33.14.16}$$

And so we have

$$-\frac{d^2u}{dr^{*2}} + Vu = 0, \tag{33.14.17}$$

where

$$V = -\omega^2 + \frac{4Marm\omega - a^2m^2 + (\mathcal{L}+a^2\omega^2+m^2)\Delta}{(r^2+a^2)^2} + \frac{\Delta^2 + 2\Delta(r-M)r}{(r^2+a^2)^3} - \frac{3r^2\Delta^2}{(r^2+a^2)^4}. \tag{33.14.18}$$

A little calculations can lead to 33.14.4.

Chapter 34

GLOBAL TECHNIQUES, HORIZONS, AND SINGULARITY THEOREMS

34.1 FLAT SPACETIME IN ψ, ξ, θ, ϕ COORDINATES

(a) Derive equation (34.2c) from (34.1) and (34.2a,b).

(b) Show that the regions I^+, I^-, I^0, $\mathscr{I}^+$, and $\mathscr{I}^-$ of flat spacetime are located at

$$\begin{aligned} I^+ &: \psi = \pi, \xi = 0, \\ I^- &: \psi = -\pi, \xi = 0, \\ I^0 &: \psi = 0, \xi = \pi, \\ \mathscr{I}^+ &: \psi + \xi = \pi, -\pi < \psi - \xi < \pi, \\ \mathscr{I}^- &: \psi - \xi = -\pi, -\pi < \psi + \xi < \pi. \end{aligned} \tag{34.1.1}$$

[see equatIons (34.2)]. These are the regions depicted in Figure 34.2.

(c) Show that in flat spacetime, in a ψ, ξ coordinate diagram (Figure 34.2), radial null lines make angles of $45°$ with the vertical axis, and nonradial null lines make angles of less than $45°$.

Solution:

(a) From (34.2a,b), we have

$$dt + dr = \frac{d\psi + d\xi}{2\cos^2\frac{1}{2}(\psi + \xi)}, \quad dt - dr = \frac{d\psi - d\xi}{2\cos^2\frac{1}{2}(\psi - \xi)}. \tag{34.1.2}$$

We can obtain

$$-dt^2 + dr^2 = -(dt - dr)(dt + dr) = \frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi + \xi)\cos^2\frac{1}{2}(\psi - \xi)}. \tag{34.1.3}$$

Now we can derive equation (34.2c).

(b) From (34.2a,b), we have

$$\psi = \arctan(t + r) + \arctan(t - r), \quad \xi = \arctan(t + r) - \arctan(t - r). \tag{34.1.4}$$

For future timelike infinity I^+, we have $t \to +\infty$ while r is finite. So we can get $\psi = \pi$ and $\xi = 0$. For past timelike infinity I^-, we have $t \to -\infty$ while r is finite. So we can get $\psi = -\pi$ and $\xi = 0$. For spacelike infinity I^0, we have $r \to +\infty$ while t is finite. So we can get $\psi = 0$ and $\xi = \pi$.

Note that

$$\psi + \xi = 2\arctan(t+r), \psi - \xi = 2\arctan(t-r). \tag{34.1.5}$$

For future null infinity $\mathscr{I}^+$, we have $t + r \to +\infty$ while $t - r$ is finite. So we can get $\psi + \xi = \pi$ and $\psi - \xi \in (-\pi, \pi)$. For past null infinity $\mathscr{I}^-$, we have $t - r \to +\infty$ while $t + r$ is finite. So we can get $\psi - \xi = \pi$ and $\psi + \xi \in (-\pi, \pi)$.

(c) For null geodesics, we have $ds^2 = 0$. From equation (34.2c), we can get

$$d\psi^2 = d\xi^2 + 4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{34.1.6}$$

and so $d\psi^2 = d\xi^2$ for radial null lines while $d\psi^2 > d\xi^2$ for non-radial null lines. So radial null lines make angles of $45°$ with the vertical axis, and nonradial null lines make angles of less than $45°$.

34.2 SCHWARZSCHILD SPACETIME IN ψ, ξ, θ, ϕ COORDINATES

(a) Derive equations (34.3c,d) from (34.3a,b) and the Kruskal-Szekeres equations (31.14).

(b) Use equations (34.3) to justify the precise form of the coordinate diagram in Figure 34.3.

Solution:

(a) The Schwarzschild geometry in Kruskal-Szekeres coordinates is

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{34.2.1}$$

Here r is to be regarded as a function of u and v defined implicitly by

$$\left(\frac{r}{2M} - 1\right)e^{r/2M} = u^2 - v^2. \tag{34.2.2}$$

From (34.3a,b), we have

$$v + u = \tan\frac{1}{2}(\psi+\xi), \quad v - u = \tan\frac{1}{2}(\psi-\xi). \tag{34.2.3}$$

So we have

$$\left(1 - \frac{r}{2M}\right)e^{r/2M} = (v+u)(v-u) = \tan\frac{1}{2}(\psi+\xi)\tan\frac{1}{2}(\psi-\xi). \tag{34.2.4}$$

Note that

$$dv + du = \frac{d\psi + d\xi}{2\cos^2\frac{1}{2}(\psi+\xi)}, \quad dv - du = \frac{d\psi - d\xi}{2\cos^2\frac{1}{2}(\psi-\xi)}. \tag{34.2.5}$$

The metric becomes

$$ds^2 = \frac{32M^3}{r}e^{-r/2M}\frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{34.2.6}$$

(b) Using 34.2.4, in $\psi - \xi$ coordinates, the horizon $r = 2M$ becomes

$$\tan\frac{1}{2}(\psi+\xi)\tan\frac{1}{2}(\psi-\xi) = 0, \tag{34.2.7}$$

i.e.

$$\psi + \xi = 0 \text{ or } \psi - \xi = 0. \tag{34.2.8}$$

The singularity $r = 0$ becomes

$$\tan\frac{1}{2}(\psi+\xi)\tan\frac{1}{2}(\psi-\xi) = 1. \tag{34.2.9}$$

Note that $1/\tan\alpha = \cot\alpha$, we have

$$\tan\frac{1}{2}(\psi+\xi) = \cot\frac{1}{2}(\psi-\xi), \tag{34.2.10}$$

i.e.

$$\frac{1}{2}(\psi+\xi) + \frac{1}{2}(\psi-\xi) = \psi = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}. \tag{34.2.11}$$

34.3 REISSNER-NORDSTROM SPACETIME

(a) Show that there exists a coordinate system in which the Reissner-Nordstrom geometry with $0 < |Q| < M$ (exercises 31.8 and 32.1) has the form

$$ds^2 = F^2(-d\psi^2 + d\xi^2) + r^2 d\Omega^2, \quad F = F(\psi,\xi),\ r = r(\psi,\xi), \tag{34.3.1}$$

and in which the honzons and infinities are as shown in Figure 34.4.

(b) Use Figure 34.4 to deduce that the Reissner-Nordstrom geometry describes a "worm-hole" or bridge, connecting two asymptotically flat spacetimes, which: (i) expands to a state of maximum circumference; (ii) recontracts toward a state of minimum circumference, and in the process disconnects its outer regions from the two I^0's (spatial infinity) and reconnects them to a pair of $r = 0$ singularities; (iii) bounces; (iv) reexpands, and in the process disconnects its "outer regions" from the two singularities and reconnects them to a pair of I^0's in two new asymptotically flat universes; (v) slows its expansion to a halt; (vi) recontracts toward a state of minimum circumference, and in the process disconnects its outer regions from the two I^0's and reconnects them to a new pair of $r = 0$ singularities; etc. *ad infinitum*.

Solution:

(a) The metric of Reissner-Nordstrom geometry in $t - r$ coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{34.3.2}$$

Define $r^*(r)$ by

$$\frac{dr^*}{dr} = f^{-1}, \tag{34.3.3}$$

where $f = (1 - 2M/r + Q^2/r^2)$. The transformation matrix of coordinate transformation

$$\begin{cases} \tilde{V} = t + r^* \\ \tilde{U} = t - r^* \end{cases} \tag{34.3.4}$$

is

$$\frac{\partial(t,r)}{\partial(\tilde{V},\tilde{U})} = \begin{pmatrix} 1/2 & 1/2 \\ f/2 & -f/2 \end{pmatrix}. \tag{34.3.5}$$

The new components of the metric are therefore

$$\begin{pmatrix} 1/2 & f/2 \\ 1/2 & -f/2 \end{pmatrix}\begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ f/2 & -f/2 \end{pmatrix}. = \begin{pmatrix} 0 & -f/2 \\ -f/2 & 0 \end{pmatrix}, \tag{34.3.6}$$

i.e.

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)d\tilde{V}d\tilde{U} + r^2 d\Omega^2. \tag{34.3.7}$$

Define

$$\tilde{u} = \mp \mathrm{sgn}(f) e^{-\tilde{U}/4M}, \quad \tilde{v} = \pm e^{\tilde{V}/4M}. \tag{34.3.8}$$

We have

$$d\tilde{U}d\tilde{V} = \mathrm{sgn}(f)16M^2 e^{(\tilde{U}-\tilde{V})/4M} d\tilde{u}d\tilde{v} = \mathrm{sgn}(f)16M^2 e^{-r^*/2M} d\tilde{u}d\tilde{v}. \tag{34.3.9}$$

Now introducing new coordinates

$$\tilde{u} = \tan\frac{1}{2}(\psi - \xi), \quad \tilde{v} = \tan\frac{1}{2}(\psi + \xi). \tag{34.3.10}$$

We have

$$d\tilde{u}d\tilde{v} = \frac{d\psi + d\xi}{2\cos^2\frac{1}{2}(\psi+\xi)} \frac{d\psi - d\xi}{-2\cos^2\frac{1}{2}(\psi-\xi)} = -\frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi-\xi)\cos^2\frac{1}{2}(\psi+\xi)}. \tag{34.3.11}$$

The metric is

$$ds^2 = F^2(-d\psi^2 + d\xi^2) + r^2 d\Omega^2, \tag{34.3.12}$$

where

$$F = 2Me^{-r^*/M} \sec\frac{1}{2}(\psi+\xi)\sec\frac{1}{2}(\psi-\xi)\left|1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right|^{1/2}. \tag{34.3.13}$$

Note that

$$f = \frac{(r-r_+)(r-r_-)}{r^2} \tag{34.3.14}$$

The integration gives

$$r^* = \frac{r_+^2 \log|r/r_+ - 1| - r_-^2 \log|r/r_- - 1|}{r_+ - r_-} + r. \tag{34.3.15}$$

So we have $r^* \to \infty$ when $r \to \infty$; $r^* \to -\infty$ when $r \to r_+$; $r^* \to \infty$ when $r \to r_-$; $r^* \to 0$ when $r \to 0$.

Also note that

$$\tilde{u}\tilde{v} = -\mathrm{sgn}(f)e^{r^*/2M},\ \tilde{v}/\tilde{u} = -\mathrm{sgn}(f)e^{t/2M},\ \tilde{v} = e^{(r^*+t)/4M},\ \tilde{u} = -\mathrm{sgn}(f)e^{(r^*-t)/4M}. \tag{34.3.16}$$

The spacelike infinity is $\tilde{u}\tilde{v} \to -\infty$ while $\tilde{v}/\tilde{u}$ is finite. In $\psi - \xi$ coordinates, we have $\psi = 2n\pi$ while $\xi = \pm\pi$. We limit $-\pi \le \xi \le \pi$ in order to separate disconnect spacetime.

The future timelike infinity is $t \to \infty$ for $\xi > 0$ and $t \to -\infty$ for $\xi < 0$, while r is finite. So we have $\tilde{v}/\tilde{u} \to -\infty$ for $\xi > 0$ and $\tilde{v}/\tilde{u} \to 0$ for $\xi < 0$ while $\tilde{v}\tilde{u}$ is finite. In $\psi - \xi$ coordinates, we can get $\psi = (2n + 1/2)\pi$ and $\xi = \pm\pi/2$. Similarly, we have $\psi = (2n - 1/2)\pi$ and $\xi = \pm\pi/2$ for past time like infinity.

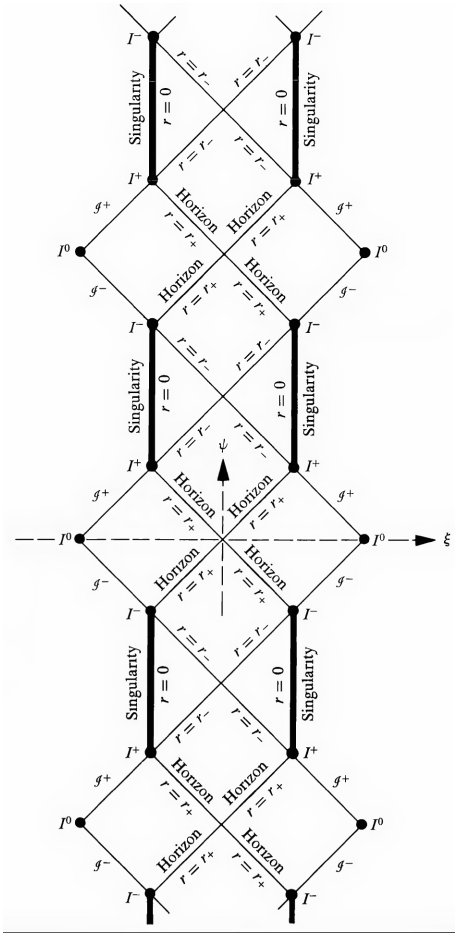
The future null infinity is $t + r \to \infty$ for $\xi > 0$ and $r - t \to \infty$ for $\xi < 0$. So we have $\tilde{v} \to \infty$ for $\xi > 0$ and $\tilde{u} \to \infty$ for $\xi < 0$. In $\psi - \xi$ coordinates, we can get $\psi + \xi = (2n+1)\pi$ for $\xi > 0$ and $\psi - \xi = (2n+1)\pi$ for $\xi < 0$ Similarly, we have $\psi - \xi = (2n+1)\pi$ for $\xi > 0$ and $\psi - \xi = (2n+1)\pi$ for $\xi < 0$ in the case of past null infinity. Note we must keep the line representing the future (past) null infinity between spacelike infinity and future (past) timelike infinite.

Horizon $r = r_-$ sits at $t + r^* \to \infty$ with $t - r^*$ finite or $r^* - t \to \infty$ with $t + r^*$ finite since horizon is null surface. So they are also lines $\psi \pm \xi = (2n+1)\pi$ in $\psi - \xi$ coordinates. But we must keep the line representing the inner horizon beyond spacelike infinity and timelike infinite.

Horizon $r = r_+$ sits at $r^* - t \to -\infty$ with $t + r^*$ infinite or $r^* + t \to \infty$ with $t - r^*$ finite, corresponding to $\tilde{v} = 0$ or $\tilde{u} = 0$. So they are lines $\psi \pm \xi = 2n\pi$.

Singularity $r = r^* = 0$ corresponds to $\tilde{u}\tilde{v} = -1$. It is easily to get $\xi = \pm\pi/2$. Note we must keep it with the inner horizon.

Put all the discussion together, we can get the following spacetime diagram copied from textbook.



(b) Consider the hypersurface $\psi =$ constant.

At $\psi = 0$, the worm whole expands to a state of maximum circumference $2\pi r_+$.

When $0 < \psi < \pi$, the worm hole recontracts. At $\psi = \pi/2$, the worm hole disconnects its outer regions from the two I^0s and reconnects them to a pair of $r = 0$ singularities.

At $\psi = \pi$, the worm hole recontracts to a state of minimum circumference $2\pi r_-$, and then bounces.

When $\pi < \psi < 2\pi$, the worm hole reexpands. At $\psi = \pi/2$, the worm hole disconnects its "outer regions" from the two singularities and reconnects them to a pair of I^0s in two new asymptotically flat universes.

At $\psi = 2\pi$, the worm hole slows its expansion to a halt and expands to a state of maximum circumference $2\pi r_+$ ...

34.4 A BLACK HOLE CAN NEVER BIFURCATE

Make plausible the theorem that no matter how hard one "zaps" a black hole, and no matter what one "zaps" it with, one can never make it bifurcate into two black holes.

Solution:

The solution is based on chapter 9.2 of the book *The large scale structure of spacetime (S.W.Hawking &*

G.F.R.Ellis)

Let $\mathscr{B}_1(\tau_1)$ be a black hole on $\mathscr{S}(\tau_1)$. Let $\mathscr{B}_2(\tau_2)$ and $\mathscr{B}_3(\tau_2)$ be black holes on a later surface $\mathscr{S}(\tau_2)$.

If $\mathscr{B}_1(\tau_1)$ bifurcates into $\mathscr{B}_2(\tau_2)$ and $\mathscr{B}_3(\tau_2)$, then $\mathscr{B}_2(\tau_2)$ and $\mathscr{B}_3(\tau_2)$ both intersect $J^+(\mathscr{B}_1(\tau_1))$, and does not intersect with each other.

Other the other hand, every future-directed inextendible timelike curve from $\mathscr{B}_1(\tau_1)$ will intersect $\mathscr{S}(\tau_2)$. Thus

$$J^+(\mathscr{B}_1(\tau_1)) \cap \mathscr{S}(\tau_2) \tag{34.4.1}$$

is connected and will be contained in a connected component of $\mathscr{B}(\tau_2)$, or the signal from $\mathscr{B}_1(\tau_1)$ can escape to future null infinite $\mathscr{I}^+$. If $\mathscr{B}_2(\tau_2)$ and $\mathscr{B}_3(\tau_2)$ both intersect $J^+(\mathscr{B}_1(\tau_1))$, then we have

$$J^+(\mathscr{B}_1(\tau_1)) \cap \mathscr{S}(\tau_2) \subset \mathscr{B}_2(\tau_2) \tag{34.4.2}$$

and

$$J^+(\mathscr{B}_1(\tau_1)) \cap \mathscr{S}(\tau_2) \subset \mathscr{B}_3(\tau_2), \tag{34.4.3}$$

which is contradictory with the fact that $\mathscr{B}_2(\tau_2)$ and $\mathscr{B}_3(\tau_2)$ does not intersect with each other.

So a black hole can never bifurcate.

Chapter 35

PROPAGATION OF GRAVITATIONAL WAVES

35.1 TRANSFORMATION OF PLANE WAVE TO TT GAUGE

Let a plane wave of the form (35.4) be given, in some arbitrary gauge of linearized theory. Exhibit explicitly the transformation that puts it into the TT gauge.

Solution:

For plane wave

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha), \tag{35.1.1}$$

where $k^\alpha k_\alpha = 0$, $A_{\mu\alpha}k^\alpha = 0$, the gauge transformation generated by

$$\xi^\mu = -iC^\mu \exp(ik_\alpha x^\alpha) \tag{35.1.2}$$

will be

$$A_{\mu\nu,\text{new}} = A_{\mu\nu,\text{old}} - C_\mu k_\nu - C_\nu k_\mu + \eta_{\mu\nu}k_\alpha C^\alpha. \tag{35.1.3}$$

Work in a Lorentz frame where the 4-velocity u^μ, of the TT gauge is $u^0 = 1$, $u^j = 0$. The the new $A_{\mu\nu}$ must satisfy $A_{\mu 0} = 0$ and $A_{kk} = 0$. We can obtain the equation

$$A_{00,\text{old}} - C_0 k_0 - C_i k_i = 0, \tag{35.1.4a}$$
$$A_{i0,\text{old}} - C_i k_0 - C_0 k_i = 0, \tag{35.1.4b}$$
$$A_{ii,\text{old}} - 3C_0 k_0 + C_i k_i = 0. \tag{35.1.4c}$$

From 35.1.4b, we have

$$k_i A_{i0,\text{old}} = k_i C_i k_0 + C_0 k_i^2 = k_i C_i k_0 + C_0 k_0^2 = k_0(k_i C_i + C_0 k_0) = k_0 A_{00,\text{old}} \tag{35.1.5}$$

using $k_0^2 = k_i^2$ and $A_{00}k_0 = A_{0i}k_i$. Noting that $k_0 \neq 0$, we conclude that 35.1.4a can be deduced from 35.1.4b. Equation 35.1.4b and 35.1.4c can be written as

$$\begin{pmatrix} k_1 & k_0 & & \\ k_2 & & k_0 & \\ k_3 & & & k_0 \\ 3k_0 & -k_1 & -k_2 & -k_3 \end{pmatrix} \cdot \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} A_{10,\text{old}} \\ A_{20,\text{old}} \\ A_{30,\text{old}} \\ A_{ii,\text{old}} \end{pmatrix} \tag{35.1.6}$$

The determine of the matrix is $-4k_0^4 \neq 0$. So we must have a solution C_μ, i.e. we can put plane wave into the TT gauge.

35.2 LIMITATION ON EXISTENCE OF TT GAUGE

Although the metric perturbation $h_{\mu\nu}$ for any gravitational wave in linearized theory can be put into the TT form (35.8), nonradiative $h_{\mu\nu}$'s cannot. Consider, for example, the external field of a rotating, spherical star, which cannot be written as a superposition of plane waves:

$$h_{00} = \frac{2M}{r}, \quad h_{jk} = \frac{2M}{r}\delta_{jk}, \quad h_{0k} = -2\epsilon_{klm}\frac{S^l x^m}{r^3}, \tag{35.2.1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, M is the star's mass and $\boldsymbol{S}$ is its angular momentum. Show that this cannot be put into TT gauge.

Solution:

Suppose the external field of a rotating can be put into TT gauge, then we have

$$h^{TT}_{jk,00} = -2R_{j0k0}. \tag{35.2.2}$$

R_{j0k0} is gauge invariant, so it can be calculated using original $h_{\mu\nu}$. To linear order, we have

$$R_{j0k0} = \Gamma_{j00,k} - \Gamma_{j0k,0} = -\frac{1}{2}h_{00,jk} = \frac{M(r^2\delta_{jk} - 3x_j x_k)}{r^5}. \tag{35.2.3}$$

Thus

$$h^{TT}_{jk,00} = \frac{2M(3x_k x_j - r^2\delta_{jk})}{r^5}. \tag{35.2.4}$$

Now we calculate R_{0xyz} in both the original gauge and the new gauge. In TT gauge, we have

$$R_{0xyz} = \Gamma_{0xz,y} - \Gamma_{0xy,z} = -\frac{1}{2}h^{TT}_{xz,0y} + \frac{1}{2}h^{TT}_{xy,0z}. \tag{35.2.5}$$

Clearly, R_{0xyz} calculated in this way must be a function of time While R_{0xyz} calculated using original $h_{\mu\nu}$ can not be a function of time. This is contradictory with the fact that $R_{\mu\nu\rho\sigma}$ is gauge invariant. So the external field of a rotating can not be put into TT gauge.

35.3 A CYLINDRICAL GRAVITATIONAL WAVE

To restore one's faith, which may have been shaken by exercise 35.2, one can consider the radiative solution whose only nonvanishing component $h_{\mu\nu}$ is

$$\bar{h}_{zz} = 4A\cos(\omega t)J_0(\omega\sqrt{x^2 + y^2}), \tag{35.3.1}$$

where J_0 is the Bessel function. This solution represents a superposition of ingoing and outgoing cylindrical gravitational waves. For this gravitational field calculate R_{j0k0} and from it infer h^{TT}_{jk}. Then calculate several other components of $R_{\alpha\beta\gamma\delta}$ (e.g., R_{xyxy}) in the original gauge and in TT gauge, and verify that the answers are the same.

Solution:

In the original gauge, we have

$$h_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} 2A\cos(\omega t)J_0(\omega\rho), \tag{35.3.2}$$

where $\rho = \sqrt{x^2 + y^2}$. So the curvature tensor

$$R_{j0k0} = \Gamma_{j00,k} - \Gamma_{j0k,0} = -\frac{1}{2}(h_{00,jk} + h_{jk,00}) = -\frac{1}{2}h^{TT}_{jk,00}. \tag{35.3.3}$$

We can infer that

$$h^{TT}_{jk} = -\omega^{-2}h_{00,jk} + h_{jk}. \tag{35.3.4}$$

Then we calculate several other components of $R_{\alpha\beta\gamma\delta}$ in the original gauge and in TT gauge. For example, in the original gauge, we have

$$R_{xyxy} = \Gamma_{xyy,x} - \Gamma_{xyx,y} = -\frac{1}{2}(h_{yy,xx} + h_{xx,yy}). \tag{35.3.5}$$

In TT gauge, we have

$$\begin{aligned} R_{xyxy} &= -\frac{1}{2}(h^{TT}_{yy,xx} + h^{TT}_{xx,yy} - 2h^{TT}_{xy,xy}) \\ &= -\frac{1}{2}(h_{yy,xx} + h_{xx,yy}) + \frac{1}{2}\omega^{-2}(h_{00,yyxx} + h_{00,xxyy} - 2h_{00,xyxy}) \\ &= -\frac{1}{2}(h_{yy,xx} + h_{xx,yy}). \end{aligned} \tag{35.3.6}$$

By similar procedure, we can verify that other components of $R_{\alpha\beta\gamma\delta}$ in the original gauge and in TT gauge are the same.

35.4 NON-TT PARTS OF METRIC PERTURBATION

From Box 35.1 establish the formula $h^T = \nabla^{-2}(h_{kk,ll} - h_{kl,kl})$; then verify the gauge invariance of h^T directly, by showing that $h_{kk,ll} - h_{kl,kl}$ is gauge-invariant. Use $\delta h_{ij} = \xi_{i,j} + \xi_{j,i}$. Show similarly that the quantities $\tilde{h}_{0k}$ defined by

$$\tilde{h}_{0k} = \bar{h}_{0k} - \nabla^{-2}(\bar{h}^{\mu}_{0,\mu k} + \bar{h}_{kl,l0}) \tag{35.4.1}$$

are gauge-invariant. Show from the gauge-invariant linearized field equations (18.5) that

$$\nabla^2 h^T = -16\pi T^{00}, \quad \nabla^2 \tilde{h}_{0k} = -16\pi T_{0k}, \tag{35.4.2}$$

so h^T and $\tilde{h}_{0k}$ must vamsh for waves in empty space.

Solution:

The projection operator to transverse direction is

$$P_{jk} = \delta_{jk} - \nabla^{-2}\partial_j\partial_k. \tag{35.4.3}$$

So we have

$$h^T \equiv P_{kl}h_{kl} = h_{kk} - \nabla^{-2}h_{kl,kl} = \nabla^{-2}(h_{kk,ll} - h_{kl,kl}). \tag{35.4.4}$$

Under gauge transformation, we have

$$\delta(h_{kk,ll} - h_{kl,kl}) = 2\xi_{k,kll} - \xi_{k,lkl} - \xi_{l,kkl} = 0. \tag{35.4.5}$$

So h^T is gauge invariant.

As for $\tilde{h}_{0k}$, we have

$$\begin{aligned}
\delta\tilde{h}_{0k} &= \delta h_{0k} - \nabla^{-2}\left[\delta h^{\mu}_{0,\mu k} - \frac{1}{2}\delta^{\mu}_{0}(-\delta h_{00,\mu k} + \delta h_{ii,\mu k}) + \delta h_{kl,l0} - \frac{1}{2}\delta_{kl}(-\delta h_{00,l0} + \delta h_{ii,l0})\right] \\
&= \delta h_{0k} - \nabla^{-2}\left[-\delta h_{00,0k} + \delta h_{i0,ik} - \frac{1}{2}(-\delta h_{00,0k} + \delta h_{ii,0k}) + \delta h_{kl,l0} - \frac{1}{2}(-\delta h_{00,k0} + \delta h_{ii,k0})\right] \\
&= \delta h_{0k} - \nabla^{-2}\left(\delta h_{i0,ik} - \delta h_{ii,0k} + \delta h_{kl,l0}\right) \\
&= \xi_{0,k} + \xi_{k,0} - \nabla^{-2}\left(\xi_{i,0ik} + \xi_{0,iik} - 2\xi_{i,i0k} + \xi_{k,ll0} + \xi_{l,kl0}\right) \\
&= \xi_{0,k} + \xi_{k,0} - \nabla^{-2}\left(\xi_{0,iik} + \xi_{k,ii0}\right) \\
&= 0.
\end{aligned} \tag{35.4.6}$$

So $\tilde{h}_{0k}$ is gauge invariant.

The gauge-invariant linearized field equations is

$$h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}) = 16\pi T_{\mu\nu}. \tag{35.4.7}$$

The 00 component of the equation is

$$\begin{aligned}
&2h_{0\alpha,0}{}^{\alpha} - h_{00,\alpha}{}^{\alpha} - h_{,00} + (h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}) \\
=&-2h_{00,00} + 2h_{0i,0i} + h_{00,00} - h_{00,ii} + h_{00,00} - h_{ii,00} + (h_{00,00} - 2h_{0i,0i} + h_{ij,ij}) \\
&+ (-h_{00} + h_{ii})_{,00} - (-h_{00} + h_{ii})_{,jj} \\
=&h_{ij,ij} - h_{ii,jj} = 16\pi T_{00}.
\end{aligned} \tag{35.4.8}$$

Note that

$$\nabla^2 h^T = h_{kk,ll} - h_{kl,kl}. \tag{35.4.9}$$

We have

$$\nabla^2 h^T = -16\pi T_{00}. \tag{35.4.10}$$

The linearized field equations can also be written as

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi T_{\mu\nu}. \tag{35.4.11}$$

The $0k$ component of the equation is

$$\begin{aligned}
&-\bar{h}_{0k,\alpha}{}^{\alpha} + \bar{h}_{0\alpha,}{}^{\alpha}{}_{k} + \bar{h}_{k\alpha,}{}^{\alpha}{}_{0} \\
=&\bar{h}_{0k,00} - \bar{h}_{0k,jj} + \bar{h}^{\alpha}_{0,\alpha k} - \bar{h}_{k0,00} + \bar{h}_{kj,j0} \\
=&-\bar{h}_{0k,jj} + \bar{h}^{\alpha}_{0,\alpha k} + \bar{h}_{kj,j0} = 16\pi T_{0k}.
\end{aligned} \tag{35.4.12}$$

Note that

$$\nabla^2\tilde{h}_{0k} = \bar{h}_{0k,jj} - (\bar{h}^{\mu}_{0,\mu k} + \bar{h}_{kl,l0}). \tag{35.4.13}$$

We have

$$\nabla^2\tilde{h}_{0k} = -16\pi T_{0k}. \tag{35.4.14}$$

35.5 ALTERNATIVE CALCULATION OF RELATIVE OSCILLATIONS

Introduce a TT coordinate system in which, at time $t = 0$, the two particles are both at rest. Use the geodesic equation to show that subsequently they both always remain at rest in the TT coordinates, despite the action of the wave. This means that the contravariant components of the separation vector are always constant in the TT coordinate frame:

$$n^j = x^j_B - x^j_A = \text{const}. \tag{35.5.1}$$

Call this constant $x^j_{B(0)}$. Transform these components to the comoving orthonormal frame; the answer should be equation (35.15).

Solution:

The geodesic equation particles is

$$\frac{d^2x^j}{d\tau^2} + \Gamma^j{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{35.5.2}$$

In TT gauge, $\Gamma^j{}_{\alpha\beta} = 0$ unless if $\alpha = 0$ or $\beta = 0$. However, $dx^\alpha/d\tau \neq 0$ unless $\alpha = 0$. If the particle is at rest at $t = 0$, we have $d^2x^j/d\tau^2 = 0$ at $t = 0$. And so it will remain at rest.

Assuming $x_A = 0$, the coordinate transformation to local Lorentz frame of A is

$$x^{\hat{i}} = x^i + \frac{1}{2}h^{TT}_{ij}x^j + O(h^2, x^2) \tag{35.5.3}$$

So we have

$$x^{\hat{i}}_B = x^i_B + \frac{1}{2}h^{TT}_{ij}x^j_B. \tag{35.5.4}$$

Supposing h^{TT}_{ij} vanish when $\tau = 0$, we have

$$x^{\hat{i}}_B(\tau) = x^{\hat{i}}_{B(0)} + \frac{1}{2}h^{TT}_{ij}x^{\hat{i}}_{B(0)}. \tag{35.5.5}$$

35.6 ROTATIONAL TRANSFORMATIONS FOR POLARIZATION STATES

Consider two Lorentz coordinate systems, one rotated by an angle θ about the z direction relative to the other:

$$t' = t,\ x' = x\cos\theta + y\sin\theta,\ y' = y\cos\theta - y'\sin\theta,\ z' = z. \tag{35.6.1}$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be quantum-mechanical states of a neutrino with spin-up and spin-down relative to the x direction; and similarly for $|\uparrow'\rangle$ and $|\downarrow'\rangle$. Let $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_{x'}$, $\boldsymbol{e}_{y'}$, be the unit polarization vectors in the two coordinate systems for an electromagnetic wave traveling in the z-direction; and similarly $\boldsymbol{e}_+$, $\boldsymbol{e}_\times$, $\boldsymbol{e}_{+'}$, $\boldsymbol{e}_{\times'}$ for a gravitational wave in linearized theory. Derive the following transformation laws:

$$\begin{gathered}
|\uparrow'\rangle = |\uparrow\rangle\cos\frac{1}{2}\theta + |\downarrow\rangle\sin\frac{1}{2}\theta; \quad |\downarrow'\rangle = -|\uparrow\rangle\sin\frac{1}{2}\theta + |\downarrow\rangle\cos\frac{1}{2}\theta; \\
\boldsymbol{e}_{x'} = \boldsymbol{e}_x\cos\theta + \boldsymbol{e}_y\sin\theta; \quad \boldsymbol{e}_{y'} = -\boldsymbol{e}_x\sin\theta + \boldsymbol{e}_y\cos\theta; \\
\boldsymbol{e}_{+'} = \boldsymbol{e}_+\cos 2\theta + \boldsymbol{e}_\times\sin 2\theta; \quad \boldsymbol{e}_{\times'} = -\boldsymbol{e}_+\sin 2\theta + \boldsymbol{e}_\times\cos 2\theta.
\end{gathered} \tag{35.6.2}$$

What is the generalization to the linear-polarization basis states for a radiation field of arbitrary spin S?

Solution:

Using $|\uparrow\rangle$ and $|\downarrow\rangle$ as base vector, the operator for spin in x and y direction is

$$S_x = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; \quad S_y = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{35.6.3}$$

So the spin operate in x' direction is

$$S_{x'} = S_x\cos\theta + S_y\sin\theta = \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta.\end{pmatrix} \tag{35.6.4}$$

The eigenvector is

$$\left(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\right) \text{ for } \lambda = \frac{1}{2}; \quad \left(-\sin\frac{1}{2}\theta, cos\frac{1}{2}\theta\right) \text{ for } \lambda = -\frac{1}{2}, \tag{35.6.5}$$

i.e.

$$|\uparrow'\rangle = |\uparrow\rangle \cos\frac{1}{2}\theta + |\downarrow\rangle \sin\frac{1}{2}\theta; \quad |\downarrow'\rangle = -|\uparrow\rangle \sin\frac{1}{2}\theta + |\downarrow\rangle \cos\frac{1}{2}\theta. \tag{35.6.6}$$

The unit polarization vectors for an electromagnetic wave is a normal vector in space. The transformation laws are identical to that of coordinates. So we have

$$\boldsymbol{e}_{x'} = \boldsymbol{e}_x \cos\theta + \boldsymbol{e}_y \sin\theta; \quad \boldsymbol{e}_{y'} = -\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_y \cos\theta. \tag{35.6.7}$$

Note that

$$h^{i'j'} = \frac{\partial x^{i'}}{\partial x^i} h^{ij} \frac{\partial x^{j'}}{\partial x^j}. \tag{35.6.8}$$

We have

$$\boldsymbol{e}_{+'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{35.6.9}$$

and

$$\boldsymbol{e}_{\times'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}, \tag{35.6.10}$$

i.e.

$$\boldsymbol{e}_{+'} = \boldsymbol{e}_+ \cos 2\theta + \boldsymbol{e}_\times \sin 2\theta; \quad \boldsymbol{e}_{\times'} = -\boldsymbol{e}_+ \sin 2\theta + \boldsymbol{e}_\times \cos 2\theta. \tag{35.6.11}$$

The generalization to the linear-polarization basis states for a radiation field of arbitrary spin S is

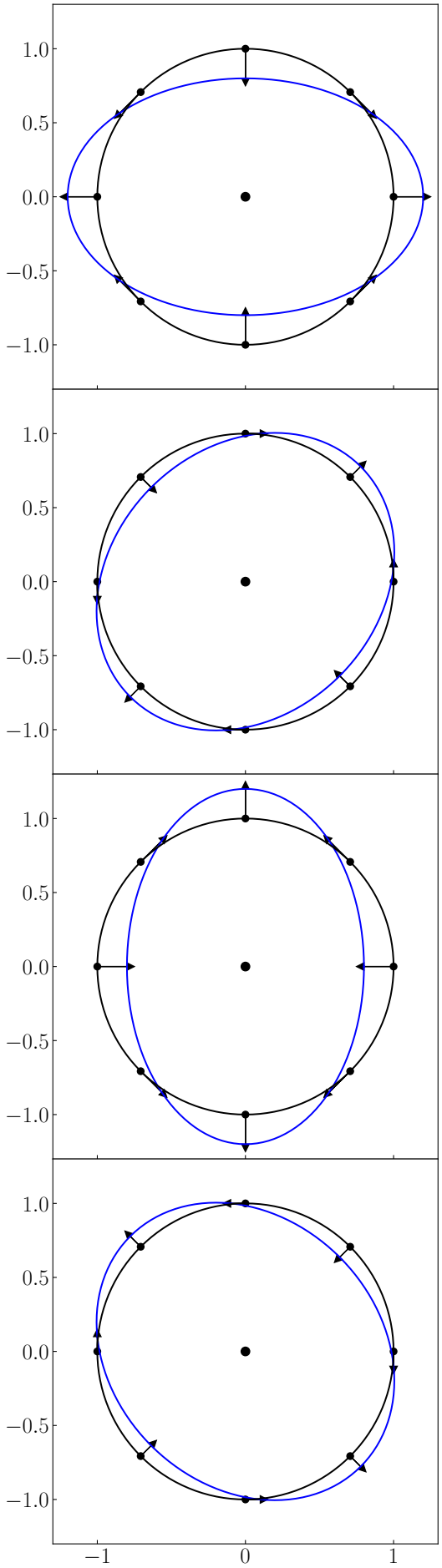
$$\boldsymbol{e}_a = \boldsymbol{e}_a \cos S\theta + \boldsymbol{e}_b \sin S\theta; \quad \boldsymbol{e}_b = -\boldsymbol{e}_a \sin S\theta + \boldsymbol{e}_b \cos S\theta. \tag{35.6.12}$$

35.7 ELLIPTICAL POLARIZATION

Discuss elliptically polarized gravitational waves in a manner analogous to the discussion of linearly and circularly polarized waves in Figure 35.2.

Solution:

For elliptical polarized waves $\boldsymbol{e} = (2\boldsymbol{e}_+ + i\boldsymbol{e}_\times)/\sqrt{5}$, deformation of a ring of test particles are shown in the following figure.


1.0
0.5
0.0
−0.5
−1.0
−1
0
1

35.8 GLOBALLY WELL-BEHAVED COORDINATES FOR PLANE WAVE

Find a coordinate transformation similar to (35.35), which puts the exact plane-wave solution (35.29a), (35.31), into the form

$$ds^2 = dX^2 + dY^2 - dUdV + (X^2 - Y^2)FdU^2, \quad F = F(U) \text{ completely arbitrary.} \tag{35.8.1}$$

This coordinate system has the advantage of no coordinate singularities anywhere; while the original coordinate system has the advantages of an easy transition to linearized theory, and easy interpretation of the action of the wave on test particles.

Solution:

The coordinate transformation can be given by

$$x = \frac{X}{Le^{\beta}}, \tag{35.8.2a}$$
$$y = \frac{y}{Le^{-\beta}}, \tag{35.8.2b}$$
$$u = U, \tag{35.8.2c}$$
$$v = V - \left(\frac{L'}{L} + \beta'\right) X^2 - \left(\frac{L'}{L} - \beta'\right) Y^2, \tag{35.8.2d}$$

where $L'' + (\beta')^2 L = 0$. Then we can get

$$dx = \frac{1}{Le^{\beta}} \left[dX - X\left(\frac{L'}{L} + \beta'\right) dU\right], \tag{35.8.3a}$$
$$dy = \frac{1}{Le^{-\beta}} \left[dY - Y\left(\frac{L'}{L} - \beta'\right) dU\right], \tag{35.8.3b}$$
$$du = dU, \tag{35.8.3c}$$
$$\begin{aligned} dv = {} & dV - 2X\left(\frac{L'}{L} + \beta'\right) dX - 2Y\left(\frac{L'}{L} - \beta'\right) dY \\ & - \left[X^2\left(\frac{L''}{L} - \frac{L'^2}{L^2} + \beta''\right) + Y^2\left(\frac{L''}{L} - \frac{L'^2}{L^2} - \beta''\right)\right] dU. \end{aligned} \tag{35.8.3d}$$

The metric of the spacetime is therefore

$$\begin{aligned} ds^2 &= L^2 e^{2\beta} dx^2 + L^2 e^{-2\beta} dy^2 - dudv \\ &= \left[dX - X\left(\frac{L'}{L} + \beta'\right) dU\right]^2 + \left[dY - Y\left(\frac{L'}{L} - \beta'\right) dU\right]^2 \\ &\quad - dUdV + 2X\left(\frac{L'}{L} + \beta'\right) dXdU + 2Y\left(\frac{L'}{L} - \beta'\right) dUdY \\ &\quad + \left[X^2\left(\frac{L''}{L} - \frac{L'^2}{L^2} + \beta''\right) + Y^2\left(\frac{L''}{L} - \frac{L'^2}{L^2} - \beta''\right)\right] dU^2 \\ &= dX^2 + dY^2 - dUdV + (X^2 - Y^2)\left(\frac{2L'\beta'}{L} + \beta''\right) dU^2. \end{aligned} \tag{35.8.4}$$

If we assume $F(U) = 2L'\beta'/L + \beta''$, then we have

$$ds^2 = dX^2 + dY^2 - dUdV + F(U)(X^2 - Y^2)\left(\frac{2L'\beta'}{L} + \beta''\right) dU^2. \tag{35.8.5}$$

35.9 GEODESIC COMPLETENESS FOR PLANE-WAVE MANIFOLD

Prove that the coordinate system (X, Y, U, V) of exercise 35.8 completely covers its spacetime manifold. More specifically, show that every geodesic can be extended in both directions for an arbitrarily large affine-parameter length without leaving the X, Y, U, V coordinate system. This property is called geodesic completeness.

[Hint: Choose an arbitrary event and an arbitrary tangent vector $d/d\lambda$ there. They determine an arbitrary geodesic. Perform a coordinate transformation that leaves the form of the metric unchanged and puts $d/d\lambda$ either in the $(\tilde{U}, \tilde{V})$ = constant 2-surface, or in the $(\tilde{X}, \tilde{Y})$ = constant 2-surface. Verify that the two coordinate systems cover the same region of spacetime. Then analyze completeness of geodesic in $(\tilde{X}, \tilde{Y}, \tilde{U}, \tilde{V})$ coordinates.]

I do not know how to find a coordinate transformation that leaves the form of the metric unchanged and puts an arbitrary vector either in the $(\tilde{U}, \tilde{V})$ = constant 2-surface, or in the $(\tilde{X}, \tilde{Y})$ = constant 2-surface. Any help is appreciated.

35.10 PLANE WAVE WITH TWO POLARIZATIONS PRESENT

The exact plane-wave solution (35.29) has polarization $\boldsymbol{e}_+$. Construct a similar solution, containing two arbitrary amplitudes, $\beta(u)$ and $\gamma(u)$, for polarizations $\boldsymbol{e}_+$ and $\boldsymbol{e}_\times$. Extend the discussions of section 35.9-35.12 to this solution.

Solution:

Suppose the metric is

$$ds^2 = L^2(e^{2\beta}dx^2 + 4\gamma dxdy + e^{-2\beta}dy^2) - dudv. \tag{35.10.1}$$

The only component of the Ricci tensor that does not vanish identically is

$$R_{uu} = -\frac{2\left[(1-4\gamma^2)L\beta'^2 - (1+4\gamma^2)L\gamma'^2 - 2\gamma(1-4\gamma^2)L\gamma'' - 4\gamma(1-4\gamma^2)\gamma' L' + (1-4\gamma^2)^2 L''\right]}{(1-4\gamma^2)^2 L}. \tag{35.10.2}$$

The field equation is

$$(1-4\gamma^2)^2 L'' - 4(1-4\gamma^2)\gamma\gamma' L' + \left[(1-4\gamma^2)\beta'^2 - (1+4\gamma^2)\gamma'^2 - 2(1-4\gamma^2)\gamma\gamma''\right] L = 0. \tag{35.10.3}$$

The linearized version of this equation is $L'' = 0$. Therefore the solution corresponding to linearized theory is

$$L = 1, \quad \beta(u) \text{ and } \gamma(u) \text{ arbitrary but small.} \tag{35.10.4}$$

The corresponding metric is

$$ds^2 = (1+2\beta)dx^2 + (1-2\beta)dy^2 + 4\gamma dxdy + dz^2 - dt^2, \quad \beta = \beta(t-z),\ \gamma = \gamma(t-z). \tag{35.10.5}$$

Notice that this is a plane wave containing two arbitrary amplitudes, $\beta(u)$ and $\gamma(u)$, for polarizations $\boldsymbol{e}_+$ and $\boldsymbol{e}_\times$.

Return attention to the exact plane wave, and focus on the case where the "wave factor" $\beta(u)$ and $\gamma(u)$ is a pulse of duration $2T$, and $|\beta'|, |\gamma'| \ll 1/T$ throughout the pulse. Then the exact solution is: (1) for $u < T$ (flat spacetime; pulse has not yet arrived),

$$\beta = \gamma = 0,\ L = 1; \tag{35.10.6}$$

(2) for $-T < u < T$ (interior of pulse),

$$\beta = \beta(u) \text{ is arbitrary, except that } |\beta'| \ll 1/T,$$
$$\gamma = \gamma(u) \text{ is arbitrary, except that } |\gamma'| \ll 1/T,$$
$$L(u) = 1 - \int_{-T}^{u} \left\{ \int_{-T}^{\bar{u}} [\beta'^2(\bar{\bar{u}}) - \gamma'^2(\bar{\bar{u}}) - 2\gamma(\bar{\bar{u}})\gamma''(\bar{\bar{u}})] d\bar{\bar{u}} \right\} d\bar{u} + O(h^4), \tag{35.10.7}$$

where $h \sim \beta$, γ, $\beta' T$, $\gamma' T$ and $\gamma'' T^2$. (3) for $u > T$ (after the pulse has passed),

$$\beta = 0,\ \gamma = 0\ L = 1 - \frac{u}{a}, \quad a \equiv \frac{1 + O(h^2)}{\int_T^T (\beta'^2 - \gamma'^2 - 2\gamma\gamma'') du}. \tag{35.10.8}$$

Perform the coordinate transformation as equation (35.35), we can show that spacetime is completely flat in regions where the "wave factor" vanishes - which is everywhere outside the pulse.

Consider a family of particles that are all at rest in the original t, x, y, z coordinate system before the pulse arrives. Then even while the pulse is passing, and after it has gone, the particles remain at rest in the coordinate system from the fact that $\Gamma^{\mu}{}_{00} = 0$.

Two particles whose separation is in the direction of propagation of the pulse (z-direction) have constant proper separation since $g_{zz} = 1$. Plane wave is completely transverse, like a plane wave of linearized theory.

Neighboring particles transverse to the propagation direction have a proper separation that wiggles as the pulse passes

$$\Delta s^2 \approx L[(1 + 2\beta)(\Delta x)^2 + (1 - 2\beta)(\Delta y)^2 + 4\gamma \Delta x \Delta y]. \tag{35.10.9}$$

Superimposed on the usual linearized-theory type of wiggling is a very small net acceleration of the particles toward each other due to L. This is an acceleration of almost Newtonian type, caused by the gravitational attraction of the energy that the gravitational wave carries between the two particles. The total effect of all the energy that passes is to convert the particles from an initial state of relative rest, to a final state of relative motion with speed

$$v_{\text{final}} = \frac{d\Delta s}{dt} = d\frac{L\Delta s_i}{dt} = -\frac{\Delta s_i}{a}, \tag{35.10.10}$$

where Δs_i is the initial particle separation.

Think of the exact gravitational plane-wave solution as ripples in the spacetime curvature, described by $\beta(u)$ and $\gamma(u)$, propagating on a very slightly curved background spacetime, characterized by $L(u)$. By similar discussion in section 12, we can show that the wavelength of gravitational wave is much smaller than the radius of curvature of background. This difference in scales enables one to separate out the background from the ripples.

The metric for the background of the gravitational plane wave is

$$ds^2 = g^{(\mathrm{B})}_{\mu\nu} dx^\mu dx^\nu = L^2(dx^2 + dy^2) - dudv. \tag{35.10.11}$$

The metric for the full spacetime is

$$ds^2 = (g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu, \quad h_{xx} = -h_{yy} = 2\beta,\ h_{xy} = h_{yx} = 2\gamma,\ \text{ all other } h_{\mu\nu} = 0. \tag{35.10.12}$$

One can think of the ripples as a transverse, traceless, $G_{uu} = -2L''/L$symmetric tensor field $h_{\mu\nu}$ propagating in the background geometry.

Suppose the background curvature is produced the gravitational wave ripples $h_{\mu\nu}$. We have

$$G^{(\mathrm{B})}_{uu} = -\frac{2L''}{L} = 8\pi T^{(\mathrm{EFF})}_{uu}. \tag{35.10.13}$$

Using equation 35.10.3, to the second order of h, we have

$$T_{uu}^{(\mathrm{EFF})} = \frac{\beta'^2 - \gamma'^2 - 2\gamma\gamma''}{4\pi}. \tag{35.10.14}$$

In averaging sense, we have

$$0 = \langle(\gamma^2)''\rangle = 2\langle\gamma'^2\rangle + 2\langle\gamma\gamma''\rangle. \tag{35.10.15}$$

So

$$\langle T_{uu}^{(\mathrm{EFF})}\rangle = \left\langle \frac{\beta'^2 + \gamma'^2}{4\pi} \right\rangle = \left\langle \frac{1}{32\pi} h_{jk,u} h_{jk,u} \right\rangle. \tag{35.10.16}$$

35.11 CONNECTION COEFFICIENTS AND CURVATURE TENSORS FOR A PERTURBED METRIC

In a specific coordinate frame of an arbitrary spacetime, write the metric coefficients in covariant representation in the form

$$g_{\mu\nu} = g_{\mu\nu}^{(\mathrm{B})} + h_{\mu\nu}. \tag{35.11.1}$$

(At the end of the calculation, one can split $h_{\mu\nu}$ into two parts, $h_{\mu\nu} \to h_{\mu\nu} + j_{\mu\nu}$ and out of this split obtain the formulas used in the text.) Assume that the typical components of $h_{\mu\nu}$ are much less than those of $g_{\mu\nu}^{(\mathrm{B})}$; so one can expand Christoffel symhols and curvature tensors in $h_{\mu\nu}$. Raise and lower indices of $h_{\mu\nu}$ with $g_{\mu\nu}^{(\mathrm{B})}$; and denote by a "|" covariant derivatives relative to $g_{\mu\nu}^{(\mathrm{B})}$ and by a ";" covariant derivatives relative to $g_{\mu\nu}$.

(a) Here $g_{\mu\nu}$ and $g_{\mu\nu}^{(\mathrm{B})}$ can be thought of as two different metrics coexisting in the spacetime manifold. Show that the difference between the corresponding covariant derivatives, $\boldsymbol{\nabla} - \boldsymbol{\nabla}^{(\mathrm{B})}$ -indeed, the difference between any two covariant derivatives!- is a tensor with components

$$S^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma}. \tag{35.11.2}$$

(b) Show that

$$g^{\mu\nu} = g_{(\mathrm{B})}^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}^{\nu} - h^{\mu\alpha}h_{\alpha}^{\beta}h_{\beta}^{\nu} + \cdots \tag{35.11.3}$$

and also that

$$g^{\mu\nu} = g_{(\mathrm{B})}^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_{\alpha}^{\nu} - h^{\mu\alpha}h_{\alpha}^{\beta}h_{\beta\gamma}g^{\gamma\nu}. \tag{35.11.4}$$

(c) By calculating in a local Lorentz frame of $g_{\mu\nu}^{(\mathrm{B})}$ and then transforming back to the original frame, show that

$$S^{\mu}{}_{\beta\gamma} = \frac{1}{2}g^{\mu\alpha}(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}), \tag{35.11.5a}$$

$$R^{\alpha}{}_{\beta\gamma\delta} - R^{(\mathrm{B})\alpha}{}_{\beta\gamma\delta} = S^{\alpha}{}_{\beta\delta|\gamma} - S^{\alpha}{}_{\beta\gamma|\delta} + S^{\alpha}{}_{\mu\gamma}S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\gamma}, \tag{35.11.5b}$$

$$R_{\beta\delta} - R^{(\mathrm{B})}{}_{\beta\delta} = S^{\alpha}{}_{\beta\delta|\alpha} - S^{\alpha}{}_{\beta\alpha|\delta} + S^{\alpha}{}_{\mu\alpha}S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\alpha}. \tag{35.11.5c}$$

(d) Show that expression 35.11.5c reduces to

$$R_{\beta\delta} = R^{(\mathrm{B})}{}_{\beta\delta} + R^{(1)}{}_{\beta\delta} + R^{(2)}{}_{\beta\delta} + \cdots \tag{35.11.6}$$

where $R^{(1)}$ and $R^{(2)}$ are defined by equations (35.58).

Solution:

(a) For any vector field $\boldsymbol{u} = u^{\alpha}\boldsymbol{e}_{\alpha}$, we have

$$(\boldsymbol{\nabla} - \boldsymbol{\nabla}^{(\mathrm{B})})\boldsymbol{u} = (u^{\alpha}{}_{;\beta} - u^{\alpha}{}_{|\beta})\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} = (\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma})u^{\gamma}\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta}. \tag{35.11.7}$$

Suppose that

$$\boldsymbol{S} \equiv S^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma = (\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma})u^\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{\omega}^\beta \otimes \boldsymbol{\omega}^\gamma. \tag{35.11.8}$$

We have

$$(\boldsymbol{\nabla} - \boldsymbol{\nabla}^{(\mathrm{B})})\boldsymbol{u} = \boldsymbol{S} \cdot \boldsymbol{u}. \tag{35.11.9}$$

Since $(\boldsymbol{\nabla} - \boldsymbol{\nabla}^{(\mathrm{B})})\boldsymbol{u}$ is a tensor, $\boldsymbol{S}\cdot\boldsymbol{u}$ is a tensor for any vector field $\boldsymbol{u}$. And so S is a tensor with components

$$S^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma}. \tag{35.11.10}$$

(b) Introduce a formal parameter ϵ in

$$g_{\mu\nu} = g^{(\mathrm{B})}_{\mu\nu} + \epsilon h_{\mu\nu}. \tag{35.11.11}$$

And we have the expansion

$$g^{\mu\nu} = g^{\mu\nu}_{(\mathrm{B})} + \epsilon x^{(1)\mu\nu} + \cdots + \epsilon^n x^{(\mathrm{n})\mu\nu} + \cdots. \tag{35.11.12}$$

Using $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$, we can obtain

$$\delta^\nu_\mu = \delta^\nu_\mu + \epsilon(h_{\mu\alpha}g^{\alpha\nu}_{(\mathrm{B})} + g^{(\mathrm{B})}_{\mu\alpha}x^{(1)\alpha\nu}) + \cdots + \epsilon^n(h_{\mu\alpha}x^{(\mathrm{n}-1)\alpha\nu} + g^{(\mathrm{B})}_{\mu\alpha}x^{(n)\alpha\nu}) + \cdots \tag{35.11.13}$$

Thus coefficients for all orders of ϵ must vanish, i.e.

$$x^{(n)\mu\nu} = -h^\mu_\alpha x^{(n-1)\alpha\nu}, \quad x^{(0)\mu\nu} = g^{\mu\nu}_{(\mathrm{B})}. \tag{35.11.14}$$

Explicitly, we have

$$x^{(1)\mu\nu} = -h^{\mu\nu}; \quad x^{(2)\mu\nu} = h^\mu_\alpha h^{\alpha\nu}; \quad x^{(3)\mu\nu} = -h^\mu_\alpha h^\alpha_\beta h^{\beta\nu}\cdots \tag{35.11.15}$$

i.e.

$$g^{\mu\nu} = g^{\mu\nu}_{(\mathrm{B})} - h^{\mu\nu} + h^{\mu\alpha}h^\nu_\alpha - h^{\mu\alpha}h^\beta_\alpha h^\nu_\beta + \cdots \tag{35.11.16}$$

If we note that

$$x^{(n)\mu\nu} = -h^\mu_\alpha x^{(n-1)\alpha\nu} = h^\mu_\alpha h^\alpha_\beta x^{(n-2)\beta\nu} = -h^\mu_\alpha h^\alpha_\beta h^\beta_\gamma x^{(n-3)\gamma\nu}, \tag{35.11.17}$$

we have

$$\begin{aligned} g^{\mu\nu} &= g^{\mu\nu}_{(\mathrm{B})} + x^{(1)\mu\nu} + x^{(2)\mu\nu} - h^\mu_\alpha h^\alpha_\beta h^\beta_\gamma (x^{(0)\gamma\nu} + x^{(1)\gamma\nu} + \cdots) \\ &= g^{\mu\nu}_{(\mathrm{B})} - h^{\mu\nu} + h^{\mu\alpha}h^\nu_\alpha - h^{\mu\alpha}h^\beta_\alpha h_{\beta\gamma}g^{\gamma\nu}. \end{aligned} \tag{35.11.18}$$

(c) In local Lorentz frame of $g^{(\mathrm{B})}_{\alpha\beta}$, at the origin we have

$$\Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma} = 0, \quad \Gamma^{\alpha}{}_{\beta\gamma} = S^{\alpha}{}_{\beta\gamma}, \quad g^{(\mathrm{B})}_{\alpha\beta,\gamma} = 0, \quad g_{\alpha\beta,\gamma} = h_{\alpha\beta,\gamma} = h_{\alpha\beta|\gamma}. \tag{35.11.19}$$

So we can get

$$S^{\mu}{}_{\beta\gamma} = \Gamma^{\mu}{}_{\beta\gamma} = \frac{1}{2}g^{\mu\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) = \frac{1}{2}g^{\mu\alpha}(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}), \tag{35.11.20a}$$

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} - R^{(\mathrm{B})\alpha}{}_{\beta\gamma\delta} &= \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} + \Gamma^{(\mathrm{B})\alpha}{}_{\beta\delta,\gamma} - \Gamma^{(\mathrm{B})\alpha}{}_{\beta\gamma,\delta} \\ &= S^{\alpha}{}_{\beta\delta,\gamma} - S^{\alpha}{}_{\beta\gamma,\delta} + S^{\alpha}{}_{\mu\gamma}S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\gamma} \\ &= S^{\alpha}{}_{\beta\delta|\gamma} - S^{\alpha}{}_{\beta\gamma|\delta} + S^{\alpha}{}_{\mu\gamma}S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\gamma}, \end{aligned} \tag{35.11.20b}$$

$$R_{\beta\delta} - R^{(\mathrm{B})}{}_{\beta\delta} = S^{\alpha}{}_{\beta\delta|\alpha} - S^{\alpha}{}_{\beta\alpha|\delta} + S^{\alpha}{}_{\mu\alpha}S^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\alpha}. \tag{35.11.20c}$$

Because the leftmost and rightmost hand of equations 35.11.20 are tensors. So they will hold in any other frames.

(d) Using equations 35.11.20a and 35.11.16, we have

$$S^{\mu}{}_{\beta\gamma} = \frac{1}{2}(g^{\mu\alpha}_{(\mathrm{B})} - h^{\mu\alpha} + \cdots)(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}) \tag{35.11.21}$$

Thus, we can get

$$\begin{aligned}
S^{\alpha}{}_{\beta\delta|\alpha} &= \frac{1}{2}\left[(g^{\alpha\rho}_{(\mathrm{B})} - h^{\alpha\rho} + \cdots)(h_{\rho\beta|\delta} + h_{\rho\delta|\beta} - h_{\beta\delta|\rho})\right]_{|\alpha} \\
&= \frac{1}{2}(h^{\alpha}{}_{\beta|\delta\alpha} + h^{\alpha}{}_{\delta|\beta\alpha} - h_{\beta\delta|\alpha}{}^{\alpha}) \\
&\quad - \frac{1}{2}h^{\alpha\rho}{}_{|\alpha}(h_{\rho\beta|\delta} + h_{\rho\delta|\beta} - h_{\beta\delta|\rho}) - \frac{1}{2}h^{\alpha\rho}(h_{\rho\beta|\delta\alpha} + h_{\rho\delta|\beta\alpha} - h_{\beta\delta|\rho\alpha}) + \cdots \qquad (35.11.22a) \\
S^{\alpha}{}_{\beta\alpha|\delta} &= \frac{1}{2}\left[(g^{\alpha\rho}_{(\mathrm{B})} - h^{\alpha\rho} + \cdots)(h_{\rho\beta|\alpha} + h_{\rho\alpha|\beta} - h_{\beta\alpha|\rho})\right]_{|\delta} \\
&= \frac{1}{2}\left[(g^{\alpha\rho}_{(\mathrm{B})} - h^{\alpha\rho} + \cdots)h_{\rho\alpha|\beta}\right]_{|\delta} \\
&= \frac{1}{2}h_{|\beta\delta} - \frac{1}{2}h^{\alpha\rho}{}_{|\delta}h_{\rho\alpha|\beta} - \frac{1}{2}h^{\alpha\rho}h_{\alpha\rho|\beta\delta} + \cdots \qquad (35.11.22b) \\
S^{\alpha}{}_{\mu\alpha}S^{\mu}{}_{\beta\delta} &= \frac{1}{4}h^{\alpha}{}_{\alpha|\mu}(h^{\mu}{}_{\beta|\delta} + h^{\mu}{}_{\delta|\beta} - h_{\beta\delta}{}^{|\mu}) + \cdots \qquad (35.11.22c) \\
S^{\alpha}{}_{\mu\delta}S^{\mu}{}_{\beta\alpha} &= \frac{1}{4}(h^{\alpha}{}_{\mu|\delta} + h^{\alpha}{}_{\mu|\delta} - h_{\mu\delta}{}^{|\alpha})(h^{\mu}{}_{\beta|\alpha} + h^{\mu}{}_{\alpha|\beta} - h_{\beta\alpha}{}^{|\mu}) + \cdots \qquad (35.11.22d)
\end{aligned}$$

It is easy to obtain that

$$R^{(1)}{}_{\beta\delta} = \frac{1}{2}(h^{\alpha}{}_{\beta|\delta\alpha} + h^{\alpha}{}_{\delta|\beta\alpha} - h_{\beta\delta|\alpha}{}^{\alpha} - h_{|\beta\delta}) \tag{35.11.23}$$

and

$$\begin{aligned}
R^{(2)}{}_{\beta\delta} &= -\frac{1}{2}h^{\alpha\rho}{}_{|\alpha}(h_{\rho\beta|\delta} + h_{\rho\delta|\beta} - h_{\beta\delta|\rho}) - \frac{1}{2}h^{\alpha\rho}(h_{\rho\beta|\delta\alpha} + h_{\rho\delta|\beta\alpha} - h_{\beta\delta|\rho\alpha}) + \frac{1}{2}h^{\alpha\rho}{}_{|\delta}h_{\rho\alpha|\beta} + \frac{1}{2}h^{\alpha\rho}h_{\alpha\rho|\beta\delta} \\
&\quad + \frac{1}{4}h_{|\mu}(h^{\mu}{}_{\beta|\delta} + h^{\mu}{}_{\delta|\beta} - h_{\beta\delta}{}^{|\mu}) - \frac{1}{4}(h^{\alpha}{}_{\mu|\delta} + h^{\alpha}{}_{\delta|\mu} - h_{\mu\delta}{}^{|\alpha})(h^{\mu}{}_{\beta|\alpha} + h^{\mu}{}_{\alpha|\beta} - h_{\beta\alpha}{}^{|\mu}). \qquad (35.11.24)
\end{aligned}$$

If we note that

$$(h^{\alpha}{}_{\mu|\delta} + h^{\alpha}{}_{\delta|\mu} - h_{\mu\delta}{}^{|\alpha})(h^{\mu}{}_{\beta|\alpha} + h^{\mu}{}_{\alpha|\beta} - h_{\beta\alpha}{}^{|\mu}) = h^{\alpha\mu}{}_{|\beta}h_{\alpha\mu|\delta} + 2h_{\beta\mu|\alpha}h_{\delta}{}^{\alpha|\mu} - 2h_{\beta\alpha|\mu}h_{\delta}{}^{\alpha|\mu}. \tag{35.11.25}$$

We can simplify equation 35.11.24 into

$$\begin{aligned}
R^{(2)}{}_{\beta\delta} &= \frac{1}{2}\left[\frac{1}{2}h_{\alpha\rho|\beta}h^{\alpha\rho}{}_{|\delta} + h^{\alpha\rho}(h_{\beta\delta|\alpha\rho} + h_{\alpha\rho|\beta\delta} - h_{\rho\beta|\delta\alpha} - h_{\rho\delta|\beta\alpha})\right. \\
&\quad \left. + h_{\delta}{}^{\alpha|\mu}(h_{\beta\alpha|\mu} - h_{\beta\mu|\alpha}) - \left(h^{\rho\alpha}{}_{|\alpha} - \frac{1}{2}h^{|\rho}\right)(h_{\rho\beta|\delta} + h_{\rho\delta|\beta} - h_{\beta\delta|\rho})\right]
\end{aligned} \tag{35.11.26}$$

35.12 GAUGE TRANSFORMATIONS IN A CURVED BACKGROUND

(a) Show that the infinitesimal coordinate transformatIon (35.65a) induces the change (35.65b) in the functional form of the metric perturbation.

(b) Discuss the relationship between this gauge transformation and the concept of a Killing vector.

Solution:

(a) The transformation law gives

$$\begin{aligned}
\bar{g}_{\mu\nu}(\bar{x}) &= \left(\delta^\alpha_\mu - \frac{\partial \xi^\alpha}{\partial \bar{x}^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial x^\beta}{\partial \bar{x}^\nu}\right) g_{\alpha\beta}(\bar{x}-\xi) \\
&\approx g_{\mu\nu}(\bar{x}-\xi) - \frac{\partial \xi^\alpha}{\partial x^\mu} g_{\alpha\nu}(\bar{x}) - \frac{\partial \xi^\beta}{\partial x^\nu} g_{\mu\beta}(\bar{x}) \\
&\approx g^{(\mathrm{B})}_{\mu\nu}(\bar{x}) - g^{(\mathrm{B})}_{\mu\nu,\rho}\xi^\rho(\bar{x}) + h_{\mu\nu}(\bar{x}) - \frac{\partial \xi^\alpha}{\partial x^\mu} g^{(\mathrm{B})}_{\alpha\nu}(\bar{x}) - \frac{\partial \xi^\alpha}{\partial x^\nu} g^{(\mathrm{B})}_{\mu\alpha}(\bar{x}) \\
&= g^{(\mathrm{B})}_{\mu\nu}(\bar{x}) - (g^{(\mathrm{B})}_{\mu\alpha}\Gamma^{\alpha(\mathrm{B})}_{\nu\rho} + g^{(\mathrm{B})}_{\alpha\nu}\Gamma^{\alpha(\mathrm{B})}_{\mu\rho})\xi^\rho(\bar{x}) + h_{\mu\nu}(\bar{x}) - \frac{\partial \xi^\alpha}{\partial x^\mu} g^{(\mathrm{B})}_{\alpha\nu}(\bar{x}) - \frac{\partial \xi^\alpha}{\partial x^\nu} g^{(\mathrm{B})}_{\mu\alpha}(\bar{x}) \\
&= g^{(\mathrm{B})}_{\mu\nu}(\bar{x}) + h_{\mu\nu}(\bar{x}) - \xi_{\mu|\nu}(\bar{x}) - \xi_{\nu|\mu}(\bar{x}). \qquad (35.12.1)
\end{aligned}$$

So we have

$$h^{\mathrm{new}}_{\mu\nu} = h_{\mu\nu} - \xi_{\mu|\nu} - \xi_{\nu|\mu}. \qquad (35.12.2)$$

(b) If ξ^μ is a Killing field, then the infinitesimal transformation $x^{\mathrm{new}\mu} = x^\mu + \epsilon\xi^\mu$ will leave $h_{\mu\nu}$ invariant.

35.13 TRANSVERSE-TRACELESS GAUGE FOR GRAVITATIONAL WAVES PROPAGATING IN A CURVED BACKGROUND

(a) Show that, in vacuum in a curved background spacetime, the gauge condition $\bar{h}_\mu{}^\alpha{}_{|\alpha} = 0$ is preserved by transformations whose generator satisfies the wave equation $\xi_\mu{}^{|\alpha}{}_\alpha = 0$.

(b) Locally (over distances much smaller than $\mathcal{R}$) linearized theory is applicable; so there exists such a transformation which makes

$$\bar{h} = 0 + \text{ error}, \quad \bar{h}_{\mu\alpha}u^\alpha = 0 + \text{ error}. \qquad (35.13.1)$$

Here u^α is a vector field that is as nearly covariantly constant as possible ($u^\alpha{}_{|\beta} = 0$); i.e., it is a constant vector in the inertial coordinates of linearized theory; and the errors are small over distances much less than $\mathcal{R}$. Show that $\bar{h} = 0$ can be imposed globally along with $\bar{h}_\mu{}^\alpha{}_{|\alpha} = 0$; i.e., show that, if it is imposed on an initial hypersurface, the propagation equation (35.68) preserves it.

(c) Show that in general, the background curvature prevents any vector field from being covariantly constant ($u^{\hat{\alpha}}{}_{|\hat{\beta}} \sim u^{\hat{\alpha}}/\mathcal{R}$ at best); and from this show that $\bar{h}_{\mu\alpha}u^\alpha = 0$ cannot be imposed globally along with $\bar{h}_\mu{}^\alpha{}_{|\alpha} = 0$.

Solution:

(a) After a gauge transformation, we have

$$\begin{aligned}
\bar{h}^{\mathrm{new}}{}_{\mu\alpha}{}^{|\alpha} &= h^{\mathrm{old}}{}_{\mu\alpha}{}^{|\alpha} - \xi_{\mu|\alpha}{}^\alpha - \xi_{\alpha|\mu}{}^\alpha - \frac{1}{2}g_{\mu\alpha}(\bar{h}^{\mathrm{old}} - 2\xi^\beta{}_{|\beta})^{|\alpha} \\
&= h^{\mathrm{old}}{}_{\mu\alpha}{}^{|\alpha} - \xi_{\mu|\alpha}{}^\alpha - \xi_{\alpha|\mu}{}^\alpha + \xi^\alpha{}_{|\alpha\mu} \\
&= h^{\mathrm{old}}{}_{\mu\alpha}{}^{|\alpha} - \xi_{\mu|\alpha}{}^\alpha - R^{(\mathrm{B})}_{\mu\nu}\xi^\nu. \qquad (35.13.2)
\end{aligned}$$

In vacuum, the only source of a nonvanishing Ricci tensor is the stress-energy carried by the gravitational waves thenlselves; hence $R^{(\mathrm{B})}_{\mu\nu} \sim \mathcal{A}^2/\lambda^2$ and can be neglected. So the gauge condition is preserved by transformations whose generator satisfies the wave equation $\xi_\mu{}^{|\alpha}{}_\alpha = 0$.

(b) Take the trace of the propagation equation (35.64) and impose the gauge condition $\bar{h}_\mu{}^\alpha{}_{|\alpha} = 0$, we have

$$\bar{h}_{|\alpha}{}^\alpha = 0. \qquad (35.13.3)$$

Choose the coordinates in which the hypersurface with $\bar{h}=0$ is $t=t_0$. Now, let us solve the wave equation

$$\xi_{\mu}{}^{|\nu}{}_{\nu}+R^{(\mathrm{B})}_{\mu\nu}\xi^{\nu}=0 \tag{35.13.4}$$

with initial conditions

$$\xi^{\mu}{}_{|\mu}=0, \quad \xi^{\mu}{}_{|\mu t}=0 \tag{35.13.5}$$

on hypersurface $t=t_0$. Equation 35.13.5 can be rewritten as

$$\xi^{t}{}_{|t}=-\xi^{i}{}_{|i}, \quad \xi^{i}{}_{|ti}=\frac{1}{2}\bar{h}_{,t}+R^{(\mathrm{B})}_{t\alpha}\xi^{\alpha}-\xi^{t}{}_{|tt}. \tag{35.13.6}$$

Note that $\xi^{t}{}_{|tt}$ will be fixed by ξ^{μ} and $\xi^{\mu}{}_{|t}$ through 35.13.4. In principle, we can find appropriate initial value of ξ^{μ} and $\xi^{\mu}{}_{|t}$ to make 35.13.5 holds. And then will can solve equation 35.13.4 throughout the whole spacetime. Using the gauge transformation generated by ξ^{μ}, we can keep $\bar{h}=0$ on hypersurface and $\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha}=0$ in whole spacetime, as well as make $\bar{h}_{,t}=0$ on this surface. Now the propagation equation 35.13.3 will ensure that $\bar{h}=0$ in whole spacetime.

(c) If there exist a vector field u^{α} whose covariant derivatives vanish everywhere, then we have $u^{\alpha}{}_{|\beta\gamma}=u^{\alpha}{}_{|\gamma\beta}$, and so we can derive that

$$R^{\alpha(\mathrm{B})}{}_{\rho\beta\gamma}u^{\rho}=0. \tag{35.13.7}$$

But this is impossible for a generic background curvature without special geometry.

35.14 BRILL-HARTLE AVERAGE

Isaacson (1968b) introduces the following averaging scheme, which he names “Brill-Hartle averaging”.

(a) In the small region, of size several times it, where the averaging occurs, there will be a unique geodesic of $g^{(\mathrm{B})}_{\mu\nu}$ connecting any two points $\mathcal{P}'$ and $\mathcal{P}$; so given a tensor $\boldsymbol{E}(\mathcal{P}')$ at $\mathcal{P}'$, one can parallel transport it along this geodesic to $\mathcal{P}'$, getting there a tensor $\boldsymbol{E}(\mathcal{P}')_{\to\mathcal{P}}$.

(b) Let $f(\mathcal{P}',\mathcal{P})$ be a weighting function that falls smoothly to zero when $\mathcal{P}'$ and $\mathcal{P}$ are separated by many wavelengths, and such that

$$\int f(\mathcal{P}',\mathcal{P})\sqrt{-g^{(\mathrm{B})}(\mathcal{P}')}d^4x'=1. \tag{35.14.1}$$

(c) Then the average of the tensor field $\boldsymbol{E}(\mathcal{P}')$ over several wavelengths about the point $\mathcal{P}$ is

$$\langle\boldsymbol{E}\rangle_{\mathcal{P}}\equiv\int\boldsymbol{E}(\mathcal{P}')_{\to\mathcal{P}}f(\mathcal{P}',\mathcal{P})\sqrt{-g^{(\mathrm{B})}(\mathcal{P}')}d^4x'. \tag{35.14.2}$$

(i) Show that there exists an entity $g^{(\mathrm{B})\alpha'}_{\mu}(\mathcal{P},\mathcal{P}')$, whose primed index transforms as a tensor at $\mathcal{P}$ and whose unprimed index transforms as a tensor at $\mathcal{P}'$, such that (for $\boldsymbol{E}$ second rank)

$$E_{\alpha\beta}(\mathcal{P}')_{\to\mathcal{P}}=g^{(\mathrm{B})\mu'}_{\alpha}g^{(\mathrm{B})\nu'}_{\beta}E_{\mu'\nu'}(\mathcal{P}'). \tag{35.14.3}$$

This entity is called the “bivector of geodesic parallel displacement”; see DeWitt and Brehme (1960) or Synge (1960a).

(ii) Rewriting expression 35.14.2 in coordinate language as

$$\langle E_{\alpha\beta}(x)\rangle=\int g^{(\mathrm{B})\mu'}_{\alpha}(x,x')g^{(\mathrm{B})\nu'}_{\beta}(x,x')E_{\mu'\nu'}(x')f(x,x')\sqrt{-g^{(\mathrm{B})}(x')}d^4x', \tag{35.14.4}$$

derive the three averaging rules cited at the beginning of section 35.15.

Solution:

(i) The matrix $g_{\mu}^{(\mathrm{B})\alpha'}(x,x')$ is defined as the solution of the equation

$$\frac{dg_{\alpha}^{(\mathrm{B})\mu'}}{d\tau} = \Gamma^{\rho}{}_{\alpha\sigma} g_{\rho}^{(\mathrm{B})\mu'} \frac{dx^{\sigma}}{d\tau} \tag{35.14.5}$$

with initial conditions that $g_{\mu}^{(\mathrm{B})\alpha'}(x,x')$ is an identical matrix when $x = x'$. Here $x^{\mu}(\tau)$ is a geodesic from x'. Now we have

$$\begin{aligned}
\frac{dE_{\alpha\beta}}{d\tau} &= \frac{d}{d\tau}\left(g_{\alpha}^{(\mathrm{B})\mu'} g_{\beta}^{(\mathrm{B})\nu'}\right) E_{\mu'\nu'} \\
&= \left(\Gamma^{\rho}{}_{\alpha\sigma} g_{\rho}^{(\mathrm{B})\mu'} \frac{dx^{\sigma}}{d\tau} g_{\beta}^{(\mathrm{B})\nu'} + \Gamma^{\rho}{}_{\beta\sigma} g_{\rho}^{(\mathrm{B})\nu'} \frac{dx^{\sigma}}{d\tau} g_{\alpha}^{(\mathrm{B})\mu'}\right) \frac{dx^{\sigma}}{d\tau} E_{\mu'\nu'} \\
&= \left(\Gamma^{\rho}{}_{\alpha\sigma} E_{\rho\beta} + \Gamma^{\rho}{}_{\beta\sigma} E_{\alpha\rho}\right) \frac{dx^{\sigma}}{d\tau},
\end{aligned} \tag{35.14.6}$$

i.e. $\boldsymbol{E}$ is parallel transport along the geodesic. Since $E_{\mu'\nu'}$ is a tensor at x' while $E_{\alpha\beta}$ is a tensor at x, it is obvious that $g_{\mu}^{(\mathrm{B})\alpha'}(x,x')$'s primed index transforms as a tensor at x and whose unprimed index transforms as a tensor at x'.

(ii) Firstly, we have

$$h_{\mu\nu|\alpha\beta} - h_{\mu\nu|\beta\alpha} = R^{\rho}{}_{\mu\alpha\beta} h_{\rho\nu} + R^{\rho}{}_{\nu\alpha\beta} h_{\mu\rho} \tag{35.14.7}$$

And the Riemann tensor is of the order $\partial^2 g^{(\mathrm{B})}$ or $(\partial g^{(\mathrm{B})})^2$. Suppose the background metric varies over the region with scale $\mathcal{R}$, then the average value of Riemann tensor will be of the order $(\mathcal{R})^{-2}$ while $\partial^2 h$ will be of the order $h\lambda\!\!\!^{-2}$, i.e. the fractional errors made by freely commuting are of the order $(\lambda\!\!\!^{-}/\mathcal{R})^2$.

Secondly, for any tensor S, the covariant derivative ∇S is of the form $(\partial + \Gamma)$. From equation 35.14.5, the matrix $g_{\mu}^{(\mathrm{B})\alpha'}(x,x')$ is of the order $1 + \lambda\!\!\!^{-}/\mathcal{R}$ and its derivative is of the order $1/\mathcal{R}$. The derivative of $g_{\alpha\beta}^{(\mathrm{B})}$ is also of the order $1/\mathcal{R}$. So we have

$$\begin{aligned}
\langle \nabla S \rangle &= \int g^{(\mathrm{B})}(x,x') \nabla S(x') f(x') \sqrt{-g^{(\mathrm{B})}} d^4x' \\
&= \int \partial S(x') f(x') \sqrt{-g^{(\mathrm{B})}} d^4x + O\left(\frac{\lambda\!\!\!^{-}}{\mathcal{R}}\right) \\
&= \int \partial\left(S f \sqrt{-g^{(\mathrm{B})}}\right) d^4x - \int S \partial f \sqrt{-g^{(\mathrm{B})}} d^4x + O\left(\frac{\lambda\!\!\!^{-}}{\mathcal{R}}\right)
\end{aligned} \tag{35.14.8}$$

The first term may be converted to a surface integral taken in the region where $f = 0$. As for second term, since $f(\mathcal{P}', \mathcal{P})$ is a weighting function that falls smoothly to zero when $\mathcal{P}'$ and $\mathcal{P}$ are separated by many wavelengths, $S\partial f$ can be neglected when compare with $f\partial S$, since ∂S varies a lot over $\lambda\!\!\!^{-}$, i.e.

$$\langle \nabla S \rangle \sim O\left(\frac{\lambda\!\!\!^{-}}{\mathcal{R}}\right). \tag{35.14.9}$$

35.15 GEOMETRIC OPTICS

Develop geometric optics for gravitational waves of small amplitude propagating in a curved background. Pattern the analysis after geometric optics for electromagnetic waves. In particular, let $\bar{h}_{\mu\nu}$ have an amplitude that varies slowly (on a scale $\ell \lesssim \mathcal{R}$) and a phase θ that varies rapidly ($\theta_{\hat{\alpha}} \sim 1/\lambda\!\!\!^{-}$). Expand the amplitude in powers of $\lambda\!\!\!^{-}/\ell$, so that

$$\bar{h}_{\mu\nu} = \mathrm{Re}\{A_{\mu\nu} + \epsilon B_{\mu\nu} + \epsilon^2 C_{\mu\nu} + \cdots\} e^{i\theta/\epsilon}. \tag{35.15.1}$$

Here ϵ is a formal expansion parameter, actually equal to unity, which reminds one that the terms attached to ϵ^n are proportional to $(\lambda/\mathcal{R})^n$. Define the following quantities (with $A^*_{\mu\nu}$ denoting the complex conjugate of $A_{\mu\nu}$):

$$\text{“wave vector”: } k_\alpha \equiv \theta_{,\alpha} \tag{35.15.2a}$$

$$\text{“scalar amplitude”: } \mathcal{A} \equiv \left(\frac{1}{2}A^*_{\mu\nu}A^{\mu\nu}\right)^{1/2} \tag{35.15.2b}$$

$$\text{“polarization”: } e_{\mu\nu} \equiv A_{\mu\nu}/\mathcal{A}. \tag{35.15.2c}$$

By inserting expression 35.15.1 into the gauge condition (35.66) and the propagation equation (35.68), derive the fundamental equations of geometrical optics as follows.

(a) The rays (curves perpendicular to surfaces of constant phase) are null geodesics; i.e.

$$k_\alpha k^\alpha = 0, \tag{35.15.3a}$$

$$k_{\alpha|\beta}k^\beta = 0. \tag{35.15.3b}$$

(b) The polarization is orthogonal to the rays and is parallel transported along them;

$$e_{\mu\alpha}k^\alpha = 0, \tag{35.15.3c}$$

$$e_{\mu\nu|\alpha}k^\alpha = 0. \tag{35.15.3d}$$

(c) The scalar amplitude decreases as the rays diverge away from each other in accordance with

$$\mathcal{A}_{,\alpha}k^\alpha = -\frac{1}{2}k^\alpha{}_{|\alpha}\mathcal{A}. \tag{35.15.3e}$$

i.e.

$$(\mathcal{A}^2 k^\alpha)_{|\alpha} = 0(\text{“conservation of gravitons”}). \tag{35.15.3f}$$

(d) The correction $B_{\mu\nu}$ to the amplitude obeys

$$B_{\mu\alpha}k^\alpha = iA_{\mu\alpha}{}^{|\alpha}, \tag{35.15.3g}$$

$$B_{\mu\nu|\alpha}k^\alpha = -\frac{1}{2}k^\alpha{}_{|\alpha}B_{\mu\nu} + \frac{1}{2}iA_{\mu\nu|\alpha}{}^\alpha + iR^{(\mathrm{B})}_{\alpha\mu\beta\nu}A^{\alpha\beta}. \tag{35.15.3h}$$

In accordance with exercise 35.13, specialize the gauge so that $h = 0$, i.e.,

$$e_\alpha{}^\alpha = 0. \tag{35.15.3i}$$

Then show that the stress-energy tensor (35.70') for the waves is

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\mathcal{A}^2 k_\mu k_\nu. \tag{35.15.3j}$$

This has the same form as the stress-energy tensor for a beam of particles with zero rest mass. Show explicitly that $T^{(\mathrm{GW})\mu\nu}{}_{|\nu} = 0$.

Solution:

Firstly, we have

$$h_{\mu\nu|\alpha} = \frac{i}{\epsilon}\theta_{,\alpha}\mathscr{A}_{\mu\nu}e^{i\theta/\epsilon} + \mathscr{A}_{\mu\nu|\alpha}e^{i\theta/\epsilon}, \tag{35.15.4}$$

where $\mathscr{A}_{\mu\nu} = A_{\mu\nu} + \epsilon B_{\mu\nu} + \epsilon^2 C_{\mu\nu} + \cdots$. Using the definition 35.15.2a and contracting ν and α, the gauge condition becomes

$$h^{\mu\nu}{}_{|\nu} = \left(\frac{i}{\epsilon}k_\nu\mathscr{A}^{\mu\nu} + \mathscr{A}^{\mu\nu}{}_{|\nu}\right)e^{i\theta/\epsilon} = 0. \tag{35.15.5}$$

Taking another derivative, we have

$$h_{\mu\nu|\alpha\beta} = \left(\frac{i}{\epsilon}\theta_{|\alpha\beta}\mathscr{A}_{\mu\nu} - \frac{1}{\epsilon^2}\theta_{,\alpha}\theta_{,\beta}\mathscr{A}_{\mu\nu} + \frac{i}{\epsilon}\theta_{,\alpha}\mathcal{A}_{\mu\nu|\beta} + \frac{i}{\epsilon}\theta_{,\beta}\mathcal{A}_{\mu\nu|\alpha} + \mathscr{A}_{\mu\nu|\alpha\beta}\right)e^{i\theta/\epsilon}. \tag{35.15.6}$$

Using the definition 35.15.2a and contracting α and β, we can arrive at

$$h_{\mu\nu|\alpha}{}^{\alpha} = \left(i\epsilon^{-1}k^{\alpha}{}_{|\alpha}\mathscr{A}_{\mu\nu} - \epsilon^{-2}k_{\alpha}k^{\alpha}\mathscr{A}_{\mu\nu} + 2i\epsilon^{-1}k^{\alpha}\mathscr{A}_{\mu\nu|\alpha} + \mathscr{A}_{\mu\nu|\alpha}{}^{\alpha}\right)e^{i\theta/\epsilon}. \tag{35.15.7}$$

The wave equation (35.68) now becomes

$$i\epsilon^{-1}k^{\alpha}{}_{|\alpha}\mathscr{A}_{\mu\nu} - \epsilon^{-2}k_{\alpha}k^{\alpha}\mathscr{A}_{\mu\nu} + 2i\epsilon^{-1}k^{\alpha}\mathscr{A}_{\mu\nu|\alpha} + \mathscr{A}_{\mu\nu|\alpha}{}^{\alpha} + 2R^{(\mathrm{B})}_{\alpha\mu\beta\nu}\mathscr{A}^{\alpha\beta} = 0. \tag{35.15.8}$$

(a) Take the term of the order ϵ^{-2} from 35.15.8, and we can get

$$k_{\alpha}k^{\alpha}A_{\mu\nu} = 0, \tag{35.15.9}$$

i.e. $k_{\alpha}k^{\alpha} = 0$.

As a result, we also have

$$k_{\alpha|\beta}k^{\beta} = \theta_{|\alpha\beta}k^{\beta} = \theta_{|\beta\alpha}k^{\beta} = k_{\beta|\alpha}k^{\beta} = \frac{1}{2}\left(k_{\beta}k^{\beta}\right)_{|\alpha} = 0. \tag{35.15.10}$$

(b) & (c) Take the term of the order ϵ^{-1} from 35.15.5, and we can get

$$k_{\nu}A^{\mu\nu} = 0, \tag{35.15.11}$$

i.e. $e_{\mu\alpha}k^{\alpha} = 0$.

Take the term of the order ϵ^{-1} from 35.15.8, and we can get

$$k^{\alpha}{}_{|\alpha}A_{\mu\nu} + 2k^{\alpha}A_{\mu\nu|\alpha} = 0. \tag{35.15.12}$$

Substitute $A_{\mu\nu} = \mathcal{A}e_{\mu\nu}$ into the equation above, we have

$$\left(k^{\alpha}{}_{|\alpha}\mathcal{A} + 2k^{\alpha}\mathcal{A}_{,\alpha}\right)e_{\mu\nu} + 2k^{\alpha}\mathcal{A}e_{\mu\nu|\alpha} = 0. \tag{35.15.13}$$

Multiplying both sides of the equation above with $e^{*\mu\nu}$, we have

$$0 = \left(k^{\alpha}{}_{|\alpha}\mathcal{A} + 2k^{\alpha}\mathcal{A}_{,\alpha}\right)e_{\mu\nu}e^{*\mu\nu} + k^{\alpha}\mathcal{A}\left(e^{*\mu\nu}e_{\mu\nu}\right)_{|\alpha} = 2\left(k^{\alpha}{}_{|\alpha}\mathcal{A} + 2k^{\alpha}\mathcal{A}_{,\alpha}\right). \tag{35.15.14}$$

i.e.

$$\mathcal{A}_{,\alpha}k^{\alpha} = -\frac{1}{2}k^{\alpha}{}_{|\alpha}\mathcal{A}. \tag{35.15.15}$$

Then we can get

$$2k^{\alpha}\mathcal{A}e_{\mu\nu|\alpha} = 0, \tag{35.15.16}$$

i.e. $e_{\mu\nu|\alpha} = 0$.

(d) Take the term of the order $O(1)$ from 35.15.5, and we can get

$$ik_{\nu}B^{\mu\nu} + A^{\mu\nu}{}_{|\nu} = 0, \tag{35.15.17}$$

i.e.

$$B_{\mu\alpha}k^{\alpha} = iA_{\mu\alpha}{}^{|\alpha}. \tag{35.15.18}$$

Take the term of the order $O(1)$ from 35.15.8, and we can get

$$ik^{\alpha}{}_{|\alpha}B_{\mu\nu} + 2ik^{\alpha}B_{\mu\nu|\alpha} + A_{\mu\nu|\alpha}{}^{\alpha} + 2R^{(\mathrm{B})}_{\alpha\mu\beta\nu}A^{\alpha\beta} = 0, \tag{35.15.19}$$

i.e.

$$B_{\mu\nu|\alpha}k^{\alpha} = -\frac{1}{2}k^{\alpha}{}_{|\alpha}B_{\mu\nu} + \frac{1}{2}iA_{\mu\nu|\alpha}{}^{\alpha} + iR^{(\mathrm{B})}_{\alpha\mu\beta\nu}A^{\alpha\beta}. \tag{35.15.20}$$

Equation (35.70') states that

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu}\rangle \text{ if } \bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = \bar{h} = 0. \tag{35.15.21}$$

To the order of ϵ^{-2}, we have

$$\bar{h}_{\alpha\beta|\mu}\bar{h}^{*\alpha\beta}{}_{|\nu} = k_\mu k_\nu A_{\alpha\beta}A^{*\alpha\beta} = 2\mathcal{A}^2 k_\mu k_\nu \tag{35.15.22}$$

Note that

$$\langle \mathrm{Re}\, X\, \mathrm{Re}\, Y\rangle = \frac{1}{2}XY^* \tag{35.15.23}$$

if $X, Y \propto e^{i\theta}$ and θ varies quickly when compared with the scale of the domain of averaging. So we have

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\mathcal{A}^2 k_\mu k_\nu. \tag{35.15.24}$$

Using equation 35.15.3b and 35.15.3e, we have

$$T^{(\mathrm{GW})\mu\nu}{}_{|\nu} = \frac{1}{32\pi}\mathcal{A}(2\mathcal{A}_{,\nu}k^\nu k^\mu + \mathcal{A}k^\mu{}_{|\nu}k^\nu + \mathcal{A}k^\mu k^\nu{}_{|\nu}) = 0. \tag{35.15.25}$$

35.16 GRAVITONS

Show that geometric optics, as developed in the preceding exercise, is equivalent to the following: "A graviton is postulated to be a particle of zero rest mass and 4-momentum $\boldsymbol{p}$, which moves along a null geodesic. It parallel transports with itself a transverse traceless polarization tensor $\boldsymbol{e}$. Geometric optics is the theory of a stream of such gravitons moving through spacetime." Exhibit the relationship between the quantities in this version of geometric optics and the quantities in the preceding version.

Solution:

A graviton is postulated to be a particle of zero rest mass and 4-momentum $\boldsymbol{p}$, which moves along a null geodesic. This is equivalent to

$$\boldsymbol{p}\cdot\boldsymbol{p} = 0, \quad \boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{p} = 0. \tag{35.16.1}$$

It parallel transports with itself a transverse traceless polarization tensor $\boldsymbol{e}$. This is equivalent to

$$\boldsymbol{e}\cdot\boldsymbol{p} = 0, \quad e^{\alpha}{}_{\alpha} = 0, \qquad \boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{e} = 0. \tag{35.16.2}$$

If we assume $\boldsymbol{p}$ is proportional to $\boldsymbol{k}$, geometric optics developed in the preceding exercise is equivalent to the statements in this exercise.

35.17 GRAVITATIONAL DEFLECTION OF GRAVITATIONAL WAVES

Show that gravitational waves of short wavelength passing through the solar system experience the same redshift and gravitational deflection as does light.

Solution:

Gravitons (in short wavelength approximation) and photons both follow null geodesic in spacetime. So gravitational waves of short wavelength passing through the solar system must experience the same redshift and gravitational deflection as does light.

35.18 GAUGE INVARIANCE OF $T^{(\mathrm{GW})}_{\mu\nu}$

Show that the stress-energy tensor $T^{(\mathrm{GW})}_{\mu\nu}$ of equation (35.70) is invariant under gauge transformations of the form (35.65).

Solution:

Under gauge transformation, we have

$$\bar{h}^{\mathrm{new}}_{\mu\nu} = \bar{h}^{\mathrm{old}}_{\mu\nu} - \xi_{\mu|\nu} - \xi_{\nu|\mu} + g^{(\mathrm{B})}_{\mu\nu}\xi^{\alpha}{}_{|\alpha}, \quad \bar{h}^{\mathrm{new}} = \bar{h}^{\mathrm{old}} + 2\xi^{\alpha}{}_{|\alpha}. \tag{35.18.1}$$

The stress-energy tensor of gravitational waves is

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\left\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} - \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} - 2\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha(\mu|\nu)} \right\rangle. \tag{35.18.2}$$

Under gauge transformation, we have

$$\begin{aligned}
\bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} \to\ & \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} \\
& - (\xi_{\alpha|\beta\mu} + \xi_{\beta|\alpha\mu} - g^{(\mathrm{B})}_{\alpha\beta}\xi^{\rho}{}_{|\rho\mu})\bar{h}^{\alpha\beta}{}_{|\nu} \\
& - \bar{h}_{\alpha\beta|\mu}(\xi^{\alpha|\beta}{}_{\nu} + \xi^{\beta|\alpha}{}_{\nu} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\sigma}{}_{|\sigma\nu}) \\
& + (\xi_{\alpha|\beta\mu} + \xi_{\beta|\alpha\mu} - g^{(\mathrm{B})}_{\alpha\beta}\xi^{\rho}{}_{|\rho\mu})(\xi^{\alpha|\beta}{}_{\nu} + \xi^{\beta|\alpha}{}_{\nu} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\sigma}{}_{|\sigma\nu})
\end{aligned} \tag{35.18.3}$$

$$\frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} \to \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} + \xi^{\rho}{}_{|\rho\mu}\bar{h}_{|\nu} + \bar{h}_{|\mu}\xi^{\sigma}{}_{|\sigma\nu} + 2\xi^{\rho}{}_{|\rho\mu}\xi^{\sigma}{}_{|\sigma\nu} \tag{35.18.4}$$

$$\begin{aligned}
\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha\mu|\nu} \to\ & \bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha\mu|\nu} \\
& - (\xi^{\alpha|\beta}{}_{\beta} + \xi^{\beta|\alpha}{}_{\beta} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\rho}{}_{|\rho\beta})\bar{h}_{\alpha\mu|\nu} \\
& - \bar{h}^{\alpha\beta}{}_{|\beta}(\xi_{\alpha|\mu\nu} + \xi_{\mu|\alpha\nu} - g^{(\mathrm{B})}_{\alpha\mu}\xi^{\sigma}{}_{|\sigma\nu}) \\
& + (\xi^{\alpha|\beta}{}_{\beta} + \xi^{\beta|\alpha}{}_{\beta} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\rho}{}_{|\rho\beta})(\xi_{\alpha|\mu\nu} + \xi_{\mu|\alpha\nu} - g^{(\mathrm{B})}_{\alpha\mu}\xi^{\sigma}{}_{|\sigma\nu})
\end{aligned} \tag{35.18.5}$$

$$\begin{aligned}
\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha\nu|\mu} \to\ & \bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha\nu|\mu} \\
& - (\xi^{\alpha|\beta}{}_{\beta} + \xi^{\beta|\alpha}{}_{\beta} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\rho}{}_{|\rho\beta})\bar{h}_{\alpha\nu|\mu} \\
& - \bar{h}^{\alpha\beta}{}_{|\beta}(\xi_{\alpha|\nu\mu} + \xi_{\nu|\alpha\mu} - g^{(\mathrm{B})}_{\alpha\nu}\xi^{\sigma}{}_{|\sigma\mu}) \\
& + (\xi^{\alpha|\beta}{}_{\beta} + \xi^{\beta|\alpha}{}_{\beta} - g^{\alpha\beta}_{(\mathrm{B})}\xi^{\rho}{}_{|\rho\beta})(\xi_{\alpha|\nu\mu} + \xi_{\nu|\alpha\mu} - g^{(\mathrm{B})}_{\alpha\nu}\xi^{\sigma}{}_{|\sigma\mu})
\end{aligned} \tag{35.18.6}$$

So we have

$$\begin{aligned}
& \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} - \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} - 2\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha(\mu|\nu)} \\
& \to \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}{}_{|\nu} - \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} - 2\bar{h}^{\alpha\beta}{}_{|\beta}\bar{h}_{\alpha(\mu|\nu)} \\
& - 2\xi_{\alpha|\beta\mu}h^{\alpha\beta}{}_{|\nu} - 2\xi_{\alpha|\beta\nu}h^{\alpha\beta}{}_{|\mu} + 2\xi_{\alpha|\beta\mu}\xi^{\alpha|\beta}{}_{\nu} + 2\xi_{\alpha|\beta\mu}\xi^{\beta|\alpha}{}_{\nu} - 4\xi^{\rho}{}_{|\rho\mu}\xi^{\sigma}{}_{|\sigma\nu} \\
& + \xi^{\alpha|\beta}{}_{\beta}\bar{h}_{\alpha\mu|\nu} + \xi^{\alpha|\beta}{}_{\alpha}\bar{h}_{\beta\mu|\nu} - \xi^{\rho}{}_{|\rho\beta}\bar{h}^{\beta}{}_{\mu|\nu} + \bar{h}^{\alpha\beta}{}_{|\beta}\xi_{\alpha|\mu\nu} + \bar{h}^{\alpha\beta}{}_{|\beta}\xi_{\mu|\alpha\nu} - \xi^{\rho}{}_{|\rho\nu}\bar{h}^{\beta}{}_{\mu|\beta} \\
& - \xi^{\alpha|\beta}{}_{\beta}\xi_{\alpha|\mu\nu} - \xi^{\alpha|\beta}{}_{\beta}\xi_{\mu|\alpha\nu} - \xi^{\alpha|\beta}{}_{\alpha}\xi_{\beta|\mu\nu} - \xi^{\alpha|\beta}{}_{\alpha}\xi_{\mu|\beta\nu} + \xi^{\rho}{}_{|\rho\beta}\xi^{\beta}{}_{|\mu\nu} + \xi^{\rho}{}_{|\rho\beta}\xi_{\mu}{}^{|\beta}{}_{\nu} + \xi^{\rho}{}_{|\rho\nu}\xi^{\beta}{}_{|\mu\beta} + \xi^{\rho}{}_{|\rho\nu}\xi_{\mu}{}^{|\beta}{}_{\beta} \\
& + \xi^{\alpha|\beta}{}_{\beta}\bar{h}_{\alpha\nu|\mu} + \xi^{\alpha|\beta}{}_{\alpha}\bar{h}_{\beta\nu|\mu} - \xi^{\rho}{}_{|\rho\beta}\bar{h}^{\beta}{}_{\nu|\mu} + \bar{h}^{\alpha\beta}{}_{|\beta}\xi_{\alpha|\nu\mu} + \bar{h}^{\alpha\beta}{}_{|\beta}\xi_{\nu|\alpha\mu} - \xi^{\rho}{}_{|\rho\mu}\bar{h}^{\beta}{}_{\nu|\beta} \\
& - \xi^{\alpha|\beta}{}_{\beta}\xi_{\alpha|\nu\mu} - \xi^{\alpha|\beta}{}_{\beta}\xi_{\nu|\alpha\mu} - \xi^{\alpha|\beta}{}_{\alpha}\xi_{\beta|\nu\mu} - \xi^{\alpha|\beta}{}_{\alpha}\xi_{\nu|\beta\mu} + \xi^{\rho}{}_{|\rho\beta}\xi^{\beta}{}_{|\nu\mu} + \xi^{\rho}{}_{|\rho\beta}\xi_{\nu}{}^{|\beta}{}_{\mu} + \xi^{\rho}{}_{|\rho\mu}\xi^{\beta}{}_{|\nu\beta} + \xi^{\rho}{}_{|\rho\mu}\xi_{\nu}{}^{|\beta}{}_{\beta}
\end{aligned} \tag{35.18.7}$$

As stated in the exercise 35.14, there are some useful rules for manipulating quantities inside the averaging brackets:

- Covariant derivatives commute;
- Gradients average out to zero.

I think it is impossible to eliminate the $\bar{h}^{\alpha\beta}{}_{|\beta}\xi_{\mu|\alpha\nu}$ because there is no other terms ever containing ξ_μ. Some other difficulties also exist. Any help is appreciated. I have posted the question on the *Stackexchange*. Please check it.

35.19 $T_{\mu\nu}^{(\mathrm{GW})}$ EXPRESSED AS THE AVERAGE OF A STRESS-ENERGY PSEUDOTENSOR

Calculate the average over several wavelengths of the Landau-Lifshitz stress-energy pseudotensor [equation (20.22)] for gravitational waves with $\lambda/\mathcal{R} \ll 1$. The result should be equal to $T_{\mu\nu}^{(\mathrm{GW})}$.

Solution:

The Landau-Lifshitz pseudotensor is

$$\begin{aligned}(-g)t^{\alpha\beta}_{\mathrm{L-L}} =& \frac{1}{16\pi}\Big\{\mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda}\mathfrak{g}^{\beta\mu}{}_{,\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}{}_{,\rho}\mathfrak{g}^{\rho\mu}{}_{,\nu} \\ & - (g^{\alpha\lambda}g_{\mu\nu}\mathfrak{g}^{\beta\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda}g_{\mu\nu}\mathfrak{g}^{\alpha\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda}) + g^{\lambda\mu}g_{\nu\rho}\mathfrak{g}^{\alpha\lambda}{}_{,\nu}\mathfrak{g}^{\beta\mu}{}_{,\rho} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\mu}\Big\},\end{aligned} \tag{35.19.1}$$

where $\mathfrak{g}^{\alpha\beta} = (-g)^{1/2}g^{\alpha\beta}$. For simplicity, we will perform the calculation in flat background and TT gauge, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ h^{\mu\nu}{}_{,\nu} = 0,\ h^{\mu}{}_{\mu} = 0. \tag{35.19.2}$$

As a result, we have

$$-g = 1 + O(h^2), \quad \mathfrak{g}^{\mu\nu}{}_{,\rho} = -h^{\mu\nu}{}_{,\rho} + O(h^2). \tag{35.19.3}$$

Up to the second order of h, we have

$$\begin{aligned}t^{\alpha\beta}_{\mathrm{L-L}} =& \frac{1}{16\pi}\Big\{+\frac{1}{2}\eta^{\alpha\beta}\eta_{\lambda\mu}h^{\lambda\nu}{}_{,\rho}h^{\rho\mu}{}_{,\nu} - (\eta^{\alpha\lambda}\eta_{\mu\nu}h^{\beta\nu}{}_{,\rho}h^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\eta_{\mu\nu}h^{\alpha\nu}{}_{,\rho}h^{\mu\rho}{}_{,\lambda}) + \eta^{\lambda\mu}\eta_{\nu\rho}h^{\alpha\lambda}{}_{,\nu}h^{\beta\mu}{}_{,\rho} \\ & + \frac{1}{4}(2\eta^{\alpha\lambda}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\lambda\mu})\eta_{\nu\rho}\eta_{\sigma\tau}h^{\nu\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\mu}\Big\} + O(h^3).\end{aligned} \tag{35.19.4}$$

When taking the average over the region of which the scale is several times that of the typical wavelength of the gravitational wave, we have

$$\langle a_{,m}b\rangle = -\langle ab_{,m}\rangle. \tag{35.19.5}$$

So we have

$$\langle h^{\lambda\nu}{}_{,\rho}h^{\rho\mu}{}_{,\nu}\rangle = -\langle h^{\lambda\nu}h^{\rho\mu}{}_{,\nu\rho}\rangle = 0, \tag{35.19.6}$$

using Lorentz gauge condition

$$\langle \eta_{\nu\rho}h^{\alpha\lambda}{}_{,\nu}h^{\beta\mu}{}_{,\rho}\rangle = -\langle \eta_{\nu\rho}h^{\alpha\lambda}{}_{,\nu\rho}h^{\beta\mu}\rangle = 0 \tag{35.19.7}$$

using field equation. Finally, we can get

$$\langle t^{\alpha\beta}_{\mathrm{L-L}}\rangle = \frac{1}{32\pi}\eta^{\alpha\lambda}\eta^{\beta\mu}\eta_{\nu\rho}\eta_{\sigma\tau}\langle h^{\nu\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\mu}\rangle = \frac{1}{32\pi}\langle h_{\rho\sigma,\alpha}h^{\rho\sigma}{}_{,\beta}\rangle = T^{\alpha\beta}_{(\mathrm{GW})}. \tag{35.19.8}$$

35.20 SHORTWAVE APPROXIMATION FROM A VARIATIONAL VIEWPOINT

Readers who have studied the variational approach to gravitation theory in Chapter 21 may find attractive the following derivation of the basic equations of the shortwave approximation. It was devised, independently, by Sandor Kovacs and Bernard Schutz, and by Bryce DeWitt (unpublished, 1971). MacCallum and Taub (1973) give a "non-Palatini" version.

(a) Define

$$g_{\mu\nu} \equiv g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu}, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g^{(\mathrm{B})}_{\mu\nu} h \tag{35.20.1a}$$

$$W^{\mu}{}_{\beta\gamma} \equiv \frac{1}{2} g^{\mu\alpha}_{(\mathrm{B})} \left(h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha} \right). \tag{35.20.1b}$$

Raise and lower indices on $h_{\mu\nu}$ and $W^{\mu}{}_{\beta\gamma}$ with the background metric. Using the results of exercise 35.11, derive the following expression for the Lagrangian of the gravitational field:

$$\mathcal{L} \equiv \frac{1}{16\pi}(-g)^{1/2} R = \mathcal{L}' + \left(\text{perfect divergence of form } \frac{\partial \mathcal{L}}{\partial x^{\alpha}} \right) + \left(\text{corrections of order } \frac{\mathcal{A}^3}{\lambda\!\!\!^{-2}}, R^{(\mathrm{B})}_{\mu\nu}\mathcal{A}, \text{ and smaller} \right), \tag{35.20.1c}$$

where

$$\mathcal{L}' \equiv \frac{1}{16\pi} \left(-g^{(\mathrm{B})}\right)^{1/2} \left[R^{(\mathrm{B})} - \bar{h}^{\mu\nu} \left(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu} \right) + g^{\mu\nu}_{(\mathrm{B})} \left(W^{\alpha}{}_{\beta\alpha} W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu} W^{\beta}{}_{\mu\alpha} \right) \right]. \tag{35.20.1d}$$

Drop the corrections of order $\mathcal{A}^3/\lambda\!\!\!^{-2}$ from $\mathcal{L}$; and, knowing in advance that the field equations will demand $R^{(\mathrm{B})}_{\mu\nu} \sim \mathcal{A}^2/\lambda\!\!\!^{-2}$, drop also the corrections of order $R^{(\mathrm{B})}_{\mu\nu}\mathcal{A}$. Knowing that a perfect divergence contributes nothing in an extremization calculation, drop the divergence term from $\mathcal{L}$. Then $\mathcal{L}'$ is the only remaining part of $\mathcal{L}$.

(b) Extremize $I \equiv \int \mathcal{L}' d^4x$ by the Palatini method (section 21.2); i.e., abandon (temporarily) definition 35.20.1b of $W^{\mu}{}_{\beta\gamma}$, and extremize I with respect to independent variations of $W^{\mu}{}_{\beta\gamma} = W^{\mu}{}_{\gamma\beta}$, $\bar{h}^{\mu\nu} = \bar{h}^{\nu\mu}$ and $g^{\mu\nu}_{(\mathrm{B})} = g^{\nu\mu}_{(\mathrm{B})}$. Show that extremization with respect to $W^{\mu}{}_{\beta\gamma}$ leads back to equation 35.20.1b for $W^{\mu}{}_{\beta\gamma}$ in terms of $h_{\mu\nu}$. Show that extremization with respect to $\bar{h}^{\mu\nu}$, when combined with equations (35.20.1a, 35.20.1b), leads to the propagation equation for gravitational waves (35.64). Show that extremization with respect to $g^{(\mathrm{B})\mu\nu}$, when combined with equations (35.20.1a, 35.20.1b) and with the propagation equation (35.64), and when averaged over several wavelengths, leads to

$$G^{(\mathrm{B})}_{\mu\nu} = 8\pi T^{(\mathrm{GW})}_{\mu\nu}, \tag{35.20.1e}$$

where $T^{(\mathrm{GW})}_{\mu\nu}$ is given by equation (35.70).

Solution:

(a) From equation 35.11.20, we have

$$S^{\mu}{}_{\beta\gamma} = \frac{1}{2} g^{\mu\alpha} (h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}), \tag{35.20.2a}$$

$$R = g^{\mu\nu} R^{(\mathrm{B})}_{\mu\nu} + g^{\mu\nu} (S^{\alpha}{}_{\mu\nu|\alpha} - S^{\alpha}{}_{\mu\alpha|\nu} + S^{\alpha}{}_{\beta\alpha} S^{\beta}{}_{\mu\nu} - S^{\alpha}{}_{\beta\nu} S^{\beta}{}_{\mu\alpha}). \tag{35.20.2b}$$

From $g_{\mu\nu} = g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu}$, we can get

$$\begin{aligned} g^{\mu\nu} &= g^{\mu\nu}_{(\mathrm{B})} - h^{\mu\nu} + O(h^2), \quad g = g^{(\mathrm{B})}(1+h) + O(h^2), \\ S^{\mu}{}_{\beta\gamma} &= W^{\mu}{}_{\beta\gamma} - \frac{1}{2} h^{\mu\alpha} (h_{\alpha\beta|\gamma} + h_{\alpha\gamma|\beta} - h_{\beta\gamma|\alpha}) + O(h^3) \end{aligned} \tag{35.20.3}$$

So

$$
\begin{aligned}
(-g)^{1/2}R &= (-g^{(\mathrm{B})})^{1/2}\left(1+\frac{h}{2}\right)R + O(h^3, R^{(\mathrm{B})}h^2) \\
&= (-g^{(\mathrm{B})})^{1/2}\left[g^{\mu\nu}_{(\mathrm{B})}R^{(\mathrm{B})}{}_{\mu\nu} + (g^{\mu\nu}_{(\mathrm{B})} - h^{\mu\nu})(S^{\alpha}{}_{\mu\nu|\alpha} - S^{\alpha}{}_{\mu\alpha|\nu} + W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha}\right] \\
&+ (-g^{(\mathrm{B})})^{1/2}\frac{1}{2}g^{\mu\nu}_{(\mathrm{B})}h\left(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu}\right) + O(h^3, R^{(\mathrm{B})}h) \\
&= (-g^{(\mathrm{B})})^{1/2}\left[R^{(\mathrm{B})} - h^{\mu\nu}(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu}) + g^{\mu\nu}_{(\mathrm{B})}(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})\right] \\
&+ (-g^{(\mathrm{B})})^{1/2}(S^{\alpha\mu}{}_{\mu|\alpha} - S^{\alpha\mu}{}_{\alpha|\mu}) \\
&+ (-g^{(\mathrm{B})})^{1/2}\frac{1}{2}g^{\mu\nu}_{(\mathrm{B})}h\left(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu}\right) + O(h^3, R^{(\mathrm{B})}h) \\
&= (-g^{(\mathrm{B})})^{1/2}\left[R^{(\mathrm{B})} - \bar{h}^{\mu\nu}(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu}) + g^{\mu\nu}_{(\mathrm{B})}(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})\right] \\
&+ \frac{\partial}{\partial x^{\alpha}}\left[(-g^{(\mathrm{B})})^{1/2}(S^{\alpha\mu}{}_{\mu} - S^{\mu\alpha}{}_{\mu})\right] + O(h^3, R^{(\mathrm{B})}h).
\end{aligned}
\tag{35.20.4}
$$

Now we can get the equation 35.20.1c and 35.20.1d.

(b) The extremization with respect to $W^{\mu}{}_{\beta\gamma}$ gives

$$
\begin{aligned}
16\pi\delta\mathcal{L}' &= -(-g^{(\mathrm{B})})^{1/2}\bar{h}^{\mu\nu}(\delta W^{\alpha}{}_{\mu\nu|\alpha} - \delta W^{\alpha}{}_{\mu\alpha|\nu}) + (-g^{(\mathrm{B})})^{1/2}g^{\mu\nu}_{(\mathrm{B})}\delta(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu} - W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha}) \\
&= (-g^{(\mathrm{B})})^{1/2}\bar{h}^{\mu\nu}{}_{|\alpha}\delta W^{\alpha}{}_{\mu\nu} - (-g^{(\mathrm{B})})^{1/2}\bar{h}^{\mu\nu}{}_{|\nu}\delta W^{\alpha}{}_{\mu\alpha} \\
&+ (-g^{(\mathrm{B})})^{1/2}g^{\mu\nu}_{(\mathrm{B})}(W^{\alpha}{}_{\beta\alpha}\delta W^{\beta}{}_{\mu\nu} + W^{\beta}{}_{\mu\nu}\delta W^{\alpha}{}_{\beta\alpha} - W^{\alpha}{}_{\beta\nu}\delta W^{\beta}{}_{\mu\alpha} - W^{\beta}{}_{\mu\alpha}\delta W^{\alpha}{}_{\beta\nu}) \\
&= (-g^{(\mathrm{B})})^{1/2}\left[\bar{h}^{\mu\nu}{}_{|\alpha} - \bar{h}^{\mu\beta}{}_{|\beta}\delta^{\nu}_{\alpha} + g^{\mu\nu}_{(\mathrm{B})}W^{\beta}{}_{\alpha\beta} + g^{\rho\sigma}_{(\mathrm{B})}W^{\mu}{}_{\rho\sigma}\delta^{\nu}_{\alpha} - g^{\mu\sigma}_{(\mathrm{B})}W^{\nu}{}_{\alpha\sigma} - g^{\rho\nu}_{(\mathrm{B})}W^{\mu}{}_{\rho\alpha}\right]\delta W^{\alpha}{}_{\mu\nu}
\end{aligned}
\tag{35.20.5}
$$

So we have the equation of motion

$$
\bar{h}^{\mu\nu}{}_{|\alpha} - \bar{h}^{\mu\beta}{}_{|\beta}\delta^{\nu}_{\alpha} + g^{\mu\nu}_{(\mathrm{B})}W^{\beta}{}_{\alpha\beta} + g^{\rho\sigma}_{(\mathrm{B})}W^{\mu}{}_{\rho\sigma}\delta^{\nu}_{\alpha} - g^{\mu\sigma}_{(\mathrm{B})}W^{\nu}{}_{\alpha\sigma} - g^{\rho\nu}_{(\mathrm{B})}W^{\mu}{}_{\rho\alpha} = 0. \tag{35.20.6}
$$

By contracting α and ν, we can get

$$
\bar{h}^{\mu\nu}{}_{|\nu} = g^{\rho\sigma}_{(\mathrm{B})}W^{\mu}{}_{\rho\sigma}\,. \tag{35.20.7}
$$

And so we have

$$
\bar{h}^{\mu\nu}{}_{|\alpha} + g^{\mu\nu}_{(\mathrm{B})}W^{\beta}{}_{\alpha\beta} - g^{\mu\sigma}_{(\mathrm{B})}W^{\nu}{}_{\alpha\sigma} - g^{\rho\nu}_{(\mathrm{B})}W^{\mu}{}_{\rho\alpha} = 0. \tag{35.20.8}
$$

By contracting μ and ν, we can get

$$
\bar{h}_{|\alpha} + 2W^{\beta}{}_{\alpha\beta} = 0. \tag{35.20.9}
$$

And so we have

$$
h^{\mu\nu}{}_{|\alpha} - g^{\mu\sigma}_{(\mathrm{B})}W^{\nu}{}_{\alpha\sigma} - g^{\rho\nu}_{(\mathrm{B})}W^{\mu}{}_{\rho\alpha} = 0, \tag{35.20.10}
$$

i.e.

$$
h_{\mu\nu|\alpha} = W_{\nu\alpha\mu} + W_{\mu\nu\alpha}\,. \tag{35.20.11}
$$

Now it is easily to derive that

$$
W^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}_{(\mathrm{B})}(h_{\beta\mu|\nu} + h_{\beta\nu|\mu} - h_{\mu\nu|\beta}). \tag{35.20.12}
$$

The extremization with respect to $\bar{h}^{\mu\nu}$ gives

$$
16\pi\delta\mathcal{L}' = -(-g^{(\mathrm{B})})^{1/2}(W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu})\delta\bar{h}^{\mu\nu}. \tag{35.20.13}
$$

So we have the equation of motion

$$
W^{\alpha}{}_{\mu\nu|\alpha} - W^{\alpha}{}_{\mu\alpha|\nu} = 0. \tag{35.20.14}
$$

Using 35.20.1b, we can get

$$g^{\alpha\beta}_{(\mathrm{B})}\left[h_{\beta\mu|\nu\alpha}+h_{\beta\nu|\mu\alpha}-h_{\mu\nu|\beta\alpha}-(h_{\beta\mu|\alpha\nu}+h_{\beta\alpha|\mu\nu}-h_{\mu\alpha|\beta\nu})\right]=0, \tag{35.20.15}$$

i.e.

$$h_{\alpha\mu|\nu}{}^{\alpha}+h_{\alpha\nu|\mu}{}^{\alpha}-h_{\mu\nu|\alpha}{}^{\alpha}-h_{|\mu\nu}=0. \tag{35.20.16}$$

The extremization with respect to $g^{\mu\nu}_{(\mathrm{B})}$ gives

$$\begin{aligned}16\pi\delta\mathcal{L}' &= \delta\left[(-g^{(\mathrm{B})})^{1/2}R^{(\mathrm{B})}\right]+(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})\delta\left[(-g^{(\mathrm{B})})^{1/2}g^{\mu\nu}_{(\mathrm{B})}\right]-\bar{h}^{\mu\nu}(W^{\alpha}{}_{\mu\nu|\alpha}-W^{\alpha}{}_{\mu\alpha|\nu})\delta(-g^{(\mathrm{B})})^{1/2}\\ &=\left\{G^{(\mathrm{B})}_{\mu\nu}+(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})-\frac{1}{2}\left[(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\rho\sigma}-W^{\alpha}{}_{\beta\sigma}W^{\beta}{}_{\rho\alpha})g^{\rho\sigma}_{(\mathrm{B})}-\bar{h}^{\rho\sigma}(W^{\alpha}{}_{\rho\sigma|\alpha}-W^{\alpha}{}_{\rho\alpha|\sigma})\right]g^{(\mathrm{B})}_{\mu\nu}\right\}\\ &\times(-g^{(\mathrm{B})})^{1/2}\delta g^{\mu\nu}_{(\mathrm{B})}.\end{aligned} \tag{35.20.17}$$

So we have the equation of motion

$$G_{\mu\nu}+(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})-\frac{1}{2}\left[(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\rho\sigma}-W^{\alpha}{}_{\beta\sigma}W^{\beta}{}_{\rho\alpha})g^{\rho\sigma}_{(\mathrm{B})}-\bar{h}^{\rho\sigma}(W^{\alpha}{}_{\rho\sigma|\alpha}-W^{\alpha}{}_{\rho\alpha|\sigma})\right]g^{(\mathrm{B})}_{\mu\nu}=0. \tag{35.20.18}$$

Note that $W^{\alpha}{}_{\rho\sigma|\alpha}-W^{\alpha}{}_{\rho\alpha|\sigma}=0$ from 35.20.14. So we can get

$$R^{(\mathrm{B})}_{\mu\nu}+(W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha})=0. \tag{35.20.19}$$

Using 35.20.1b, we have

$$W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha}=\frac{1}{4}h^{|\beta}(h_{\beta\mu|\nu}+h_{\beta\nu|\mu}-h_{\mu\nu|\beta})-\frac{1}{4}\left[h^{\alpha\beta}{}_{|\mu}h_{\alpha\beta|\nu}+2h_{\mu\beta|\alpha}h_{\nu}{}^{\alpha|\beta}-2h_{\mu\alpha|\beta}h_{\nu}{}^{\alpha|\beta}\right]. \tag{35.20.20}$$

If we use the TT gauge and take the average, it is easy to derive that

$$\langle W^{\alpha}{}_{\beta\alpha}W^{\beta}{}_{\mu\nu}-W^{\alpha}{}_{\beta\nu}W^{\beta}{}_{\mu\alpha}\rangle=-\frac{1}{4}\langle h^{\alpha\beta}{}_{|\mu}h_{\alpha\beta|\nu}\rangle. \tag{35.20.21}$$

We can also find that $\langle h^{\alpha\beta|\mu}h_{\alpha\beta|\mu}=0$. So we have

$$G^{(\mathrm{B})}_{\mu\nu}=\frac{1}{32\pi}\langle h^{\alpha\beta}{}_{|\mu}h_{\alpha\beta|\nu}\rangle. \tag{35.20.22}$$

Chapter 36

GENERATION OF GRAVITATIONAL WAVES

36.1 GRAVITATIONAL WAVES FROM ROTATING BEAM

A long steel beam of length l and mass M rotates end over end with angular velocity ω. Show that the power it radiates as gravitational waves is

$$L_{\rm GW} = \frac{2}{45} M^2 l^4 \omega^6. \tag{36.1.1}$$

Use this formula to verify that the rod described in the text radiates 2.2×10^{-22} ergs/sec.

Solution:

Set the mass center of the beam as the origin and the line perpendicular to the rotation plain as z axis. The second moment of the mass distribution of the beam is

$$I_{xx} \equiv \int_{-l/2}^{l/2} \frac{M}{l} l' \cos\theta l' \cos\theta \, \mathrm{d}l' = \frac{1}{12} M l^2 \cos^2\theta,$$

$$I_{xy} = I_{yx} = \frac{1}{12} M l^2 \sin\theta\cos\theta, \quad I_{yy} = \frac{1}{12} M l^2 \sin^2\theta, \quad I_{xz} = I_{yz} = 0. \tag{36.1.2}$$

I_{zz} is a constant can be neglected when calculating power of GW wave. So its trace free part is

$$\text{Ŧ}_{jk} = \frac{1}{12} M l^2 \begin{pmatrix} \cos^2\theta - \frac{1}{3} & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & \sin^2\theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{36.1.3}$$

Thus

$$\dddot{\text{Ŧ}}_{jk} = \frac{1}{3} M l^2 \omega^3 \begin{pmatrix} \sin 2\theta & -\cos 2\theta & 0 \\ \cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36.1.4}$$

We then have

$$L_{\rm GW} = \frac{1}{5} \left\langle \dddot{\text{Ŧ}}_{jk} \dddot{\text{Ŧ}}_{jk} \right\rangle = \frac{2}{45} M^2 l^4 \omega^6 = \frac{2G}{45c^5} M^2 l^4 \omega^6 \approx 2.26 \times 10^{-22} \mathrm{erg\,s^{-1}}. \tag{36.1.5}$$

36.2 GRAVITATIONAL WAVES FROM MATTER FALLING INTO A BLACK HOLE

A lump of matter with mass m falls into a black hole of mass M. Show that a burst of gravitational waves is emitted with duration M and power $L_{\rm GW} \sim (m/M)^2 L_o$, so that the total energy radiated is given in crude order of magnitude by equation (36.13).

Solution:

The non spherical part of kinetic energy is $E_{\rm int} \sim Mm/R$. The period of the system is $T \sim (M/R^3)^{-1/2}$. So we have

$$L_{\rm GM} \sim \left(\frac{E_{\rm int}}{T}\right)^2 L_o \sim \frac{M^3 m^2}{R^5} L_o. \tag{36.2.1}$$

The infalling matter will radiate only weakly when it is far from the gravitational radius; but as it falls through the gravitational radius, i.e. $R \sim M$,it should emit a strong burst. And so we have

$$L_{\rm GM} \sim \frac{m^2}{M^2} L_o. \tag{36.2.2}$$

The total energy emitted is

$$E_{\text{radiated}} \sim m^2/M. \tag{36.2.3}$$

So the burst of gravitational waves is emitted with duration $\tau \sim E_{\text{radiated}}/L_{\rm GM} \sim M$.

36.3 GRAVITATIONAL WAVES FROM A VIBRATING NEUTRON STAR

Idealize a neutron star as a sphere of incompressible fluid of mass M and radius R, with structure governed by Newton's laws of gravity. Let the star pulsate in its fundamental quadrupole mode. Using Newtonian theory, calculate: the angular frequency of pulsation, ω; the energy of pulsation $E_{\rm puls}$; the quantity $\left\langle \dddot{\boldsymbol{I}}^2 \right\rangle /5$, which, according to equation (36.1), is the power radiated in gravitational waves, $L_{\rm GW}$; and the e-folding time, $\tau = E_{\rm puls}/L_{\rm GW}$ for radiating away the energy of the pulsations. Compare the answers with equations (36.15) - which are based on a much cruder approximation-and with the results in Box 36.1, which are based on much better approximations.

Solution:

The fundamental equations in Newtonian fluid mechanics are

$$\frac{{\rm d}\rho}{{\rm d}t} = -\rho \boldsymbol{\nabla}\cdot\boldsymbol{v}, \quad \rho\frac{{\rm d}\boldsymbol{v}}{{\rm d}t} = -\rho\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}p, \quad \nabla^2\phi = 4\pi\rho, \tag{36.3.1}$$

where ${\rm d}/{\rm d}t$ is the time derivative comoving with the matter

$$\frac{\rm d}{{\rm d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}. \tag{36.3.2}$$

For incompressible fluid, we have

$$\frac{{\rm d}\rho}{{\rm d}t} = 0. \tag{36.3.3}$$

The spherically symmetric and static solution for an incompressible fluid is

$$\rho_0(r) = \begin{cases} \rho_s & \text{if} \quad r < R \\ 0 & \text{if} \quad r > R \end{cases} \quad p_0(r) = \begin{cases} \frac{2\pi}{3}\rho_s^2(R^2 - r^2) & \text{if} \quad r < R \\ 0 & \text{if} \quad r > R \end{cases} \quad \phi_0(r) = \begin{cases} \frac{2\pi}{3}\rho_s r^2 - 2\pi\rho_s R^2 & \text{if} \quad r < R \\ -\frac{4\pi\rho_s R^3}{3r} & \text{if} \quad r > R \end{cases} \tag{36.3.4}$$

In the limit of small oscillations, we will apply first order perturbation theory and neglect all higher terms. Suppose that the perturbation of velocity, density, pressure and gravitational potential are $\delta\boldsymbol{v}$, $\delta\rho$, δp and $\delta\phi$ respectively. The first order perturbations of the equation 36.3.1 and 36.3.3 are

$$\frac{\partial\delta\rho}{\partial t}+\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho_0=-\rho_0\boldsymbol{\nabla}\cdot\delta\boldsymbol{v}=0, \tag{36.3.5}$$

$$\rho_0\frac{\partial\delta\boldsymbol{v}}{\partial t}=-\rho_0\boldsymbol{\nabla}\delta\phi-\delta\rho\boldsymbol{\nabla}\phi_0-\boldsymbol{\nabla}\delta p, \tag{36.3.6}$$

$$\nabla^2\delta\phi=4\pi\delta\rho. \tag{36.3.7}$$

Suppose the material particles are moved from their equilibrium position $\boldsymbol{r}_0$ by a displacement $\boldsymbol{s}(\boldsymbol{r}_0,t)$, and $\|\boldsymbol{s}\|\ll 1$. To the first order, we have

$$\delta\boldsymbol{v}=\frac{\partial\boldsymbol{s}}{\partial t}. \tag{36.3.8}$$

For a point inside the star ($r<R$), the perturbation of density vanish because the fluid is imcompressible. For a point at the surface of the star ($r=R$), the mass of the bump at a surface element $\Delta\boldsymbol{S}$ is $\Delta m=\rho_s\Delta\boldsymbol{S}\cdot\boldsymbol{s}=\rho_s s_r\Delta S$, causing a surface density perturbation $\delta\sigma=\rho_s s_r$. We can conclude that

$$\delta\rho=\rho_s s_r\delta(r-R). \tag{36.3.9}$$

It is compatible with

$$\frac{\partial\delta\rho}{\partial t}+\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho_0=0, \tag{36.3.10}$$

noting that $\boldsymbol{\nabla}\rho_0=\rho_s\delta(r-R)\hat{\boldsymbol{r}}$. As a result, we have

$$\nabla^2\delta\phi=4\pi\rho_s s_r\delta(r-R). \tag{36.3.11}$$

Taking the divergence of equation 36.3, we have

$$\nabla^2\delta p=0\quad\text{for}\quad r<R. \tag{36.3.12}$$

Because the pressure vanish at the surface of the star, we have

$$\delta p+\frac{\partial p_0}{\partial r}s_r=0\quad\text{at}\quad r=R. \tag{36.3.13}$$

For $r<R$, equation can be written as

$$\rho_s\frac{\partial^2\boldsymbol{s}}{\partial t^2}=-\rho_s\boldsymbol{\nabla}\delta\phi-\boldsymbol{\nabla}\delta p. \tag{36.3.14}$$

So we also have $\boldsymbol{\nabla}\times\boldsymbol{s}=0$ when $r<R$.

$\boldsymbol{s}$ can be expanded using vector spherical harmonics,

$$\boldsymbol{s}=\sum_{l,m}\left(U_{lm}(r)\boldsymbol{Y}_{lm}+V_{lm}(r)\boldsymbol{\Psi}_{lm}+W_{lm}(r)\boldsymbol{\Phi}_{lm}\right)e^{-i\omega t}, \tag{36.3.15}$$

where

$$\boldsymbol{Y}_{lm}=Y_{lm}\hat{\boldsymbol{r}},\quad \boldsymbol{\Psi}_{lm}=r\boldsymbol{\nabla}Y_{lm},\quad \boldsymbol{\Phi}_{lm}=\boldsymbol{r}\times\boldsymbol{\nabla}Y_{lm}. \tag{36.3.16}$$

Using $\boldsymbol{\nabla}\cdot\boldsymbol{s}=0$ and $\boldsymbol{\nabla}\times\boldsymbol{s}=0\,(r<R)$, we have

$$\frac{\mathrm{d}U_{lm}}{\mathrm{d}r}+\frac{2}{r}U_{lm}-\frac{l(l+1)}{r}V_{lm}=0,\quad W_{lm}=0\quad(r<R). \tag{36.3.17}$$

δp and $\delta\rho$ can be expanded using spherical harmonics,

$$\delta p=\sum_{lm}P_{lm}(r)Y_{lm}e^{-i\omega t},\quad \delta\phi=\sum_{lm}\Phi_{lm}(r)Y_{lm}e^{-i\omega t}. \tag{36.3.18}$$

Substituting 36.3.15 and 36.3.18 into 36.3.14, we can obtain

$$\rho_s\omega^2 U_{lm} = \rho_s\frac{\mathrm{d}\Phi_{lm}}{\mathrm{d}r} + \frac{\mathrm{d}P_{lm}}{\mathrm{d}r}, \quad \rho_s\omega^2 V_{lm} = \rho_s\frac{\Phi_{lm}}{r} + \frac{P_{lm}}{r} \quad (r < R) \tag{36.3.19}$$

Because $\nabla^2\delta p = 0$ and $\nabla^2\delta\phi = 0$ when $r \neq R$, we have

$$\Phi_{lm}(r) = \begin{cases} A_{lm}r^l & \text{if} \quad r < R \\ B_{lm}r^{-l-1} & \text{if} \quad r > R \end{cases} \qquad P_{lm}(r) = \begin{cases} C_{lm}r^l & \text{if} \quad r < R \\ 0 & \text{if} \quad r > R \end{cases} \tag{36.3.20}$$

Note that

$$\left.\frac{\partial\delta\phi}{\partial r}\right|_{R^-}^{R^+} = 4\pi\rho_s s_r, \quad \left.\delta\phi\right|_{R^-}^{R^+} = 0. \tag{36.3.21}$$

We have

$$-(l+1)B_{lm}R^{-l-2} - lA_{lm}R^{l-1} = 4\pi\rho_s U_{lm}(R), \quad A_{lm}R^l = B_{lm}R^{-l-1}. \tag{36.3.22}$$

The solution is

$$A_{lm} = -\frac{4\pi\rho_s U_{lm}(R)}{(2l+1)R^{l-1}}, \quad B_{lm} = -\frac{4\pi\rho_s U_{lm}(R)}{(2l+1)}R^{l+2}. \tag{36.3.23}$$

Using equation 36.3.13, we have

$$C_{lm}R^l - \frac{4\pi}{3}\rho_s^2 R U_{lm}(R). \tag{36.3.24}$$

The solution is

$$C_{lm} = \frac{4\pi\rho_s^2 U_{lm}(R)}{3R^{l-1}}. \tag{36.3.25}$$

Note that quation 36.3.19 and 36.3.20 gives

$$\rho_s\omega^2 U_{lm}(r) = (\rho_s A_{lm} + C_{lm})lr^{l-1}, \quad \rho_s\omega^2 V_{lm}(r) = (\rho_s A_{lm} + C_{lm})r^{l-1}. \tag{36.3.26}$$

We can get

$$\omega^2 = \frac{8l(l-1)\pi\rho_s}{3(2l+1)}, \quad U_{lm}(r) = U_{lm}(R)\left(\frac{r}{R}\right)^{l-1}, \quad V_{lm}(r) = \frac{1}{l}U_{lm}(R)\left(\frac{r}{R}\right)^{l-1}. \tag{36.3.27}$$

It is compatible with 36.3.17. Now we have

$$\boldsymbol{s} = \sum_{l,m}\left(\boldsymbol{Y}_{lm} + \frac{1}{l}\boldsymbol{\Psi}_{lm}\right)\left(\frac{r}{R}\right)^{l-1} U_{lm}(R)e^{-i\omega t}, \tag{36.3.28}$$

Define ξ as the root mean squared fractional departure of the surface point from the original sphere, average over the sphere. We have

$$\xi_{\text{peak}}^2 = \frac{1}{4\pi}\int \mathrm{d}\Omega\,\frac{s_{r,\text{peak}}^2(R)}{R^2} = \frac{U_{lm}^2(R)}{4\pi R^2} \tag{36.3.29}$$

The pulsing energy of the star

$$\frac{1}{2}\omega^2\int \rho_s|\boldsymbol{s}_{\text{peak}}|^2\,\mathrm{d}V = \frac{\rho_s U_{lm}^2(R)\omega^2}{2}\int_0^R \mathrm{d}r\,r^2\left(\frac{r}{R}\right)^{2l-2}\int \mathrm{d}\Omega\left(\boldsymbol{Y}_{lm}\cdot\boldsymbol{Y}_{lm}^* + \frac{1}{l^2}\boldsymbol{\Psi}_{lm}\cdot\boldsymbol{\Psi}_{lm}^*\right) \tag{36.3.30}$$

$$= \frac{2}{l}\pi\rho_s\omega^2 R^5\xi_{\text{peak}}^2 \tag{36.3.31}$$

$$= \frac{16(l-1)}{3(2l+1)}\pi^2\rho_s^2 R^5\xi_{\text{peak}}^2 \tag{36.3.32}$$

The reduced second quadrupole moment of the star is

$$\boldsymbol{\mathcal{I}} = \int \mathrm{d}V\, \delta\rho \left(\boldsymbol{r} \otimes \boldsymbol{r} - \frac{1}{3} r^2 \mathbf{I} \right) = \rho_s R^4 U_{lm}(R) e^{-i\omega t} \int \mathrm{d}\Omega\, Y_{lm} \left(\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} - \frac{1}{3}\mathbf{I} \right) \equiv \rho_s R^4 U_{lm}(R) e^{-i\omega t} \mathbf{X} \tag{36.3.33}$$

The radiation power of the gravitational wave is

$$L_{\mathrm{GW}} = \frac{1}{5} \langle \dddot{\mathcal{I}}_{ij}\, \dddot{\mathcal{I}}_{ij} \rangle = \frac{2\pi}{5} \rho_s^2 \omega^6 R^{10} \xi_{\mathrm{peak}}^2 \,\mathrm{Tr}\, \boldsymbol{X}\boldsymbol{X}^\dagger. \tag{36.3.34}$$

We can show that

$$\mathrm{Tr}\, \boldsymbol{X}\boldsymbol{X}^\dagger = \frac{8\pi}{15} \delta_{l2} \tag{36.3.35}$$

So, for quadrupole oscillations, $l = 2$, we have

$$\omega^2 = \frac{16}{15}\pi\rho_s, \quad E_{\mathrm{puls}} = \frac{16}{15}\pi^2 \rho_s^2 R^5 \xi_{\mathrm{peak}}^2, \quad L_{\mathrm{GW}} = \frac{65536}{253125}\pi^5 \rho_s^5 R^{10} \xi_{\mathrm{peak}}^2, \quad \tau = \frac{16875}{4096}\pi^{-3}\rho_s^{-3} R^{-5}. \tag{36.3.36}$$

They can also be written in terms of M and R,

$$\omega = \sqrt{\frac{4M}{5R}} \frac{1}{R}, \quad E_{\mathrm{puls}} = \frac{3}{5} \left(\frac{M}{R} \right)^2 R \xi_{\mathrm{peak}}^2, \quad L_{\mathrm{GW}} = \frac{192}{3125} \left(\frac{M}{R} \right)^5 \xi_{\mathrm{peak}}^2, \quad \tau = \frac{625}{64} \left(\frac{M}{R} \right)^{-3} R. \tag{36.3.37}$$

Suppose $M = 1.1 \times 10^{30}\mathrm{g}$, $R = 1.414 \times 10^6 \mathrm{cm}$, $\xi_{\mathrm{peak}} = 0.1$. We have

$$\omega = 4.557 \times 10^3 \mathrm{sec}^{-1}, \quad E_{\mathrm{puls}} = 3.426 \times 10^{50} \mathrm{erg}, \quad L_{\mathrm{GW}} = 1.433 \times 10^{50} \mathrm{erg\,s}^{-1}, \quad \tau = 2.391 \mathrm{s}. \tag{36.3.38}$$

36.4 PULSAR SLOWDOWN

The pulsar NP 0532 in the Crab Nebula has a period of 0.033 seconds and is slowing down at the rate $\mathrm{d}P/\mathrm{d}t = 1.35 \times 10^{-5} \mathrm{sec/yr}$. Assuming the pulsar is a typical neutron star, calculate the rate at which it is losing rotational energy. If this energy loss is due primarily to gravitational radiation reaction, what is the magnitude of the star's nonaxial deformation?

Solution:

For an ellipsoid with principal axes a, b and c, the reduced second quadrupole moment is

$$\mathcal{I}_{ij} = \frac{M}{15} \begin{pmatrix} 2a^2 - b^2 - c^2 & & \\ & 2b^2 - c^2 - a^2 & \\ & & 2c^2 - a^2 - b^2 \end{pmatrix} \tag{36.4.1}$$

Applying the rotation operator $R(z)$ around the z-axis and differentiating three times with respect to time, we obtain

$$\dddot{\mathcal{I}}_{ij} = \frac{4M}{5}\omega^3 (a^2 - b^2) \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & \\ -\sin 2\omega t & -\sin 2\omega t & \\ & & 0 \end{pmatrix} \tag{36.4.2}$$

The power of gravitational wave radiation will be

$$L_{\mathrm{GW}} = \frac{32}{125} M^2 \omega^6 (a^2 - b^2)^2. \tag{36.4.3}$$

If we assume $a \approx b \approx r$ and $(a - b)/r = \epsilon \ll 1$, then we have

$$L_{\mathrm{GW}} = \frac{128}{125} M^2 \omega^6 R^4 \epsilon^2 = \frac{32}{5} \epsilon^2 I^2 \omega^6, \tag{36.4.4}$$

where $I = 2MR^2/5$ is the moment of inertia of the ellipsoid around the z-axis. The energy conservation equation gives

$$\frac{I}{2}\frac{\mathrm{d}\omega^2}{\mathrm{d}t} = -\frac{32}{5}\epsilon^2 I^2\omega^6 \tag{36.4.5}$$

Putting back c and G, we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{32G}{5c^5}\epsilon^2 I\omega^5. \tag{36.4.6}$$

Since

$$\omega = \frac{2\pi}{P} = 190\mathrm{sec}^{-1}, \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{2\pi}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t} = -2.47\times10^{-9}\mathrm{sec}^{-2}, \tag{36.4.7}$$

if we assume $M = 1.1\times10^{30}\mathrm{g}$, $R = 1.414\times10^{6}\mathrm{cm}$, we can get $\epsilon = 0.025$.

36.5 ENERGY AND ANGULAR MOMENTUM LOSSES DUE TO RADIATION REACTION

Derive equations (36.31) for the rate at which gravitational radiation damping saps energy and angular momentum from a slow-motion source. Base the derivation on equations (36.26b) and (36.30).

Solution:

Firstly, we have

$$\Phi^{\mathrm{react}} = \frac{1}{5}\frac{\mathrm{d}^5 I_{jk}}{\mathrm{d}t^5}x_j x_k \tag{36.5.1}$$

and

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \rho\Phi^{\mathrm{react}}_{,j} v_j \,\mathrm{d}^3x\,, \quad \frac{\mathrm{d}J_j}{\mathrm{d}t} = -\int \epsilon_{jkl}x_k\rho\Phi^{\mathrm{react}}_{,l}\,\mathrm{d}^3x\,. \tag{36.5.2}$$

We can derive that

$$\Phi^{\mathrm{react}}_{,k} = \frac{2}{5}\frac{\mathrm{d}^5 I_{jk}}{\mathrm{d}t^5}x_k. \tag{36.5.3}$$

Thus,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2}{5}\frac{\mathrm{d}^5 I_{jk}}{\mathrm{d}t^5}\int \rho x_j v_k\,\mathrm{d}^3x = -\frac{1}{5}\frac{\mathrm{d}^5 I_{jk}}{\mathrm{d}t^5}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho x_j x_k\,\mathrm{d}^3x\right) = -\frac{1}{5}\frac{\mathrm{d}^5 I_{jk}}{\mathrm{d}t^5}\,\mathrm{d}I_{jk}\,t. \tag{36.5.4}$$

After taking average, we can integrate by parts, and so we can get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{5}\left\langle \frac{\mathrm{d}^3 I_{jk}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 I_{jk}}{\mathrm{d}t^3}\right\rangle. \tag{36.5.5}$$

As for angular momentum, we have

$$\frac{\mathrm{d}J_j}{\mathrm{d}t} = -\frac{2}{5}\frac{\mathrm{d}^5 I_{li}}{\mathrm{d}t^5}\int \epsilon_{jkl}x_k\rho x_i\,\mathrm{d}^3x = -\frac{2}{5}\epsilon_{jkl}\frac{\mathrm{d}^5 I_{li}}{\mathrm{d}t^5}I_{ki}. \tag{36.5.6}$$

After taking average, we can integrate by parts, and so we can get

$$\frac{\mathrm{d}J_j}{\mathrm{d}t} = -\frac{2}{5}\epsilon_{jkl}\left\langle \frac{\mathrm{d}^3 I_{li}}{\mathrm{d}t^3}\frac{\mathrm{d}^2 I_{ki}}{\mathrm{d}t^2}\right\rangle. \tag{36.5.7}$$

36.6 GRAVITATIONAL WAVES FROM BINARY STAR SYSTEMS

Apply the full formalism of section 36.7 and 36.8 to a binary star system with circular orbits. Calculate the angular distribution of the gravitational waves; the total power radiated; the total angular momentum radiated; the radiation-reaction forces; and the loss of energy and angular momentum due to radiation reaction. Compare the answers with the results quoted in section 36.6.

Solution:

We choose the center of mass as the origin of the system. If the distance of between two stars is a, we have

$$\begin{aligned}\rho = {} & M\delta\left(x - \frac{m}{M+m}a\cos\omega t\right)\delta\left(y - \frac{m}{M+m}a\sin\omega t\right)\delta(z) \\ & + m\delta\left(x - \frac{M}{M+m}a\cos\omega t\right)\delta\left(y - \frac{M}{M+m}a\sin\omega t\right)\delta(z).\end{aligned} \tag{36.6.1}$$

The second momentum of the system is

$$I_{ij} = \int \rho x^i x^j \,\mathrm{d}^3x = \frac{1}{2}\mu a^2 \begin{pmatrix} \cos 2\omega t + 1 & \sin 2\omega t & 0 \\ \sin 2\omega t & 1 - \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36.6.2}$$

As a result, we have

$$\ddot{I}_{ik} = -2\mu\omega^2 a^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \dddot{I}_{ik} = 4\mu\omega^3 a^2 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36.6.3}$$

The gravitational wave is given by

$$h_{ij}^{\mathrm{TT}} = \frac{2}{r}\ddot{I}_{ik}^{\mathrm{TT}}(t-r), \tag{36.6.4}$$

where

$$\ddot{I}_{ik}^{\mathrm{TT}} = P_{ja}\ddot{I}_{ab}^{\mathrm{TT}}P_{bk} - \frac{1}{2}P_{jk}P_{ab}\ddot{I}_{ab}^{\mathrm{TT}}, \quad P_{ab} = \delta_{ab} - \frac{x_a x_b}{r^2}. \tag{36.6.5}$$

The total power radiated is

$$L_{\mathrm{GW}} = \frac{1}{5}\left\langle \dddot{I}_{ik}\dddot{I}_{ik}\right\rangle = \frac{32}{5}\mu^2\omega^6 a^4. \tag{36.6.6}$$

Using $\omega^2 a^3 = m_1 + m_2 \equiv M$, we have

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = L_{\mathrm{GW}} = \frac{32\mu^2 M^3}{5a^5}. \tag{36.6.7}$$

Note that

$$\left\langle \ddot{I}_{ka}\dddot{I}_{al}\right\rangle = 8\mu^2\omega^5 a^4 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36.6.8}$$

The total angular momentum radiated is

$$\frac{2}{5}\epsilon^{jkl}\left\langle \ddot{I}_{ka}\dddot{I}_{al}\right\rangle = \frac{32}{5}\mu^2\omega^5 a^4(0,0,1), \tag{36.6.9}$$

i.e.,

$$-\frac{\mathrm{d}L_z}{\mathrm{d}t} = \frac{a^{3/2}}{M^{1/2}}L_{\mathrm{GW}} \tag{36.6.10}$$

Note that

$$E = -\frac{\mu M}{2a}, \quad L_z = \mu M^{1/2}a^{1/2}. \tag{36.6.11}$$

Equation 36.6.7 and 36.6.10 are compatible.

Radiation reaction force is

$$F_k/m = -\Phi^{\rm react}_{,k} = -\frac{2}{5}\frac{{\rm d}^5 \mathcal{I}_{jk}}{{\rm d}t^5}x_k = \frac{32\mu\omega^5 a^2}{5}\begin{pmatrix} x\sin 2\omega t - y\cos 2\omega t \\ -x\cos 2\omega t - y\sin 2\omega t \\ 0 \end{pmatrix}. \tag{36.6.12}$$

The position and velocity of star are

$$\boldsymbol{r} = r(\cos\omega t, \sin\omega t, 0), \quad \boldsymbol{v} = \omega r(-\sin\omega t, \cos\omega t, 0). \tag{36.6.13}$$

We have

$$F_k v_k = -\frac{32\mu m\omega^6 a^2 r^2}{5} \tag{36.6.14}$$

And so

$$\frac{{\rm d}E}{{\rm d}t} = -\frac{32\mu\omega^6 a^2}{5}(m_1 r_1^2 + m_2 r_2^2) = -\frac{32\mu^2\omega^6 a^4}{5}. \tag{36.6.15}$$

We also have

$$M_z = xF_y - yF_x = -\frac{32\mu\omega^5 a^2}{5}mr^2. \tag{36.6.16}$$

And so

$$\frac{{\rm d}J_z}{{\rm d}t} = -\frac{32\mu\omega^5 a^2}{5}(m_1 r_1^2 + m_2 r_2^2) = -\frac{32\mu^2\omega^5 a^2}{5}. \tag{36.6.17}$$

Equation 36.6.15 and 36.6.17 are identical with equation 36.6.7 and 36.6.10.

36.7 MAGNITUDE OF $t^{\mu\nu}$

Consider a slow-motion source of gravitational waves. Show that far from the source, but in the near zone $(R \ll r \ll \lambda\!\!\!^{-})$ the components of the "stress-energy pseudotensor" $t^{\mu\nu}$ die out as $1/r^4$, but in the radiation zone $(r \gg \lambda\!\!\!^{-})$ they die out only as $1/r^2$.

Solution:

From the definition of $t^{\mu\nu}$, we have $t \sim (\partial h)^2$. The gravitation wave originating from the source is

$$\bar{h}^{\mu\nu}(t, x^j) = 4\int \frac{[T^{\mu\nu} + t^{\mu\nu}]_{\rm ret}}{|\boldsymbol{x} - \boldsymbol{x}'|}\,{\rm d}^3x'\,. \tag{36.7.1}$$

Note that $\partial T \sim \frac{T}{\lambda\!\!\!^{-}}$ and $\partial(1/r) \sim 1/r^2$. So we have

$$\partial h \sim \frac{1}{r}h + \frac{1}{\lambda\!\!\!^{-}}h \quad \text{where} \quad h \propto \frac{1}{r} \tag{36.7.2}$$

If $r \ll \lambda\!\!\!^{-}$, we have $\partial h \sim h/r$ and so $t \sim 1/r^4$. If $r \gg \lambda\!\!\!^{-}$, we have $\partial h \sim h/\lambda$ and so $t \sim 1/r^2$.

36.8 PROOF THAT THE TRANVERSE TRACELESS PARTS OF

Prove by direct computation that the TT parts of I_{jk} (36.42b) and $\mathcal{I}_{jk}$ (36.46) are identical, no matter where the observer is who does the TT projection (i.e., no matter what the unit vector $\boldsymbol{n}$ in the projection operator may be).

Solution:

Note that

$$\text{Ɨ}_{jk} = I_{jk} - \frac{1}{3}\delta_{jk} I. \tag{36.8.1}$$

We have

$$\begin{aligned}
\text{Ɨ}^{\rm TT}_{jk} &= P_{ja}\text{Ɨ}_{ab}P_{bk} - \frac{1}{2}P_{jk}P_{ab}\text{Ɨ}_{ab} \\
&= P_{ja}\left(I_{ab} - \frac{1}{3}\delta_{ab}I\right)P_{bk} - \frac{1}{2}P_{jk}P_{ab}\left(I_{ab} - \frac{1}{3}\delta_{ab}I\right) \\
&= I^{\rm TT}_{jk} - \frac{1}{3}IP_{ja}P_{ak} + \frac{1}{6}IP_{jk}\delta_{ab}P_{ab} \\
&= I^{\rm TT}_{jk}.
\end{aligned} \tag{36.8.2}$$

36.9 ENERGY AND ANGULAR MOMENTUM RADIATED

(a) For the gravitational waves in asymptotically flat spacetime described by equation (36.47), calculate the smeared-out stress-energy tensor $T^{\mu\nu}_{\rm GW}$) of equation (35.23).

(b) Perform the integrals of equations (36.23) and (36.25) to obtain the total power and angular momentum radiated.

Solution:

(a) Firstly, we have

$$h^{\rm TT}_{jk} = \frac{2}{r}\frac{{\rm d}^2 I^{\rm TT}_{jk}(t-r)}{{\rm d}t^2}, \quad T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi}\left\langle h^{\rm TT}_{jk,\mu} h^{\rm TT}_{jk,\nu}\right\rangle, \tag{36.9.1}$$

and

$$I^{\rm TT}_{jk} = \text{Ɨ}^{\rm TT}_{jk} = P_{ja}\text{Ɨ}_{jk}P_{bk} - \frac{1}{2}P_{jk}P_{ab}\text{Ɨ}_{ab}. \tag{36.9.2}$$

We can derive that

$$\begin{aligned}
I^{\rm TT}_{jk} I^{\rm TT}_{jk} &= \left(P_{ja}\text{Ɨ}_{ab}P_{bk} - \frac{1}{2}P_{jk}P_{ab}\text{Ɨ}_{ab}\right)\left(P_{jc}\text{Ɨ}_{cd}P_{dk} - \frac{1}{2}P_{jk}P_{cd}\text{Ɨ}_{cd}\right) \\
&= P_{ac}P_{bd}\text{Ɨ}_{ab}\text{Ɨ}_{cd} - \frac{1}{2}P_{ab}P_{cd}\text{Ɨ}_{ab}\text{Ɨ}_{cd} \\
&= \left[(\delta_{ac} - n_a n_c)(\delta_{bd} - n_b n_d) - \frac{1}{2}(\delta_{ab} - n_a n_b)(\delta_{cd} - n_c n_d)\right]\text{Ɨ}_{ab}\text{Ɨ}_{cd} \\
&= \left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}n_b n_d + \delta_{ab}n_c n_d + \frac{1}{2}n_a n_b n_c n_d\right)\text{Ɨ}_{ab}\text{Ɨ}_{cd} \\
&= \text{Ɨ}_{ab}\text{Ɨ}_{ab} - 2\text{Ɨ}_{ab}\text{Ɨ}_{ad}n_b n_d + \frac{1}{2}n_a n_b n_c n_d \text{Ɨ}_{ab}\text{Ɨ}_{cd}.
\end{aligned} \tag{36.9.3}$$

Note that

$$h^{\rm TT}_{jk,0} = -h^{\rm TT}_{jk,r}. \tag{36.9.4}$$

We have

$$\begin{aligned}
T^{\rm GW}_{00} = -T^{\rm GW}_{0r} = T^{\rm GW}_{rr} &= \frac{1}{32\pi}\left\langle h^{\rm TT}_{jk,0} h^{\rm TT}_{jk,0}\right\rangle = \frac{1}{8\pi r^2}\left\langle \dddot{I}^{\rm TT}_{jk}\dddot{I}^{\rm TT}_{jk}\right\rangle \\
&= \frac{1}{8\pi r^2}\left\langle \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{jk} - 2n_i\dddot{\text{Ɨ}}_{ij}\text{Ɨ}_{jk}n_k + \frac{1}{2}(n_j\dddot{\text{Ɨ}}_{jk}n_k)^2\right\rangle.
\end{aligned} \tag{36.9.5}$$

(b)

$$L_{\rm GW}(t,r)=\int T^{\rm GW}_{rr}r^2\,{\rm d}\Omega=\frac{1}{8\pi}\left\langle \dddot{\mathit{I}}_{ij}\dddot{\mathit{I}}_{kl}\right\rangle\int\left(\delta_{ik}\delta_{jl}-2n_in_k\delta_{jl}+\frac{1}{2}n_in_jn_kn_l\right){\rm d}\Omega\,. \tag{36.9.6}$$

Using the fact that

$$\int{\rm d}\Omega\,n_in_j=\frac{4\pi}{3}\delta_{ij},\quad \int{\rm d}\Omega\,n_in_jn_kn_l=\frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{36.9.7}$$

we can derive that

$$L_{\rm GW}(t,r)=\frac{1}{5}\left\langle \dddot{\mathit{I}}_{ij}\dddot{\mathit{I}}_{ij}\right\rangle. \tag{36.9.8}$$

Similarly, for

$$\mathcal{J}^i=\frac{1}{8\pi r^2}\epsilon^{ijk}\left\langle -6n_j\ddot{\mathit{I}}_{km}\dddot{\mathit{I}}_{mp}n_p+9n_j\ddot{\mathit{I}}_{km}n_mn_p\dddot{\mathit{I}}_{pq}n_q\right\rangle, \tag{36.9.9}$$

we have

$$\int{\rm d}\Omega\,\mathcal{J}_jr^2=\frac{1}{2}\epsilon^{ijk}\left[\left\langle -2\ddot{\mathit{I}}_{km}\dddot{\mathit{I}}_{mj}\right\rangle+\left\langle\frac{3}{5}(\ddot{\mathit{I}}_{kq}\dddot{\mathit{I}}_{jq}+\ddot{\mathit{I}}_{kp}\dddot{\mathit{I}}_{pj})\right\rangle\right]=-\frac{2}{5}\epsilon^{ijk}\left\langle\ddot{\mathit{I}}_{km}\dddot{\mathit{I}}_{mj}\right\rangle=\frac{2}{5}\epsilon^{ijk}\left\langle\ddot{\mathit{I}}_{jm}\dddot{\mathit{I}}_{mk}\right\rangle. \tag{36.9.10}$$

Chapter 37

DETECTION OF GRAVITATIONAL WAVES

37.1 GENERAL PLANE WAVE IN TT GAUGE

Show that the most general linearized plane wave can be described in the transverse-traceless gauge of linearized theory by expressions (37.1).

Solution:

Under a gauge transformation generated by ξ^μ, we have

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\rho}{}_{,\rho}. \tag{37.1.1}$$

Suppose that ξ^μ is the solution of

$$\xi^{\mu}{}_{,\nu}{}^{\nu} = \bar{h}^{\mu\nu}{}_{,\nu}. \tag{37.1.2}$$

After the gauge transformation, we have

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0, \tag{37.1.3}$$

called Lorentz gauge.

In Lorentz gauge, a plane wave propagated along z direction $\bar{h}_{\mu\nu}(t-z)$ can be expressed as a superposition of monochromatic plane waves,

$$A_{\mu\nu}\exp(ik_\mu x^\mu) \quad \text{where} \quad A^{\mu\nu}k_\nu = 0, \quad k_\mu = k(-1,0,0,1), \tag{37.1.4}$$

or explicit $A^{\mu 0} = A^{\mu z}$.

Consider a gauge transformation generated by

$$\xi^\mu = -iC^\mu \exp(ik_\mu x^\mu). \tag{37.1.5}$$

Note that $\xi_{\mu,\nu} + \xi_{\nu,\mu} = (C_\mu k_\nu + C_\nu k_\mu)\exp[-ik(t-z)]$. We have the transformation

$$A_{\mu\nu} \to A_{\mu\nu} - C_\mu k_\nu - C_\nu k_\mu. \tag{37.1.6}$$

Now solve the equation for C_μ

$$A^{i0} - C^i k - C^0 k^i = 0, \quad A^{\mu}{}_{\mu} - 2C^\mu k_\mu = 0 \tag{37.1.7}$$

In new gauges, we have $A^{i0}=0$, $A^{\mu}{}_{\mu}=0$. Since transformation generated by 37.1 will keep the wave in Lorentz gauge, we have $A^{\mu 0}=A^{\mu z}$. Now we can get $A^{0\mu}=A^{z\mu}=0$ and $A^{\mu}{}_{\mu}=0$.

If we make gauge transformation for each mode of plane wave, the resulting wave must satisfy that

$$\bar{h}^{\mu\nu}{}_{,\nu}=0, \quad \bar{h}^{0\mu}=\bar{h}^{z\mu}=0, \quad \bar{h}=0. \tag{37.1.8}$$

So the most general linearized plane wave can be described in the transverse-traceless gauge of linearized theory by

$$h^{\mathrm{TT}}_{xx}=-h^{\mathrm{TT}}_{yy}=A_{+}(t-z), \quad h^{\mathrm{TT}}_{xy}=h^{\mathrm{TT}}_{yx}=A_{\times}(t-z) \tag{37.1.9}$$

In the TT gauge, the time-space components

$$R_{j0k0}=R_{0j0k}=-R_{j00k}=-R_{0jk0} \tag{37.1.10}$$

of the Riemann curvature tensor have an especially simple form (see exercise 18.4)

$$R_{j0k0}=-\frac{1}{2}h^{\mathrm{TT}}_{jk,00} \tag{37.1.11}$$

So we have

$$R_{x0x0}=-R_{y0y0}=-\frac{1}{2}\ddot{A}_{+}(t-z), \quad R_{x0y0}=R_{y0x0}=-\frac{1}{2}\ddot{A}_{\times}(t-z). \tag{37.1.12}$$

In the TT gauge, the stress energy tensor of gravitational wave is

$$T^{\mathrm{GW}}_{\alpha\beta}=\frac{1}{32\pi}\left\langle h_{\mu\nu,\alpha}h^{\mu\nu}{}_{,\beta}\right\rangle. \tag{37.1.13}$$

So we can get

$$T^{\mathrm{GW}}_{00}=T^{\mathrm{GW}}_{zz}=-T^{\mathrm{GW}}_{0z}=\frac{1}{16\pi}\left\langle \dot{A}^2_{+}+\dot{A}^2_{\times}\right\rangle. \tag{37.1.14}$$

37.2 TEST-PARTICLE MOTION IN PROPER REFERENCE FRAME

Show that a slowly moving test particle, falling freely through the proper reference frame of equation (37.2), obeys the equation of motion (geodesic equation)

$$\frac{\mathrm{d}^2x^{\hat{j}}}{\mathrm{d}t^2}=-a_{\hat{j}}+\mathcal{O}\left(\left|x^{\hat{k}}\right|\right). \tag{37.2.1}$$

Thus, one can interpret $-a_{\hat{j}}$ as the "local acceleration of gravity".

Solution:

The metric in the "proper reference frame of the detector" is

$$\mathrm{d}s^2=-\left(1+2a_{\hat{j}}x^{\hat{j}}\right)\left(\mathrm{d}x^{\hat{0}}\right)^2+\delta_{\hat{j}\hat{k}}\,\mathrm{d}x^{\hat{j}}\,dx^{\hat{k}}+\mathcal{O}\left(\left|x^{\hat{j}}\right|^2\right)\mathrm{d}x^{\hat{\alpha}}\,\mathrm{d}x^{\hat{\beta}} \tag{37.2.2}$$

So we have

$$\mathrm{d}s=\left[-\left(\mathrm{d}x^{\hat{0}}\right)^2+\delta_{\hat{j}\hat{k}}\,\mathrm{d}x^{\hat{j}}\,dx^{\hat{k}}\right]\left[1+\mathcal{O}\left(\left|x^{\hat{k}}\right|\right)\right] \tag{37.2.3}$$

and

$$\Gamma^{\hat{i}}{}_{00}=a_{\hat{i}}+\mathcal{O}\left(\left|x^{\hat{k}}\right|\right) \tag{37.2.4}$$

For a slowly moving test particle, we have $\mathrm{d}\tau = -\,\mathrm{d}s \approx \mathrm{d}t$. So the equation of motion

$$\frac{\mathrm{d}^2 x^{\hat{j}}}{\mathrm{d}\tau^2} + \Gamma^{\hat{i}}{}_{\hat{\alpha}\hat{\beta}} \frac{\mathrm{d}x^{\hat{\alpha}}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\hat{\beta}}}{\mathrm{d}\tau} = 0 \tag{37.2.5}$$

will become

$$\frac{\mathrm{d}^2 x^{\hat{j}}}{\mathrm{d}t^2} = -a_{\hat{j}} + \mathcal{O}\Big(\big|x^{\hat{k}}\big|\Big). \tag{37.2.6}$$

37.3 CONNECTION COEFFICIENTS IN PROPER REFERENCE FRAME

(a) Calculate $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$ for the metric 37.2.2, ignoring corrections of $\mathcal{O}\Big(|x^{\hat{j}}|\Big)$.

(b) Calculate $R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}$ using the standard formula (8.44), and leaving spatial derivatives of the connection coefficients unevaluated because of the unknown corrections of $\mathcal{O}\Big(|x^{\hat{j}}|\Big)$.

(c) Use the answer to part (b) to evaluate the $\mathcal{O}\Big(|x^{\hat{j}}|\Big)$ corrections to $\Gamma^{\hat{j}}{}_{\hat{0}\hat{0}}$.

Solution:

(a) Firstly, we have

$$g_{\hat{0}\hat{0},\hat{i}} = -2a_{\hat{i}} + \mathcal{O}\Big(|x^{\hat{j}}|\Big) \tag{37.3.1}$$

and

$$g_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = \mathcal{O}\Big(|x^{\hat{j}}|\Big) \tag{37.3.2}$$

for other components.

Using

$$\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = \frac{1}{2} g^{\hat{\gamma}\hat{\mu}}(g_{\hat{\mu}\hat{\beta},\hat{\gamma}} + g_{\hat{\mu}\hat{\gamma},\hat{\beta}} - g_{\hat{\beta}\hat{\gamma},\hat{\mu}}), \tag{37.3.3}$$

we can get

$$\Gamma^{\hat{0}}{}_{\hat{0}\hat{i}} = \Gamma^{\hat{0}}{}_{\hat{i}\hat{0}} = \Gamma^{\hat{i}}{}_{\hat{0}\hat{0}} = -\frac{1}{2} g_{\hat{0}\hat{0},\hat{i}} + \mathcal{O}\Big(|x^{\hat{j}}|\Big) = a_{\hat{i}} + \mathcal{O}\Big(|x^{\hat{j}}|\Big) \tag{37.3.4}$$

and

$$\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = \mathcal{O}\Big(|x^{\hat{j}}|\Big) \tag{37.3.5}$$

for other components.

(b)

$$\begin{aligned} R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} &= \Gamma^{\hat{j}}{}_{\hat{0}\hat{0},\hat{k}} - \Gamma^{\hat{j}}{}_{\hat{0}\hat{k},\hat{0}} + \Gamma^{\hat{j}}{}_{\hat{\alpha}\hat{k}}\Gamma^{\hat{\alpha}}{}_{\hat{0}\hat{0}} - \Gamma^{\hat{j}}{}_{\hat{\alpha}\hat{0}}\Gamma^{\hat{\alpha}}{}_{\hat{0}\hat{k}} \\ &= \Gamma^{\hat{j}}{}_{\hat{0}\hat{0},\hat{k}} - \Gamma^{\hat{j}}{}_{\hat{0}\hat{0}}\Gamma^{\hat{0}}{}_{\hat{0}\hat{k}} + \mathcal{O}\Big(|x^{\hat{j}}|\Big) \\ &= \Gamma^{\hat{j}}{}_{\hat{0}\hat{0},\hat{k}} - a_{\hat{j}}a_{\hat{k}} + \mathcal{O}\Big(|x^{\hat{j}}|\Big). \end{aligned} \tag{37.3.6}$$

(c)

$$\begin{aligned} \Gamma^{\hat{j}}{}_{\hat{0}\hat{0}} &= \Gamma^{\hat{j}}{}_{\hat{0}\hat{0}}(x^{\hat{i}} = 0) + \Gamma^{\hat{j}}{}_{\hat{0}\hat{0},\hat{k}}(x^{\hat{i}} = 0)x^{\hat{k}} + \mathcal{O}\Big(|x^{\hat{j}}|^2\Big) \\ &= a_{\hat{j}} + (R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} + a_{\hat{j}}a_{\hat{k}})x^{\hat{k}} + \mathcal{O}\Big(|x^{\hat{j}}|^2\Big) \end{aligned} \tag{37.3.7}$$

37.4 WHY THE $a \cdot x$?

Explain the origin of the $a \cdot x$ correction in equation (5b) of Box 37.1.

Solution:

Take the viewpoint of an observer at rest at the spatial origin who watches two freely falling particles respond to the inertial force. Suppose the force is at $\hat{\boldsymbol{z}}$ direction.

At time $\hat{t} = 0$, put the particle A at the origin and the particle B at $(0, 0, z_0)$. As time passes, the separation of the particles in their common Lorentz frame remains fixed; so there develops a Lorentz contraction in $\hat{\boldsymbol{z}}$ direction from the viewpoint of the observer fixed at the origin. So we have

$$z_{\rm B} - z_{\rm A} = z_0 \left(1 - v^2\right)^{1/2}, \tag{37.4.1}$$

where $v = at$.

Take the derivative with respect to time. We have

$$\frac{{\rm d}^2 z_{\rm B}}{{\rm d}\hat{t}^2} - \frac{{\rm d}^2 z_{\rm A}}{{\rm d}\hat{t}^2} = -\frac{z_0 a^2}{(1-v^2)^{3/2}} \approx -z_0 a^2 \quad \text{when} \quad \hat{t} \approx 0. \tag{37.4.2}$$

Because particle A is at the origin, we have

$$\frac{{\rm d}^2 z_{\rm A}}{{\rm d}\hat{t}^2} = a. \tag{37.4.3}$$

As a result,

$$\frac{{\rm d}^2 z_{\rm B}}{{\rm d}\hat{t}^2} = -a(1 + a z_0). \tag{37.4.4}$$

Or more generally, we have

$$\frac{{\rm d}^2 x^{\hat{j}}}{{\rm d}\hat{t}^2} = -a^{\hat{j}}(1 + \boldsymbol{a} \cdot \boldsymbol{x}). \tag{37.4.5}$$

37.5 ORIENTATION OF POLARIZATION DIAGRAM

Derive equation (2) of Box 37.2.

Solution:

New coordinates relate with old coordinates through

$$\hat{x} = \hat{x}' \cos\phi_0 - \hat{y}' \sin\phi_0, \quad \hat{y} = \hat{x}' \sin\phi_0 + \hat{y}' \cos\phi_0. \tag{37.5.1}$$

The equation of motion in will be

$$\frac{{\rm d}^2 \hat{x}' \cos\phi_0 - \hat{y}' \sin\phi_0}{{\rm d}\hat{t}^2} = \frac{1}{2}\left[\ddot{A}_+(\hat{x}' \cos\phi_0 - \hat{y}' \sin\phi_0) + \ddot{A}_\times(\hat{x}' \sin\phi_0 + \hat{y}' \cos\phi_0)\right] \tag{37.5.2}$$

and

$$\frac{{\rm d}^2 \hat{x}' \sin\phi_0 + \hat{y}' \cos\phi_0}{{\rm d}\hat{t}^2} = \frac{1}{2}\left[-\ddot{A}_+(\hat{x}' \sin\phi_0 + \hat{y}' \cos\phi_0) + \ddot{A}_\times(\hat{x}' \cos\phi_0 - \hat{y}' \sin\phi_0)\right]. \tag{37.5.3}$$

It can be simplified as

$$\frac{\mathrm{d}^2\hat{x}'}{\mathrm{d}\hat{t}^2} = \frac{1}{2}\left[(\ddot{A}_+\cos 2\phi_0 + \ddot{A}_\times \sin 2\phi_0)\hat{x}' + (-\ddot{A}_+\sin 2\phi_0 + \ddot{A}_\times\cos 2\phi_0)\hat{y}'\right] \tag{37.5.4}$$

and

$$\frac{\mathrm{d}^2\hat{y}'}{\mathrm{d}\hat{t}^2} = \frac{1}{2}\left[-(\ddot{A}_+\cos 2\phi_0 + \ddot{A}_\times \sin 2\phi_0)\hat{y}' + (-\ddot{A}_+\sin 2\phi_0 + \ddot{A}_\times\cos 2\phi_0)\hat{x}'\right] \tag{37.5.5}$$

We will get the same pattern in new coordinates if

$$-\ddot{A}_+\sin 2\phi_0 + \ddot{A}_\times\cos 2\phi_0 = 0. \tag{37.5.6}$$

We have the solution

$$\phi_0 = \frac{1}{2}\arctan\left(\frac{\ddot{A}_\times}{\ddot{A}_+}\right). \tag{37.5.7}$$

37.6 RELATIVE MOTION OF FREELY FALLING BODIES AS A DETECTOR OF GRAVITATIONAL WAVES

Consider two test bodies initially at rest with respect to each other in flat, empty spacetime. (The case where other, gravitating bodies are nearby can be treated without too much more difficulty; but this exercise concerns only the simplest example!) A plane, nearly monochromatic gravitational wave, with angular frequency ω and polarization $\boldsymbol{e}_+$, impinges on the bodies, coming from the $-z$ direction. As shown in exercise 35.5, the bodies remain forever at rest in those TT coordinates that constituted the bodies' global inertial frame before the wave arrived. Calculate, for arbitrary separations $(\Delta x, \Delta y, \Delta z)$ of the test bodies, the redshift and the round-trip travel time of photons going back and forth between them. Compare the answer, for large Δx, Δy, Δz, with the answer one would have obtained by using (without justification!) the equation of geodesic deviation. Physically, why does the correct answer oscillate with increasing separation? Discuss the feasibility and the potential sensitivity of such a detector using current technology.

Solution:

We can decompose the geodesic of the photon into a background path plus a perturbation,

$$x^\mu(\lambda) = x^\mu_{(0)}(\lambda) + x^\mu_{(1)}(\lambda), \tag{37.6.1}$$

where $x^\mu_{(0)}$ solves the geodesic equation in the background. For convenience we denote the wave vector of the background path as k_μ and the derivative of the deviation vector as l_μ. The condition that a path be null is of course

$$(\eta_{\mu\nu} + h_{\mu\nu})(l^\mu + k^\mu)(l^\nu + k^\nu) = 0. \tag{37.6.2}$$

At zeroth order we simply have $\eta_{\mu\nu}k^\mu k^\nu = 0$. We can obtain that $k^\mu = k(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ and ϕ are spherical coordinates of $(\Delta x, \Delta y, \Delta z)$. At first order we obtain

$$2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu = 0. \tag{37.6.3}$$

We now turn to the perturbed geodesic equation. The zeroth-order geodesic equation simply tells us that $x^\mu_{(0)}(\lambda)$ is a straight trajectory, while at first order we have

$$\frac{\mathrm{d}l^\mu}{\mathrm{d}\lambda} = -\Gamma^\mu{}_{\rho\sigma}k^\rho k^\sigma. \tag{37.6.4}$$

Particularly, we have for $\mu = 0$

$$\frac{\mathrm{d}l^0}{\mathrm{d}\lambda} = -(\Gamma^0{}_{xx}k^x k^x + \Gamma^0{}_{yy}k^y k^y), \tag{37.6.5}$$

where

$$\Gamma^0{}_{xx} = -\Gamma^0{}_{yy} = \frac{-i\omega}{2}A_+ e^{-i\omega(t-z)}. \tag{37.6.6}$$

Note that

$$x^\mu_{(0)}(\lambda) = k^\mu \lambda. \tag{37.6.7}$$

We have

$$\frac{\mathrm{d}l^0}{\mathrm{d}\lambda} = \frac{i\omega}{2}A_+ e^{-i\omega\lambda(k-k^z)}[(k^x)^2 - (k^y)^2] \tag{37.6.8}$$

The integration gives

$$\Delta l^0 = -\frac{A_+}{2}\frac{(k^x)^2 - (k^y)^2}{k - k^z}(e^{-i\omega\Delta\lambda(k-k^z)} - 1)e^{i\phi_0} = -\frac{kA_+}{2}\frac{\sin^2\theta\cos 2\phi}{1-\cos\theta}(e^{-i\omega\Delta r(1-\cos\theta)} - 1)e^{i\phi_0}, \tag{37.6.9}$$

where $\phi_0 = -\omega(t_0 - z_0)$ is the initial phase. The redshift of the photon is

$$z_1 = -\frac{l^0}{k} = \frac{A_+}{2}\cos 2\phi(1+\cos\theta)(e^{-i\omega\Delta r(1-\cos\theta)} - 1)e^{i\phi_0}. \tag{37.6.10}$$

For photon propagate back, we have

$$z_2 = -\frac{l^0}{k} = \frac{A_+}{2}\cos 2\phi(1-\cos\theta)(e^{-i\omega\Delta r(1+\cos\theta)} - 1)e^{-i\omega\Delta r(1-\cos\theta)}e^{i\phi_0}. \tag{37.6.11}$$

The total redshift is $z = z_1 + z_2$.

If $\omega\Delta r \ll 1$, we have

$$z = A_+\cos 2\phi\sin^2\theta\omega\Delta r e^{i(\phi_0 - \pi/2)}. \tag{37.6.12}$$

The correct answer oscillate with increasing separation because the spacetime is oscillate when the photon propagates.

37.7 EARTH-MOON SEPARATION AS A GRAVITATIONAL-WAVE DETECTOR

In the early 1970's one can monitor the Earth-moon separation using laser ranging to a precision of 10 cm, with successive observations separated by at least one round-trip travel time. Suppose that no oscillations in round-trip travel time are observed except those (of rather long periods) to be expected from the Earth-moon-sun-planets gravitational interaction. What limits can one then place on the energy flux of gravitational waves that pass the Earth? The mathematical formula for the answer should yield numerically

$$\text{Flux} \lesssim 10^{18}\mathrm{erg/cm^2\,sec} \text{ for } 0.3\,\mathrm{cycle/sec} \lesssim \nu \lesssim 1\,\mathrm{cycle/sec} \tag{37.7.1}$$

corresponding to a limit on the mass density in gravitational waves of

$$\text{Density} \lesssim 10^{-13}\mathrm{g/cm^3}. \tag{37.7.2}$$

Why is this an uninteresting limit?

Solution:

The Earth-moon separation is about 3.84×10^{10} cm. With a 10 cm precision, the upper limit of the amplitude of gravitational wave is

$$h = \frac{10}{3.84 \times 10^{10}} = 2.6 \times 10^{-10}. \tag{37.7.3}$$

The corresponding energy flux is

$$\frac{1}{32\pi} \langle h_{jk,\mu} h_{jk,\nu} \rangle \sim \frac{\pi \nu^2 h^2 c^3}{16G} \sim 10^{18} \mathrm{erg/cm^2\,sec}. \tag{37.7.4}$$

The corresponding density is

$$\rho \sim \frac{\pi \nu^2 h^2}{16G} \sim 10^{-13} \mathrm{g/cm^3}. \tag{37.7.5}$$

The energy flux of solar radiation is $1.368 \times 10^6 \mathrm{erg/cm^2\,sec}$. So this upper limit is too high to be interesting.

37.8 POWER RERADIATED

The idealized gravitational wave detector of Figure 37.4 vibrates with angular frequency ω. Show that the power it radiates as gravitational waves is given by equation (37.37).

Solution:

Put the detector on the x axis. We have

$$\rho = M\delta(x - \xi - L)\delta(y)\delta(z) + M\delta(x + \xi + L)\delta(y)\delta(z). \tag{37.8.1}$$

The second moment of mass distribution is

$$I_{xx} = 2M(L + \xi)^2 \tag{37.8.2}$$

and all other components vanish. The reduced quadruple is

$$\text{Ŧ}_{xx} = \frac{4}{3}M(L + \xi)^2, \quad \text{Ŧ}_{yy} = -\frac{2}{3}M(L + \xi)^2, \quad \text{Ŧ}_{zz} = -\frac{2}{3}M(L + \xi)^2. \tag{37.8.3}$$

The power of radiation is

$$P = \frac{1}{5} \langle \dddot{\text{Ŧ}}_{ij} \dddot{\text{Ŧ}}_{ij} \rangle = \frac{8M^2}{15} \left\langle \frac{\mathrm{d}^3 (L + \xi)^2}{\mathrm{d}t^3} \frac{\mathrm{d}^3 (L + \xi)^2}{\mathrm{d}t^3} \right\rangle \tag{37.8.4}$$

If ξ is mush smaller of L, we have

$$P = \frac{32M^2L^2}{15} \left\langle \frac{\mathrm{d}^3 \xi}{\mathrm{d}t^3} \frac{\mathrm{d}^3 \xi}{\mathrm{d}t^3} \right\rangle = \frac{32M^2L^2\omega^6}{15} \left\langle \xi^2 \right\rangle. \tag{37.8.5}$$

37.9 CROSS SECTIONS CALCULATED BY DETAILED BALANCE

Use the principle of detailed balance to derive the cross sections (37.41) for a vibrating, resonant detector of any size, shape, or mass (e.g., for the vibrating Earth, or Weber's vibrating cylinder, or the idealized detector of Figure 37.4).

Solution:

Let the detector be in thermal equilibrium with a bath of blackbody gravitational waves. Then it must be losing energy by reradiation as rapidly as it is absorbing it from the waves. Internal damping can be ignored because, in true thermal equilibrium, energy loss by internal damping will match energy gain from random internal Brownian forces. In detail, the balance of energy in and out reads

$$[4\pi I_\nu(\nu_0)]_{\text{blackbody}} \times \int \langle\sigma\rangle_{\text{all directions}}\, \mathrm{d}\nu = A_{\text{GW}} \times (\text{Energy in normal mode of detector}). \tag{37.9.1}$$

Using the familiar form of the Planck spectrum and the fact that gravitational waves have two independent states of polarization, we have

$$I_\nu = \frac{2h\nu^3}{\exp[h\nu_i/kT]-1}. \tag{37.9.2}$$

On the other hand, the average number of quanta (phonon) for a normal mode with frequency ν_i of detector is given by

$$n_i = \frac{1}{\exp[h\nu/kT]-1}. \tag{37.9.3}$$

Thus, the energy in normal mode with frequency ν_i of detector is

$$E_i = \frac{h\nu_i}{\exp[h\nu_i/kT]-1}. \tag{37.9.4}$$

Put them together, we can obtain

$$\int \langle\sigma\rangle_{\text{all directions}}\, \mathrm{d}\nu = \frac{A_{\text{GW}}}{8\pi\nu_0^2} = \frac{\pi}{2}\lambda\!\!\!^{-2}_0 A_{\text{GW}}. \tag{37.9.5}$$

37.10 NORMAL-MODE ANALYSIS OF VIBRATING, RESONANT DETECTORS

Derive all the results for vibrating, resonant detectors quoted in Box 37.4. Pattern the derivation after the treatment of the idealized detector in section 37.6.

Solution:

(a) Let the detector be driven by the polarized waves of equation

$$h_{jk} = A(t - \boldsymbol{n}\cdot\boldsymbol{x})e_{jk}, \quad e_{jk}n_k = 0, \quad e_{jj} = 0, \quad e_{jk}e_{jk} = 2, \tag{37.10.1}$$

and let it be wave-dominated ($E_{\text{vibration}} \ll kT$). The displacements $\delta\boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{x}, t)$ of its mass elements satisfy that

$$\ddot{\boldsymbol{\xi}} + \frac{\dot{\boldsymbol{\xi}}}{\tau_0} + \omega_0^2\boldsymbol{\xi} = \frac{1}{2}\ddot{A}\mathbf{e}\cdot\boldsymbol{x}. \tag{37.10.2}$$

We decompose the displacements into normal modes of the detector as

$$\boldsymbol{\xi} = \sum_n B_n(t)\boldsymbol{u}_n(\boldsymbol{x}). \tag{37.10.3}$$

Using the orthonormalization condition

$$\int \rho\boldsymbol{u}_m\cdot\boldsymbol{u}_n \mathrm{d}^3x = M\delta_{mn}, \tag{37.10.4}$$

multiplying 37.10.2 by $\rho\boldsymbol{u}_n$ and integrating over the whole space, we can get

$$\ddot{B}_n + \frac{\dot{B}_n}{\tau_n} + \omega_n^2 B_n = \frac{1}{2}\ddot{A}\int \frac{\rho}{M}u_n^i e_{ij}x^j\, \mathrm{d}^3x\,. \tag{37.10.5}$$

Note that $\omega_0 = \omega_n$ and $\tau_0 = \tau_n$ for nth mode. Using the definition of the moment of inertia factor for the nth normal mode

$$I_{(n)jk} \equiv \int -(\rho u^l_n)_{,l} x^j x^k \mathrm{d}^3x = \int \rho(u^j_n x^k + u^k_n x^j)\,\mathrm{d}^3x\,, \tag{37.10.6}$$

and corresponding reduced quadrupole factor for the nth normal mode

$$Ⱶ_{(n)jk} \equiv I_{(n)jk} - \frac{1}{3} I_{(n)ll}\delta_{jk}, \tag{37.10.7}$$

equation 37.10.5 can be written as

$$\ddot{B}_n + \frac{\dot{B}_n}{\tau_n} + \omega_n^2 B_n = \frac{1}{4M}\ddot{A} I_{(n)jk} e_{ij} = \frac{1}{4M}\ddot{A} Ⱶ_{(n)ij} e_{ij}. \tag{37.10.8}$$

(b) Fourier-analyze the amplitudes of the detector and waves,

$$B_n(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \tilde{B}_n(\omega) e^{-i\omega t}\,\mathrm{d}\omega\,, \quad A(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \tilde{A}(\omega) e^{-i\omega t}\,\mathrm{d}\omega\,, \tag{37.10.9}$$

and substitute 37.10.9 into 37.10.8. We can derive that

$$-\omega^2\tilde{B}_n - i\frac{\omega}{\tau_n}\tilde{B}_n + \omega_n^2\tilde{B}_n = -\frac{1}{4M}\omega^2\tilde{A}Ⱶ_{(n)ij}e_{ij}. \tag{37.10.10}$$

The solution is

$$\tilde{B}_n = \frac{\omega^2\tilde{A}Ⱶ_{(n)ij}e_{ij}}{4M(\omega^2 + i\omega/\tau_n - \omega_n^2)}. \tag{37.10.11}$$

If $|\omega \pm \omega_n| \ll \omega_n$, we have the approximation

$$\tilde{B}_n = \frac{\omega_n\tilde{A}Ⱶ_{(n)ij}e_{ij}}{8M(|\omega| + i/2\tau_n - \omega_n)}. \tag{37.10.12}$$

(c) The total energy deposited in the detector can be calculated by integrating

$$(\text{energy deposited}) = \int (\textbf{Force per unit volume})\cdot(\textbf{Velocity})\,\mathrm{d}^3x\,\mathrm{d}t\,. \tag{37.10.13}$$

For nth mode, the force per unit volume is

$$f_i = \frac{1}{2}\ddot{A}\rho e_{ij}x^j. \tag{37.10.14}$$

The velocity of a small volume is

$$v_i = \dot{B}_n u^i_n. \tag{37.10.15}$$

It follows that

$$(\text{energy deposited in } n\text{th normal mode}) = \frac{1}{2}\int \ddot{A}\rho e_{ij}x^j\dot{B}_n u^i_n\,\mathrm{d}^3x\,\mathrm{d}t = \frac{1}{4}Ⱶ_{(n)ij}e_{ij}\int \ddot{A}\dot{B}_n\,\mathrm{d}t\,. \tag{37.10.16}$$

(d) Apply Parseval's theorem to 37.10.16, we have

$$\int \ddot{A}\dot{B}_n\,\mathrm{d}t = \int -\omega^2\tilde{A}^*(-i\omega\tilde{B})\,\mathrm{d}\omega\,. \tag{37.10.17}$$

Using 37.10.12, we can obtain

$$\begin{aligned}
&(\text{energy deposited in } n\text{th normal mode})\\
&= \frac{1}{4}Ⱶ_{(n)ij}e_{ij}\int i\omega^3\frac{\omega_n|\tilde{A}|^2Ⱶ_{(n)ij}e_{ij}}{8M(|\omega| + i/2\tau_n - \omega_n)}\,\mathrm{d}\omega\\
&= \frac{1}{4}|Ⱶ_{(n)ij}e_{ij}|^2\int i\omega^3\frac{\omega_n|\tilde{A}|^2(|\omega| - \omega_n - i/2\tau_n)}{8M[(|\omega| - \omega_n)^2 + (1/2\tau_n)^2]}\,\mathrm{d}\omega\\
&\approx \frac{1}{4}|Ⱶ_{(n)ij}e_{ij}|^2\int i\frac{\omega_n^4|\tilde{A}(\omega_n)|^2(|\omega| - \omega_n - i/2\tau_n)}{8M[(|\omega| - \omega_n)^2 + (1/2\tau_n)^2]}\,\mathrm{d}\omega\,.
\end{aligned} \tag{37.10.18}$$

Since

$$\int \frac{|\omega| - \omega_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}\,\mathrm{d}\omega \tag{37.10.19}$$

must vanish because of symmetry consiseration, we have

$$(\text{energy deposited in } n\text{th normal mode}) = \int \frac{\pi}{4}\frac{|\mathcal{H}_{(n)ij}e_{ij}|^2}{M}\frac{\omega_n^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}\frac{\omega^2|\tilde{A}|^2}{8}\,\mathrm{d}\nu\,, \tag{37.10.20}$$

where $\nu = \omega/2\pi$.

(e) The energy flux of the gravitational wave is

$$T^{tn} = \frac{1}{16\pi}\left\langle \dot{A}^2 \right\rangle_{\text{time avg}}. \tag{37.10.21}$$

The total energy per unit area is

$$\mathcal{E} = \int T^{tn}\,\mathrm{d}t = \frac{1}{16\pi}\int \dot{A}^2\,\mathrm{d}t\,. \tag{37.10.22}$$

Using Parseval theorem, we have

$$\mathcal{E} = \frac{1}{16\pi}\int \omega^2|\tilde{A}|^2\,\mathrm{d}\omega = \int \frac{\omega^2|\tilde{A}|^2}{8}\,\mathrm{d}\nu\,. \tag{37.10.23}$$

Thus the total energy per unit area per unit frequency carried by the waves past the detector is

$$\mathcal{F}_\nu = \frac{\omega^2|\tilde{A}|^2}{8}. \tag{37.10.24}$$

From the definition of scattering cross section, we have

$$(\text{energy deposited in } n\text{th normal mode}) = \int \sigma_n(\nu)\mathcal{F}_\nu\,\mathrm{d}\nu\,. \tag{37.10.25}$$

Using equation 37.10.20 and 37.10.24, we can conclude that

$$\sigma_n(\nu) = \frac{\pi}{4}\frac{|\mathcal{H}_{(n)ij}e_{ij}|^2}{M}\frac{\omega_n^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}. \tag{37.10.26}$$

It follows that

$$\int_{\text{resonance}} \sigma_n(\nu)\,\mathrm{d}\nu \approx \frac{1}{4}\frac{|\mathcal{H}_{(n)ij}e_{ij}|^2\omega_n^2}{M}\int \frac{\mathrm{d}x}{x^2+1} = \frac{\pi}{4}\frac{|\mathcal{H}_{(n)ij}e_{ij}|^2\omega_n^2}{M}. \tag{37.10.27}$$

(f) For simplicity, we suppose the $\boldsymbol{n}$ is in the z direction. For unpolarized radiation (random mixture of polarizations) with propagation direction $\boldsymbol{n}$, we have

$$e_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{37.10.28}$$

with θ uniformly distributed within $[0, 2\pi)$. Thus,

$$\begin{aligned}\left\langle |\mathcal{H}_{(n)ij}e_{ij}|^2 \right\rangle &= \left\langle |\mathcal{H}_{(n)xx}\cos\theta - \mathcal{H}_{(n)yy}\cos\theta + 2\mathcal{H}_{(n)xy}\sin\theta|^2 \right\rangle \\ &= \frac{|\mathcal{H}_{(n)xx} - \mathcal{H}_{(n)yy}|^2 + 4|\mathcal{H}_{(n)xy}|^2}{2}.\end{aligned} \tag{37.10.29}$$

Recall that

$$\mathcal{H}^{\mathrm{TT}}_{(n)} = P\mathcal{H}_{(n)}P - \frac{1}{2}P\,\mathrm{Tr}\big(P\mathcal{H}_{(n)}\big), \quad P_{jk} = \delta_{jk} - n_j n_k. \tag{37.10.30}$$

It follows that

$$\mathcal{H}^{\mathrm{TT}}_{(n)xx} = -\mathcal{H}^{\mathrm{TT}}_{(n)yy} = \frac{1}{2}(\mathcal{H}_{(n)xx} - \mathcal{H}_{(n)yy}), \quad \mathcal{H}^{\mathrm{TT}}_{(n)xy} = \mathcal{H}_{(n)xy}, \tag{37.10.31}$$

and all other components vanish. Thus, we have

$$\left\langle |\mathcal{I}_{(n)ij} e_{ij}|^2 \right\rangle = |\mathcal{I}^{\mathrm{TT}}_{(n)ij}|^2 \tag{37.10.32}$$

and so

$$\sigma_n(\nu) = \frac{\pi}{4} \frac{|\mathcal{I}^{\mathrm{TT}}_{(n)ij}|^2}{M} \frac{\omega_n^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}, \quad \int_{\text{resonance}} \sigma_n(\nu)\,\mathrm{d}\nu = \frac{\pi}{4} \frac{|\mathcal{I}^{\mathrm{TT}}_{(n)ij}|^2 \omega_n^2}{M}. \tag{37.10.33}$$

(g) From equation 37.10.30, we have

$$\begin{aligned}
\mathcal{I}^{\mathrm{TT}}_{ij} \mathcal{I}^{\mathrm{TT}}_{ij} &= \left(P_{ia} \mathcal{I}_{ab} P_{bj} - \frac{1}{2} P_{ij} P_{ab} \mathcal{I}_{ab} \right) \left(P_{ic} \mathcal{I}_{cd} P_{dj} - \frac{1}{2} P_{ij} P_{cd} \mathcal{I}_{cd} \right) \\
&= P_{ac} P_{bd} \mathcal{I}_{ab} \mathcal{I}_{cd} - \frac{1}{2} P_{ab} P_{cd} \mathcal{I}_{ab} \mathcal{I}_{cd} \\
&= \left[(\delta_{ac} - n_a n_c)(\delta_{bd} - n_b n_d) - \frac{1}{2} (\delta_{ab} - n_a n_b)(\delta_{cd} - n_c n_d) \right] \mathcal{I}_{ab} \mathcal{I}_{cd} \\
&= \left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac} n_b n_d + \delta_{ab} n_c n_d + \frac{1}{2} n_a n_b n_c n_d \right) \mathcal{I}_{ab} \mathcal{I}_{cd} \\
&= \mathcal{I}_{ab} \mathcal{I}_{ab} - 2 \mathcal{I}_{ab} \mathcal{I}_{ad} n_b n_d + \frac{1}{2} n_a n_b n_c n_d \mathcal{I}_{ab} \mathcal{I}_{cd}.
\end{aligned} \tag{37.10.34}$$

When averaged over all directions, we have

$$\left\langle \mathcal{I}^{\mathrm{TT}}_{ij} \mathcal{I}^{\mathrm{TT}}_{ij} \right\rangle_{\text{direction avg}} = \frac{1}{4\pi} \mathcal{I}_{ij} \mathcal{I}_{kl} \int \left(\delta_{ik}\delta_{jl} - 2 n_i n_k \delta_{jl} + \frac{1}{2} n_i n_j n_k n_l \right) \mathrm{d}\Omega\,. \tag{37.10.35}$$

Using the fact that

$$\frac{1}{4\pi} \int \mathrm{d}\Omega\, n_i n_j = \frac{1}{3}\delta_{ij}, \quad \frac{1}{4\pi} \int \mathrm{d}\Omega\, n_i n_j n_k n_l = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{37.10.36}$$

we can derive that

$$\left\langle \mathcal{I}^{\mathrm{TT}}_{ij} \mathcal{I}^{\mathrm{TT}}_{ij} \right\rangle_{\text{direction avg}} = \frac{2}{5} \mathcal{I}_{ij} \mathcal{I}_{ij} \tag{37.10.37}$$

For unpolarized radiation, averaged over all directions, cross sections are

$$\sigma_n(\nu) = \frac{\pi}{10} \frac{|\mathcal{I}_{(n)ij}|^2}{M} \frac{\omega_n^2/\tau_n}{(|\omega| - \omega_n)^2 + (1/2\tau_n)^2}. \tag{37.10.38}$$

For nth normal mode, we have

$$\delta I_{jk} = I_{(n)jk} B_n(t), \quad \delta \boldsymbol{x} = B_n(t) \boldsymbol{u}_n. \tag{37.10.39}$$

The power of radiated gravitational waves are

$$L_{\mathrm{GW}} = \frac{1}{5} \left\langle \dddot{\delta\mathcal{I}}_{ij} \dddot{\delta\mathcal{I}}_{ij} \right\rangle_{\text{time avg}} = \frac{1}{5} |\mathcal{I}_{(n)ij}|^2 \omega_n^6 \left\langle B_n^2 \right\rangle_{\text{time avg}}. \tag{37.10.40}$$

The kinetic energy of the detector is

$$E_{\mathrm{kin}} = \frac{1}{2} \int \rho (\delta \dot{\boldsymbol{x}})^2\, \mathrm{d}^3 x = \frac{1}{2} \omega_n^2 B_n^2 \int \rho \boldsymbol{u}_n \cdot \boldsymbol{u}_n\, \mathrm{d}^3 x = \frac{1}{2} M \omega_n^2 B_n^2. \tag{37.10.41}$$

The total energy of the detector is

$$E_{\mathrm{det}} = 2 \left\langle E_{\mathrm{kin}} \right\rangle = M \omega_n^2 \left\langle B_n^2 \right\rangle_{\text{time avg}}. \tag{37.10.42}$$

The Einstein A coefficients for gravitational wave radiation is

$$A_{\mathrm{GW}} = \frac{L_{\mathrm{GW}}}{E_{\mathrm{det}}} = \frac{1}{5} \frac{|\mathcal{I}_{(n)ij}|^2}{M} \omega_n^4. \tag{37.10.43}$$

The Einstein A coefficients for internal damping is

$$A_{\rm diss} = \frac{1}{\tau_0}. \tag{37.10.44}$$

Now we can write down

$$\sigma_n(\nu) = \frac{\pi}{2}\lambda\!\!\!^{-2}\frac{A_{\rm GW}A_{\rm diss}}{(|\omega| - \omega_n)^2 + (A_{\rm diss}/2)^2}. \tag{37.10.45}$$

and

$$\int_{\rm resonance} \sigma_n(\nu)\,{\rm d}\nu = \frac{\pi}{2}\lambda\!\!\!^{-2}A_{\rm GW}. \tag{37.10.46}$$

37.11 SPECTRUM OF ENERGY RADIATED BY A SOURCE

Derive the results quoted in the last section of Box 37.4.

Solution:

The component of polarization e_{ij} of a gravitational wave $h_{ij}^{\rm TT}$ is

$$\frac{1}{2}\,{\rm Tr}(h^{\rm TT}e)e_{ij}. \tag{37.11.1}$$

Noting that

$$h^{\rm TT} = PhP - \frac{1}{2}P\,{\rm Tr}(Ph), \quad P_{jk} = \delta_{jk} - n_jn_k \tag{37.11.2}$$

and

$$e_{jk}n_k = 0, \quad e_{jj} = 0, \quad e_{jk}e_{jk} = 2, \tag{37.11.3}$$

we have

$${\rm Tr}(h^{\rm TT}e) = {\rm Tr}(PhPe) - \frac{1}{2}\,{\rm Tr}(Ph)\,{\rm Tr}(Pe) = {\rm Tr}(Phe) - \frac{1}{2}\,{\rm Tr}(Ph)\,{\rm Tr}\,e = {\rm Tr}(ePh) = {\rm Tr}\{he\}. \tag{37.11.4}$$

The gravitational wave generated by the source is given by

$$h_{ij} = \frac{2}{r}\ddot{\mathcal{I}}_{ij}. \tag{37.11.5}$$

So the energy flux of gravitational wave with polarization tensor e_{ij} is

$$-T_{tr} = -\frac{1}{32\pi}\left\langle {\rm Tr}(he)_{,t}\,{\rm Tr}(he)_{,r}\right\rangle \frac{e_{ij}e_{ij}}{4} = \frac{1}{16\pi r^2}\left\langle {\rm Tr}\big(\dddot{\mathcal{I}}e\big)\,{\rm Tr}\big(\dddot{\mathcal{I}}e\big)\right\rangle, \tag{37.11.6}$$

i.e.,

$$\frac{{\rm d}E}{{\rm d}t\,{\rm d}\Omega} = \frac{1}{16\pi}\left\langle |\dddot{I}_{ij}e_{ij}|^2\right\rangle. \tag{37.11.7}$$

Using Parseval's theorem, we can obtain

$$\frac{{\rm d}E}{{\rm d}\Omega} = \frac{1}{16\pi}\int |\dddot{I}_{ij}e_{ij}|^2\,{\rm d}t = \frac{1}{16\pi}\int |\tilde{I}_{ij}e_{ij}|^2\omega^6\,{\rm d}\omega\,. \tag{37.11.8}$$

So the total energy per unit frequency ($\nu > 0$) radiated over all time, into a unit solid angle about the direction $\boldsymbol{n}$, and with polarization tensor $\boldsymbol{e}$, is

$$\frac{{\rm d}E}{{\rm d}\nu\,{\rm d}\Omega} = \frac{1}{8}\sum_{\omega=\pm 2\pi\nu} |\tilde{I}_{ij}e_{ij}|^2\omega^6. \tag{37.11.9}$$

Summed over polarizations, we have

$$-T_{tr} = -\frac{1}{32\pi}\left\langle h^{\mathrm{TT}}_{ij,t} h^{\mathrm{TT}}_{ij,r}\right\rangle = \frac{1}{8\pi r^2}\left\langle \dddot{I}^{\mathrm{TT}}_{ij}\dddot{I}^{\mathrm{TT}}_{ij}\right\rangle. \tag{37.11.10}$$

By similar method, we have

$$\frac{\mathrm{d}E}{\mathrm{d}\nu\,\mathrm{d}\Omega} = \frac{1}{4}\sum_{\omega=\pm 2\pi\nu,i,j} |\tilde{I}^{\mathrm{TT}}_{ij}|^2\omega^6 = \frac{1}{2} = \frac{1}{2}\sum_{i,j}|\tilde{I}^{\mathrm{TT}}_{ij}|^2\omega^6. \tag{37.11.11}$$

Using equation 37.10.5, the total energy radiated per unit frequency, integrated over all directions, is

$$\frac{\mathrm{d}E}{\mathrm{d}\nu} = \frac{1}{2}\sum_{i,j} 4\pi \times \frac{2}{5}|\tilde{I}_{ij}|^2\omega^6 = \frac{4\pi}{5}\sum_{i,j}|\tilde{I}_{ij}|^2\omega^6. \tag{37.11.12}$$

37.12 PATTERNS OF EMISSION AND ABSORPTION

The elementary dumbbell oscillator of Figure 37.4, initially unexcited, has a cross section for absorption of unpolarized gravitational radiation proportional to $\sin^4\theta$, and when excited radiates with an intensity also proportional to $\sin^4\theta$ (Chapter 36). The patterns of emission and absorption are identical. Any other dumb bell oscillator gives the same pattern, apart from a possible difference of orientation. Consider a nonrotating oscillator of general shape undergoing free vibrations in a single nondegenerate (and therefore nonrotatory) mode, or excited from outside by unpolarized radiation.

(a) Show that its pattern of emission is identical with its pattern of absorption. [Hint: Make the comparisons suggested in the last few parts of Box 37.4.]

(b) Show that this emission pattern (absorption pattern), apart from three Euler angles that describe the orientation of this pattern in space, and apart from a fourth parameter that determines total intensity, is uniquely fixed by a single ("fifth") parameter.

(c) Construct diagrams for the pattern of intensity for the two extreme values of this parameter and for a natural choice of parameter intermediate between these two extremes.

(d) Define the parameter in question in terms of a certain dimensionless combination of the principal moments of the reduced quadrupole tensor.

Solution:

(a) For unpolarized radiation with propagation direction $\boldsymbol{n}$, the cross section of absorption of gravitational waves is

$$\sigma_n(\nu) = \frac{\pi}{4}\frac{|I^{\mathrm{TT}}_{(n)ij}|^2}{M}\frac{\omega_n^2/\tau_n}{(|\omega|-\omega_n)^2 + (1/2\tau_n)^2}. \tag{37.12.1}$$

The total energy per unit frequency ($\nu > 0$) radiated over all time, into a unit solid angle about the direction $\boldsymbol{n}$, and summed over polarizations, is

$$\frac{\mathrm{d}E}{\mathrm{d}\nu\,\mathrm{d}\Omega} = \frac{1}{2}|\tilde{I}^{\mathrm{TT}}_{ij}|^2\omega^6. \tag{37.12.2}$$

Since

$$\delta I_{jk} = I_{(n)jk}\mathcal{B}_n e^{-i\omega_n t - t/(2\tau_n)} \quad \text{for} \quad t > 0, \tag{37.12.3}$$

we have

$$\tilde{I}_{ij} = I_{(n)jk}\mathcal{B}_n \frac{1}{1/(2\tau_n) - i(\omega-\omega_n)}. \tag{37.12.4}$$

and so

$$|\tilde{\mathcal{I}}_{ij}^{\rm TT}|^2 = |\mathcal{I}_{(n)ij}^{\rm TT}|^2 |\mathcal{B}_n|^2 \frac{1}{(\omega-\omega_n)^2 + (1/2\tau_n)^2}. \tag{37.12.5}$$

Thus, its pattern of emission is identical with its pattern of absorption.

(b) The emission pattern is determined by $\mathcal{I}_{(n)jk}$. Adjusting three Euler angles that describe the orientation of this pattern in space properly, we can bring $\mathcal{I}_{(n)jk}$ into diagonal form as

$$\mathcal{I}_{(n)jk} \propto \mathcal{J} = \begin{pmatrix} -x & & \\ & x-1 & \\ & & 1 \end{pmatrix} \tag{37.12.6}$$

where $0 \le x \le 1/2$. Apart from a fourth parameter that determines total intensity, the pattern is uniquely fixed by a single parameter a.

(c) The pattern of the emission is determined by $|\mathcal{J}_{ij}^{\rm TT}|^2$. Suppose $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We have

$$\begin{aligned} |\mathcal{J}_{ij}^{\rm TT}|^2 = {} & \frac{1}{2}\left[\left(4\cos^4\phi - 4\cos^2\phi + 1\right)\sin^4\theta - 4\sin^2\theta + 4\right] x^2 \\ & - \left[\left(2\cos^4\phi - 5\cos^2\phi + 2\right)\sin^4\theta + \left(6\cos^2\phi - 5\right)\sin^2\theta + 2\right] x \\ & + \frac{1}{2}\left(\cos^4\phi - 4\cos^2\phi + 4\right)\sin^4\theta + \left(3\cos^2\phi - 2\right)\sin^2\theta + \frac{1}{2} \end{aligned} \tag{37.12.7}$$

For $x = 0$, we have

$$|\mathcal{J}_{ij}^{\rm TT}|^2 = \frac{1}{2}\left(\cos^4\phi - 4\cos^2\phi + 4\right)\sin^4\theta + \left(3\cos^2\phi - 2\right)\sin^2\theta + \frac{1}{2}. \tag{37.12.8}$$

For $x = 1/2$, we have

$$|\mathcal{J}_{ij}^{\rm TT}|^2 = \frac{9}{8}\sin^4\theta. \tag{37.12.9}$$

(d) The parameter can be defined as

$$x = \frac{\min(|I_i|)}{\max(|I_i|)}, \tag{37.12.10}$$

where I_1, I_2 and I_3 is the principal moments of the reduced quadrupole tensor.

37.13 MULTIMODE DETECTOR

Consider a cylindrical bar of length very long compared to its diameter. Designate the fundamental mode of end-to-end vibration of the bar as $n = 1$, and call the mode with $n - 1$ nodes in its eigenfunction the "nth mode". Show that the cross section for the interception of unpolarized gravitational waves at the nth resonance, integrated over that resonance, and averaged over direction, is given by the formula [Ruffini and Wheeler (1971b)]

$$\int_{n\text{th resonance, random}} \sigma(\nu)\,\mathrm{d}\nu = \frac{32}{15\pi}\frac{v^2}{c^2}\frac{M}{n^2} \text{ for } n \text{ odd (zero for even } n), \tag{37.13.1}$$

where v is the speed of sound in the bar expressed in the same units as the speed of light, c; and M is the mass of the bar (geometric units; multiply the righthand side by the factor $G/c = 2.22 \times 10^{-18}\mathrm{cm^2\,Hz/g}$ when employing conventional units). Show that this expression gives $\int \sigma\,\mathrm{d}\nu = 1.0 \times 10^{-21}\mathrm{cm^2\,Hz}$ for the lowest mode of Weber's bar. Multimode detectors are (1973) under construction by William Fairbank and William Hamilton, and by David Douglass and John A. Tyson.

Solution:

For nth mode, we have

$$u_n^x = \sqrt{2}\cos\left[\left(\frac{x}{L}+\frac{1}{2}\right)n\pi\right], \quad \omega_n = \frac{\pi n v}{L}. \tag{37.13.2}$$

The "moment of inertia factor for the nth normal mode" is

$$I_{(n)xx} = \frac{2M}{L}\int_{-L/2}^{L/2}\sqrt{2}\cos\left[\left(\frac{x}{L}+\frac{1}{2}\right)n\pi\right]x\,\mathrm{d}x = 4\sqrt{2}ML\begin{cases}\frac{-1}{n^2\pi^2} & \text{for} \quad n \text{ odd}\\ 0 \quad \text{for} \quad n \text{ even}\end{cases} \tag{37.13.3}$$

So for n odd, we have

$$\int_{n\text{th resonance, random}}\sigma(\nu)\,\mathrm{d}\nu = \frac{\pi}{10}\frac{|\text{Ɨ}_{jk}|^2}{M}\omega_n^2 = \frac{\pi}{15}\frac{I_{(n)xx}^2}{M}\omega_n^2 = \frac{32Mv^2}{15\pi n^2}. \tag{37.13.4}$$

For Weber's bar, we have

$$M = 1.4\times10^6\mathrm{g}, \quad v = \nu\lambda = 153\mathrm{cm}\times2\times1660\mathrm{Hz} = 5.0796\times10^5\mathrm{cm/s}. \tag{37.13.5}$$

So we obtain

$$\int\sigma\,\mathrm{d}\nu = \frac{32}{15\pi}\frac{GMv^2}{c^3} = 6\times10^{-22}\mathrm{cm}^2\,\mathrm{Hz}. \tag{37.13.6}$$

37.14 CROSS SECTION OF IDEALIZED MODEL OF EARTH FOR ABSORPTION OF GRAVITATIONAL RADIATION

The observed period of quadrupole vibration of the earth is 54 minutes [see, e.g., Bolt (1964) or Press (1965) for survey and bibliography]. To analyze that mode of vibration, with all due allowance for elasticity and the variation of density in the earth, is a major enterprise. Therefore, for a first estimate of the cross section of the earth for the absorption of quadrupole radiation, treat it as a globe of fluid of uniform density held in the shape of a sphere by gravitational forces alone (zero rigidity). Let the surface be displaced from $r = a$ to

$$r = a + a\alpha P_2(\cos\theta), \tag{37.14.1}$$

where θ is polar angle measured from the North Pole and a is the fractional elongation of the principal axis. The motion of lowest energy compatible with this change of shape is described by the velocity field

$$\xi^x = -\frac{1}{2}\alpha x, \quad \xi^y = -\frac{1}{2}\alpha y, \quad \xi^z = \alpha z \tag{37.14.2}$$

(zero divergence, zero curl).

(a) Show that the sum of the kinetic energy and the gravitational potential energy is

$$E = -\frac{3}{5}\frac{M^2}{a}\left(1-\frac{\alpha^2}{5}\right)+\frac{3}{20}Ma^2\dot{\alpha}^2. \tag{37.14.3}$$

(b) Show that the angular frequency of the free quadrupole vibration is

$$\omega = \left(\frac{16\pi}{5}\right)^{1/2}\rho^{1/2}. \tag{37.14.4}$$

(c) Show that the reduced quadrupole moments are

$$\text{Ɨ}_{xx} = \text{Ɨ}_{yy} = -\frac{Ma^2\alpha}{5}, \quad \text{Ɨ}_{zz} = \frac{2Ma^2\alpha}{5}. \tag{37.14.5}$$

(d) Show that the rate of emission of vibrational energy, averaged over a period, is

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \frac{3}{125} M^2 a^4 \omega^6 \alpha_{\mathrm{peak}}^2. \tag{37.14.6}$$

(e) Show that the exponential rate of decay of energy by reason of gravitational wave dampIng, or "gravitational radiation line broadening", is

$$A_{\mathrm{GW}} = \frac{4}{25} M a^2 \omega^4. \tag{37.14.7}$$

(f) Show that the resonance integral of the absorption cross section for radiation incident from random directions with random polarization is

$$\int_{\text{resonance}} \langle \sigma(\nu) \rangle \, \mathrm{d}\nu = \frac{2\pi}{25} \frac{Ma^2}{\lambda^2}. \tag{37.14.8}$$

(g) Evaluate this resonance integral.

Solution:

(a) The kinetic energy is

$$K = \frac{1}{2} \int \rho \dot{\xi}^2 \mathrm{d}^3 x = \frac{\rho}{2} \dot{\alpha}^2 \int \left(\frac{1}{4} x^2 + \frac{1}{4} y^2 + z^2 \right) \mathrm{d}^3 x = \frac{\rho}{2} \dot{\alpha}^2 \int_0^a 2\pi r^4 \, \mathrm{d}r = \frac{1}{5} \pi \rho a^5 \dot{\alpha}^2 = \frac{3}{20} M a^2 \dot{\alpha}^2. \tag{37.14.9}$$

The potential energy of the earth without viberation is

$$V_0 = \frac{1}{2} \int \rho \phi_0 \, \mathrm{d}^3 x = \frac{1}{2} \int_0^a \rho \left(\frac{2\pi}{3} \rho r^2 - 2\pi \rho a^2 \right) 4\pi r^2 \, \mathrm{d}r = -\frac{16}{15} \pi^2 a^5 \rho^2 = -\frac{3}{5} \frac{M^2}{a}. \tag{37.14.10}$$

From Exercise 36.3, we have

$$\delta\rho = \rho \boldsymbol{\xi} \cdot \hat{\boldsymbol{r}} \delta(r - a) \quad \text{where} \quad \boldsymbol{\xi} \cdot \hat{\boldsymbol{r}} = \frac{\alpha r}{2} \left(\frac{3z^2}{r^2} - 1 \right) \tag{37.14.11}$$

I can not figure out the correct answer for the variation of potential energy. I thought

$$\delta V = -\frac{1}{2} \int \frac{\delta\rho(\boldsymbol{r}) \, \delta\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 x \, \mathrm{d}^3 x', \tag{37.14.12}$$

but only to find that

$$\delta V = -\frac{8}{25} \pi^2 \alpha^2 a^2 \rho^2. \tag{37.14.13}$$

The correct answer should be

$$\delta V = \frac{16}{75} \pi^2 \alpha^2 a^2 \rho^2. \tag{37.14.14}$$

Any help is appreciated.

(b) Since

$$E = V_0 + \frac{3}{25} \frac{M^2}{a} \alpha^2 + \frac{3}{20} M a^2 \dot{\alpha}^2, \tag{37.14.15}$$

we have

$$\omega = \left(\frac{3M^2}{25a} \frac{20}{3Ma^2} \right)^{1/2} = \left(\frac{12M}{5a^3} \right)^{1/2} = \left(\frac{16\pi}{5} \right)^{1/2} \rho^{1/2}. \tag{37.14.16}$$

(c)

$$\delta \boldsymbol{I} = \int \delta\rho\, \boldsymbol{r} \otimes \boldsymbol{r}\, \mathrm{d}^3x = \rho a^4 \int (\boldsymbol{\xi}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}\, \mathrm{d}\Omega = \frac{1}{5} M a^2 \alpha \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}. \tag{37.14.17}$$

It is easy to see that $\delta \text{Ŧ}_{ij} = \delta I_{ij}$.

(d) The rate of emission of vibrational energy, averaged over a period, is

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \frac{1}{5} \left\langle \dddot{\text{Ŧ}}_{ij} \dddot{\text{Ŧ}}_{ij} \right\rangle = \frac{6}{125} M^2 a^4 \omega^6 \left\langle \alpha^2 \right\rangle = \frac{3}{125} M^2 a^4 \omega^6 \alpha_{\text{peak}}^2. \tag{37.14.18}$$

(e) The average viberating energy of the earth in a period is

$$\langle E \rangle = 2 \left\langle \frac{3}{20} M a^2 \dot{\alpha}^2 \right\rangle = \frac{3}{20} M a^2 \omega^2 \alpha_{\text{peak}}^2. \tag{37.14.19}$$

So “gravitational radiation line broadening” is

$$A_{\text{GW}} = \frac{-\langle \mathrm{d}E/\mathrm{d}t \rangle}{\langle E \rangle} = \frac{4}{25} M a^2 \omega^4 \alpha_{\text{peak}}^2. \tag{37.14.20}$$

(f) The resonance integral of the absorption cross section for radiation incident from random directions with random polarization is

$$\int_{\text{resonance}} \langle \sigma(\nu) \rangle \, \mathrm{d}\nu = \frac{\pi}{2} ƛ^2 A_{\text{GW}} = \frac{2\pi}{25} M a^2 \omega^2 = \frac{2\pi}{25} \frac{M a^2}{ƛ^2}. \tag{37.14.21}$$

(g) This resonance integral is

$$\frac{2\pi}{25} \frac{GMa^2}{c^3} \frac{16\pi G\rho}{5} = \frac{24\pi G^2 M^2}{125 a c^3} = 5.58 \text{cm}^2\,\text{Hz}. \tag{37.14.22}$$

Chapter 39

OTHER THEORIES OF GRAVITY AND POST-NEWTONIAN APPROXIMATION

39.1 ORDERS OF MAGNITUDE IN GRAVITATIONALLY BOUND SYSTEMS

Use Newtonian theory to derive conditions (39.7) for any gravitationally bound system. [Hint: Such concepts as orbital velocities, the speeds of sound and shear waves, the virial theorem, and hydrostatic equilibrium are relevant.]

Solution:

For circular orbits in solar system, we have

$$\frac{1}{2}v^2 = \frac{1}{2}|U| \tag{39.1.1}$$

and so $\epsilon^2 \gtrsim v^2$. From hydrostatic equilibrium, we have

$$\boldsymbol{\nabla} p = \rho_0 \boldsymbol{\nabla} U, \tag{39.1.2}$$

and so $\epsilon^2 \gtrsim p/\rho_0$. Stress-energy tensor of gravity is

$$t_{ij} \sim U_{,i}U_{,j} \sim UU_{,ij} \sim U\rho_0, \tag{39.1.3}$$

and so $\epsilon^2 \gtrsim |t_{ij}|/\rho_0$. Finally, we have

$$\Pi \equiv \frac{\rho - \rho_0}{\rho_0} \sim v^2, \frac{p}{\rho_0}, \frac{|t_{ij}}{\rho_0}, \tag{39.1.4}$$

and so $\epsilon^2 \gtrsim \Pi$.

39.2 PATTERN OF TERMS IN POST-NEWTONIAN EXPANSION

Verify the statements in the paragraph following equation (39.11). In particular, suppose that one wishes to evaluate the coordinate acceleration, $\mathrm{d}^2 x_j/\mathrm{d}t^2$, to accuracy $\epsilon^{2N} U_{,j}$ for some integer N. Show that this undertaking requires a knowledge of g_{00} to accuracy ϵ^{2N+2}, of g_{0k} to ϵ^{2N+1}, and of g_{jk} to ϵ^{2N}. Also

suppose that one knows T^{00} to accuracy $\rho_0\epsilon^{2N}$, T^{0j} to $\rho_0\epsilon^{2N+1}$, and T_{jk} to $\rho_0\epsilon^{2N+2}$ Show that to calculate $T^{0\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+1}\rho_{0,j}$ and $T^{j\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+2}\rho_{0,j}$, one must know g_{00} to ϵ^{2N+2}, g_{0k} to ϵ^{2N+1}, and g_{jk} to ϵ^{2N}. This dictates the pattern of Box

Solution:

Firstly, from equation (39.11), we have

$$\frac{\mathrm{d}^2x_j}{\mathrm{d}t^2} = U_{,j} + \text{terms of order } \{\epsilon g_{0k,j}; \epsilon^2 g_{kl,j}\} + \text{terms of higher order in } g_{00,j}. \tag{39.2.1}$$

Notice that $U_{,j}$ is of the order ϵ^2. If one wishes to evaluate $\mathrm{d}^2x_j/\mathrm{d}t^2$ to accuracy $\epsilon^{2N}U_{,j}$. We must require a knowledge of g_{00} to accuracy ϵ^{2N+2}, of g_{0k} to ϵ^{2N+1}, and of g_{jk} to ϵ^{2N}.

Secondly, we have

$$\begin{aligned}
T^{0\alpha}{}_{;\alpha} &= T^{0\alpha}{}_{,\alpha} + \Gamma^0{}_{\beta\alpha}T^{\beta\alpha} + \Gamma^\alpha{}_{\beta\alpha}T^{0\beta} \\
&= T^{00}{}_{,0} + T^{0i}{}_{,i} + \Gamma^0{}_{00}T^{00} + 2\Gamma^0{}_{0i}T^{0i} + \Gamma^0{}_{ij}T^{ij} \\
&+ \Gamma^0{}_{00}T^{00} + \Gamma^0{}_{j0}T^{0j} + \Gamma^i{}_{0i}T^{00} + \Gamma^i{}_{ji}T^{0j} \\
&\sim (g_{00,0} + g_{0i,i} + g_{ij,0})\rho_0 + (g_{00,i} + g_{i0,0} + g_{ij,k})\epsilon\rho_0 + (g_{0i,j} + g_{ij,0})\epsilon^2\rho_0 \\
&\sim (\epsilon g_{00} + g_{0i} + \epsilon g_{ij})\rho_{0,i} + (g_{00} + \epsilon g_{i0} + g_{ij})\epsilon\rho_{0,i} + (g_{0i} + \epsilon g_{ij})\epsilon^2\rho_{0,i}.
\end{aligned} \tag{39.2.2}$$

To calculate $T^{0\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+1}\rho_{0,j}$, one must know g_{00} to ϵ^{2N}, g_{0k} to ϵ^{2N+1}, and g_{jk} to ϵ^{2N}.

Thirdly, we have

$$T^{j\alpha}{}_{;\alpha} = T^{j\alpha}{}_{,\alpha} + \Gamma^j{}_{\beta\alpha}T^{\beta\alpha} + \Gamma^\alpha{}_{\beta\alpha}T^{j\beta} \tag{39.2.3}$$

Notice that $\Gamma^j{}_{00}T^{00} \sim (g_{00,j} + g_{0j,0})\rho_0 \sim (g_{00} + \epsilon g_{0j})\rho_{0,j}$. To calculate $T^{j\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+2}\rho_{0,j}$, one must know g_{00} to ϵ^{2N+2} and g_{0j} to ϵ^{2N+1}.

In conclusion, to calculate $T^{0\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+1}\rho_{0,j}$ and $T^{j\alpha}{}_{;\alpha}$ with accuracy $\epsilon^{2N+2}\rho_{0,j}$, one must know g_{00} to ϵ^{2N+2}, g_{0k} to ϵ^{2N+1}, and g_{jk} to ϵ^{2N}.

39.3 NEWTONIAN APPROXIMATION

(a) Derive equations (39.13) for the components of the stress-energy tensor in the PPN coordinate frame.

(b) Show that, in the PPN coordinate frame, $T^{0\alpha}{}_{;\alpha}$ reduces to equation (39.15a), and $T^{j\alpha}{}_{;\alpha}$, when combined with (39.15a), reduces to equation (39.15b).

Solution:

(a) In the rest frame of the matter, we have $T_{\hat{0}\hat{0}} = \rho$, $T_{\hat{0}\hat{i}} = 0$ and $T_{\hat{i}\hat{j}} = t_{\hat{i}\hat{j}}$. Lorentz-transform these components by a pure boost with ordinary velocity $-v_j$ to obtain $T_{\alpha\beta}$. The boost matrix is

$$\begin{pmatrix} \gamma & \gamma v_i \\ \gamma v_i & \delta_{ij} + (\gamma - 1)n_i n_j \end{pmatrix} \tag{39.3.1}$$

So we have

$$T^{\alpha\beta} = \begin{pmatrix} \gamma & \gamma v n_i \\ \gamma v n_i & \delta_{ij} + (\gamma-1) n_i n_j \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & t_{\hat{i}\hat{j}}. \end{pmatrix} \begin{pmatrix} \gamma & \gamma v n_i \\ \gamma v n_i & \delta_{ij} + (\gamma-1) n_i n_j \end{pmatrix}$$
$$= \begin{pmatrix} \gamma^2(\rho + v^2 n_i t_{\hat{i}\hat{j}} n_j) & \gamma^2 \rho v n_i + \gamma v n_k t_{\hat{k}\hat{i}} + \gamma(\gamma-1) v n_k t_{\hat{k}\hat{l}} n_l n_i \\ \gamma^2 \rho v n_i + \gamma v n_k t_{\hat{k}\hat{i}} + \gamma(\gamma-1) v n_k t_{\hat{k}\hat{l}} n_l n_i & \gamma^2 \rho v^2 n_i n_j + t_{\hat{i}\hat{j}} + 2(\gamma-1) n_i n_k t_{\hat{k}\hat{j}} + (\gamma-1)^2 n_k t_{\hat{k}\hat{l}} n_l n_i n_j \end{pmatrix} \tag{39.3.2}$$

Notice that

$$\rho = \rho_0 + \mathcal{O}(\rho_0 \epsilon^2), \quad v \sim \mathcal{O}(\epsilon), \quad \gamma = 1 + \frac{1}{2} v^2 + \mathcal{O}(\epsilon^4), \quad t_{\hat{i}\hat{j}} \sim \mathcal{O}(\rho_0 \epsilon^2). \tag{39.3.3}$$

We can get

$$T^{00} = \rho_0 + \mathcal{O}(\rho_0 \epsilon^2), \quad T^{0i} = \rho_0 v_i + \mathcal{O}(\rho_0 \epsilon^3), \quad T^{ij} = t_{\hat{i}\hat{j}} + \rho_0 v_i v_j + \mathcal{O}(\rho_0 \epsilon^4). \tag{39.3.4}$$

(b) To the order of $\mathcal{O}(\epsilon^2)$, we have

$$\begin{aligned} T^{0\alpha}{}_{;\alpha} &= T^{00}{}_{,0} + T^{0i}{}_{,i} + 2\Gamma^0{}_{00} T^{00} + 2\Gamma^0{}_{0i} T^{0i} + \Gamma^0{}_{ij} T^{ij} + \Gamma^0{}_{j0} T^{0j} + \Gamma^i{}_{0i} T^{00} + \Gamma^i{}_{ji} T^{0j} \\ &= T^{00}{}_{,0} + T^{0i}{}_{,i} + 2\Gamma^0{}_{00} T^{00} + + \Gamma^i{}_{0i} T^{00} + \mathcal{O}(\epsilon^2) \\ &= \frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_0 v_i}{\partial x^i} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{39.3.5}$$

To the order of $\mathcal{O}(\epsilon^3)$, we also have

$$\begin{aligned} T^{j\alpha}{}_{;\alpha} &= T^{j\alpha}{}_{,\alpha} + \Gamma^j{}_{\beta\alpha} T^{\beta\alpha} + \Gamma^\alpha{}_{\beta\alpha} T^{j\beta} \\ &= T^{j0}{}_{,0} + T^{ji}{}_{,i} + \Gamma^j{}_{00} T^{00} + \mathcal{O}(\epsilon^3) \\ &= T^{j0}{}_{,0} + T^{ji}{}_{,i} - \frac{1}{2} g_{00,j} T^{00} + \mathcal{O}(\epsilon^3) \\ &= \frac{\partial \rho_0 v_j}{\partial t} + \frac{\partial t_{\hat{j}\hat{i}} + \rho_0 v_i v_j}{\partial x^i} - \rho_0 \frac{\partial U}{\partial x^j} + \mathcal{O}(\epsilon^3) \\ &= v_j \left(\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_0 v_i}{\partial x^i} \right) + \rho_0 \frac{\partial v_j}{\partial t} + \rho_0 v_i \frac{\partial v_j}{\partial x^i} + \frac{\partial t_{\hat{j}\hat{i}}}{\partial x^i} - \rho_0 \frac{\partial U}{\partial x^j} + \mathcal{O}(\epsilon^3) \\ &= \rho_0 \frac{\mathrm{d} v_j}{\mathrm{d} t} + \frac{\partial t_{\hat{j}\hat{i}}}{\partial x^i} - \rho_0 \frac{\partial U}{\partial x^j} + \mathcal{O}(\epsilon^3). \end{aligned} \tag{39.3.6}$$

39.4 A USEFUL FORMULA

Derive from equations (39.15) the following useful formula, valid for any function $f(\boldsymbol{x}, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho_0(\boldsymbol{x}, t) f(\boldsymbol{x}, t) \, \mathrm{d}^3 x = \int \rho_0(\boldsymbol{x}, t) \frac{\mathrm{d} f(\boldsymbol{x}, t)}{\mathrm{d} t} \, \mathrm{d}^3 x + \text{fractional errors of } \mathcal{O}(\epsilon^2). \tag{39.4.1}$$

Here both integrals are extended over all of space; and $\mathrm{d}f/\mathrm{d}t$ is the derivative following the matter (39.16).

Solution:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_0(\boldsymbol{x},t) f(\boldsymbol{x},t)\,\mathrm{d}^3x \\
&= \int \left[\frac{\partial \rho_0(\boldsymbol{x},t)}{\partial t} f(\boldsymbol{x},t) + \rho_0(\boldsymbol{x},t)\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right]\mathrm{d}^3x \\
&= \int \left[-\frac{\partial \rho_0(\boldsymbol{x},t) v^i}{\partial x^i} f(\boldsymbol{x},t) + \rho_0(\boldsymbol{x},t)\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right]\mathrm{d}^3x + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right) \\
&= \int \left[-\frac{\partial \rho_0(\boldsymbol{x},t) f(\boldsymbol{x},t) v^i}{\partial x^i} + \rho_0(\boldsymbol{x},t) f(\boldsymbol{x},t)\frac{\partial v^i}{\partial x^i} + \rho_0(\boldsymbol{x},t)\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right]\mathrm{d}^3x + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right) \\
&= \int \rho_0(\boldsymbol{x},t)\frac{\mathrm{d}f(\boldsymbol{x},t)}{\mathrm{d}t}\,\mathrm{d}^3x - \oint_{r_S\to\infty} \rho_0(\boldsymbol{x},t) f(\boldsymbol{x},t)\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right) \\
&= \int \rho_0(\boldsymbol{x},t)\frac{\mathrm{d}f(\boldsymbol{x},t)}{\mathrm{d}t}\,\mathrm{d}^3x + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right). \qquad (39.4.2)
\end{aligned}
$$

39.5 STRESS TENSOR FOR NEWTONIAN GRAVITATIONAL FIELD

Define a "stress tensor for the Newtonian gravitational field U" as follows:

$$
t_{jk} = \frac{1}{4\pi}\left(U_{,j}U_{,k} - \frac{1}{2}\delta_{jk}U_{,l}U_{,l}\right). \qquad (39.5.1)
$$

Show that the equations of motion for the matter (39.15b) can be rewritten in the forms

$$
\rho_0\frac{\mathrm{d}v_j}{\mathrm{d}t} = -\frac{\partial t_{jk} + t_{\hat{j}\hat{i}}}{\partial x^k} + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right) \qquad (39.5.2)
$$

$$
(\rho_0 v_j)_{,t} + (t_{jk} + t_{\hat{j}\hat{k}} + \rho_0 v_j v_k)_{,k} = 0 + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right). \qquad (39.5.3)
$$

Solution:

Firstly, we have

$$
t_{jk,k} = \frac{1}{4\pi}\left(U_{,jk}U_{,k} + U_{,j}U_{,kk} - \delta_{jk}U_{,lk}U_{,l}\right) = \frac{1}{4\pi}U_{,j}U_{,kk} = -\rho_0 U_{,j}. \qquad (39.5.4)
$$

So

$$
\rho_0\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{\partial t_{\hat{j}\hat{k}}}{\partial x^k} - \rho_0\frac{\partial U}{\partial x^j} = \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right) \qquad (39.5.5)
$$

is equivalent to 39.5.2. From exercise 39.3, we have

$$
\frac{\partial \rho_0 v_j}{\partial t} + \frac{\partial t_{\hat{j}\hat{i}} + \rho_0 v_i v_j}{\partial x^i} = \rho_0\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{\partial t_{\hat{j}\hat{i}}}{\partial x^i} + \mathcal{O}\left(\epsilon^3\right). \qquad (39.5.6)
$$

So we can get 39.5.3.

39.6 NEWTONIAN VIRIAL THEOREMS

(a) From equation 39.5.3 show that

$$
\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2\int (t_{jk} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_k)\,\mathrm{d}^3x + \text{fractional errors of } \mathcal{O}\left(\epsilon^2\right), \qquad (39.6.1)
$$

where I_{jk} is the second moment of the system's mass distribution,

$$I_{jk} = \int \rho_0 x_j x_k \, \mathrm{d}^3x \,. \tag{39.6.2}$$

This is called the "time-dependent tensor virial theorem".

(b) From this infer that, if $\langle \quad \rangle_{\text{long time}}$ longtime denotes an average over a long period of time, then

$$\left\langle \int (t_{jk} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_k) \, \mathrm{d}^3x \right\rangle_{\text{long time}} = \mathcal{O}\left(\int \rho_0 \epsilon^4 \, \mathrm{d}^3x \right). \tag{39.6.3}$$

This is called the "tensor virial theorem".

(c) By contraction of indices and use of equations (39.18), (39.14a), and (39.5e), derive the (ordinary) virial theorems:

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \int \rho_0 \boldsymbol{v}^2 \, \mathrm{d}^3x - \int \frac{1}{2}\rho_0 U \, \mathrm{d}^3x + 3\int p \, \mathrm{d}^3x + \text{fractional errors of } \mathcal{O}(\epsilon^2), \tag{39.6.4}$$

where I is the trace of the second moment of the mass distribution

$$I = I_{jj} = \int \rho_0 r^2 \, \mathrm{d}^3x \,; \tag{39.6.5}$$

and

$$\left\langle \int \rho_0 \boldsymbol{v}^2 \, \mathrm{d}^3x - \int \frac{1}{2}\rho_0 U \, \mathrm{d}^3x + 3\int p \, \mathrm{d}^3x \right\rangle_{\text{long time}} = \mathcal{O}\left(\int \rho_0 \epsilon^4 \, \mathrm{d}^3x \right). \tag{39.6.6}$$

Solution:

(a) Firstly, we have

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\int \dot{\rho}_0 x_j x_k \, \mathrm{d}^3x \,. \tag{39.6.7}$$

Secondly, we have

$$\begin{aligned}\int \dot{\rho}_0 x_j x_k \, \mathrm{d}^3x &= -\int \frac{\partial \rho_0 v_i}{\partial x_i} x_j x_k \, \mathrm{d}^3x = \int \rho_0 v_i \frac{\partial x_j}{\partial x_i} x_k \, \mathrm{d}^3x + \int \rho_0 v_i \frac{\partial x_k}{\partial x_i} x_j \, \mathrm{d}^3x \\ &= \int \rho_0 v_j x_k \, \mathrm{d}^3x + \int \rho_0 v_k x_j \, \mathrm{d}^3x \end{aligned} \tag{39.6.8}$$

Thus we have

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2\int (\rho_0 v_j)_{,t} x_k \, \mathrm{d}^3x + 2\int (\rho_0 v_k)_{,t} x_j \, \mathrm{d}^3x \,. \tag{39.6.9}$$

Using 39.5.3, we have

$$\begin{aligned}\int (\rho_0 v_j)_{,t} x_k \, \mathrm{d}^3x &= -\int (t_{ji} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_i)_{,i} x_k \, \mathrm{d}^3x = \int (t_{ji} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_i) x_{k,i} \, \mathrm{d}^3x \\ &= \int (t_{jk} + t_{\hat{j}\hat{k}} + \rho_0 v_j v_k) \, \mathrm{d}^3x \end{aligned} \tag{39.6.10}$$

Now we obtain equation 39.6.1.

(b) Since $t_{jk} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_k \sim \rho_0 \epsilon^2$, equation 39.6.1 is equivalent to

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2\int \left[t_{jk} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_k + \mathcal{O}(\rho_0 \epsilon^4) \right] \mathrm{d}^3x \,. \tag{39.6.11}$$

So we have

$$\left\langle \int (t_{jk} + t_{\hat{j}\hat{i}} + \rho_0 v_j v_k)\, \mathrm{d}^3x \right\rangle_{\text{long time}} = \left\langle \frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} + \mathcal{O}\left(\int \rho_0 \epsilon^4 \right) \mathrm{d}^3x \right\rangle_{\text{long time}} = \mathcal{O}\left(\int \rho_0 \epsilon^4 \right) \mathrm{d}^3x \tag{39.6.12}$$

(c)Firstly, we have

$$t_{ii} = \frac{1}{4\pi}\left(U_{,i}U_{,i} - \frac{1}{2}\delta_{ii}U_{,l}U_{,l} \right) = -\frac{1}{8\pi}U_{,i}U_{,i} = \frac{1}{8\pi}UU_{,ii} - \frac{1}{8\pi}(UU_{,i})_{,i} = -\frac{1}{2}U\rho_0 - \frac{1}{8\pi}(UU_{,i})_{,i} \tag{39.6.13}$$

and

$$t_{\hat{i}\hat{i}} = 3p. \tag{39.6.14}$$

By contracting equation 39.6.1, we have

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2\int (-\frac{1}{2}U\rho_0 + 3p + \rho_0 \boldsymbol{v}^2)\, \mathrm{d}^3x + \text{fractional errors of } \mathcal{O}(\epsilon^2), \tag{39.6.15}$$

i.e., equation 39.6.4. By contracting equation 39.6.12, we have 39.6.6.

39.7 PULSATION FREQUENCY FOR NEWTONIAN STAR

Use the ordinary, time-dependent virial theorem 39.6.4 to derive the following equation for the fundamental angular frequency of pulsation of a nonrotating, Newtonian star:

$$\omega^2 = (3\bar{\Gamma}_1 - 4)\frac{|\text{star's self-gravitational energy}|}{(\text{trace of second moment of star's mass distribution})}; \tag{39.7.1}$$

$$\bar{\Gamma}_1 = \text{pressure-weihted average of adiabatic index} \equiv \frac{\int \Gamma_1 p\, \mathrm{d}^3x}{\int p\, \mathrm{d}^3x}. \tag{39.7.2}$$

In the derivation assume that the pulsations are "homologous" -i.e., that a fluid element with equilibrium position x^j (relative to center of mass $x^j = 0$) gets displaced to $x^j + \xi^j(x,t)$, where

$$\xi^j = (\text{small constant}) x^j e^{-i\omega t}. \tag{39.7.3}$$

Assume nothing else.

Solution:

Suppose $\boldsymbol{\xi} = s\boldsymbol{r}\cos\omega t$. For an mass element at r, it will move to $r(1 + s\cos\omega t)$. The density of the mass element will be $\rho_0(1+s\cos\omega t)^{-3}$. The gravitational potential at the mass element will be $U(1+s\cos\omega t)^{-1}$.

For adiabatic process, we have $p \propto \rho^{\Gamma_1}$. So the pressure at the mass element will become $p(1+s\cos\omega t)^{-3\Gamma_1}$.

For a static star, we have $\mathrm{d}^2 I/\mathrm{d}t^2 = 0$. Thus

$$3\int p\, \mathrm{d}^3x = \int \frac{1}{2}\rho_0 U\, \mathrm{d}^3x \equiv |\Omega|, \tag{39.7.4}$$

where Ω is star's self-gravitational energy.

For a pulsing star, the trace of second moment of star's mass distribution is

$$I' = \int \rho_0(1 + s\cos\omega t)^{-3}(1 + s\cos\omega t)^5 r^4\, \mathrm{d}r\, \mathrm{d}\Omega = (1 + s\cos\omega t)^2 I \approx (1 + 2s\cos\omega t)I \tag{39.7.5}$$

Star's self-gravitational energy is

$$\Omega' = \int \rho_0(1+s\cos\omega t)^{-3}U(1+s\cos\omega t)^{-1}(1+s\cos\omega t)^{-3}r^2\,\mathrm{d}r\,\mathrm{d}\Omega = -(1+s\cos\omega t)^{-1}|\Omega| \approx -(1-s\cos\omega t)|\Omega|. \tag{39.7.6}$$

The pressure integral is

$$3\int p(1+s\cos\omega t)^{-3\Gamma_1}(1+s\cos\omega t)^3 r^2\,\mathrm{d}r\,\mathrm{d}\Omega = (1+s\cos\omega t)^{3-3\bar{\Gamma}_1}|\Omega| \approx [1+(3-3\bar{\Gamma}_1)s\cos\omega t]|\Omega|. \tag{39.7.7}$$

Note that kinetic energy $\int \rho_0 \boldsymbol{v}^2\,\mathrm{d}^3r$ vanishes to the first order of s. So we have

$$-\omega^2 I s\cos\omega t = |\Omega| s\cos\omega t + (3-3\bar{\Gamma}_1)s\cos\omega t. \tag{39.7.8}$$

It follows that

$$\omega^2 = (3\bar{\Gamma}_1 - 4)\frac{|\Omega|}{I}. \tag{39.7.9}$$

39.8 ABSENCE OF "METRIC-GENERATES-METRIC" TERMS IN POST-NEWTONIAN LIMIT

In writing down the post-Newtonian metric corrections, one might be tempted to include terms that are generated by the Newtonian potential acting alone, without any direct aid from the matter. After all, general relativity and other metric theories are nonlinear; so the two-step process (matter) $\to U \to$ (post-Newtonian metric corrections) seems quite natural. Show that such terms are not needed, because the equations (39.14)-(39.16) of the Newtonian approximation enable one to reexpress them in terms of direct integrals over the matter distribution. In particular, show that

$$\int \frac{\partial^2 U(\boldsymbol{x}',t)}{\partial x'_j \partial t}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} = 2\pi[V_j(\boldsymbol{x},t) - W_j(\boldsymbol{x},t)], \tag{39.8.1}$$

where V_j and W_j are defined by equations (39.23c,d); also show that

$$\int \frac{\partial U(\boldsymbol{x}',t)}{\partial x'_j}\frac{\partial U(\boldsymbol{x}',t)}{\partial x'_j}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = -2\pi[U(\boldsymbol{x},t)]^2 + 4\pi\int \frac{\rho_0(\boldsymbol{x}',t)U(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,. \tag{39.8.2}$$

Note that the terms on the righthand sides of 39.8.1 and 39.8.2 are already included in the expressions (39.23e,f) for g_{0j} and g_{00}.

Solution:

(a) Firstly, we have

$$\begin{aligned}\int \frac{\partial^2 U(\boldsymbol{x}',t)}{\partial x'_j \partial t}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} &= \int \frac{\partial^2}{\partial x'_j \partial t}\left(\int \frac{\rho_0(\boldsymbol{x}'',t)}{|\boldsymbol{x}'-\boldsymbol{x}''|}\,\mathrm{d}^3x''\right)\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} \\ &= -\int \frac{\partial \rho_0(\boldsymbol{x}'',t)v_k(\boldsymbol{x}'',t)}{\partial x''_k}\frac{\partial}{\partial x'_j}\left(\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x''\,\mathrm{d}^3x' \\ &= -\int \rho_0(\boldsymbol{x}'',t)v_k(\boldsymbol{x}'',t)\frac{\partial^2}{\partial x'_k \partial x'_j}\left(\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x''\,\mathrm{d}^3x'\end{aligned} \tag{39.8.3}$$

Secondly, we notice that

$$
\begin{aligned}
&\int_0^R r^2\,\mathrm{d}r\,\mathrm{d}\Omega\,\frac{1}{r|\boldsymbol{r}-\boldsymbol{r}_a|}\\
&=\int_0^{r_a}\mathrm{d}r\,\mathrm{d}\Omega\,\frac{r}{|\boldsymbol{r}-\boldsymbol{r}_a|}+\int_{r_a}^R\mathrm{d}r\,\mathrm{d}\Omega\,\frac{r}{|\boldsymbol{r}-\boldsymbol{r}_a|}\\
&=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{r_a}\mathrm{d}r\,\frac{r^{l+1}}{r_a^{l+1}}\int\mathrm{d}\Omega\,\frac{4\pi}{2l+1}Y_{lm}^*(\theta,\phi)Y_{lm}(\theta_a,\phi_a)\\
&+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{r_a}^R\mathrm{d}r\,\frac{r_a^l}{r^l}\int\mathrm{d}\Omega\,\frac{4\pi}{2l+1}Y_{lm}^*(\theta,\phi)Y_{lm}(\theta_a,\phi_a)\\
&=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{r_a}\mathrm{d}r\,\frac{r^{l+1}}{r_a^{l+1}}4\pi\delta_{l0}+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{r_a}^R\mathrm{d}r\,\frac{r_a^l}{r^l}4\pi\delta_{l0}\\
&=4\pi\int_0^{r_a}\mathrm{d}r\,\frac{r}{r_a}+4\pi\int_{r_a}^R\mathrm{d}r\\
&=4\pi R-2\pi r_a.
\end{aligned}
\tag{39.8.4}
$$

So we have

$$
\begin{aligned}
&\int\frac{\partial^2}{\partial x_k'\partial x_j'}\left(\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'=\int\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\frac{\partial^2}{\partial x_k'\partial x_j'}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\mathrm{d}^3x'\\
&=\frac{\partial^2}{\partial x_k\partial x_j}\int\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}''|}\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3x'=\frac{\partial^2}{\partial x_k\partial x_j}\int\frac{1}{|\boldsymbol{x}'|}\frac{1}{|\boldsymbol{x}'-(\boldsymbol{x}-\boldsymbol{x}'')|}\,\mathrm{d}^3x'\\
&=\frac{\partial^2}{\partial x_k\partial x_j}(-2\pi|\boldsymbol{x}-\boldsymbol{x}''|)=-2\pi\left(\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{x}''|}-\frac{(x_j-x_j'')(x_k-x_k'')}{|\boldsymbol{x}-\boldsymbol{x}''|^3}\right)
\end{aligned}
\tag{39.8.5}
$$

It follows that

$$
\begin{aligned}
\int\frac{\partial^2U(\boldsymbol{x}',t)}{\partial x_j'\partial t}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}&=2\pi\int\frac{\rho_0(\boldsymbol{x}'',t)v_j(\boldsymbol{x}'',t)}{|\boldsymbol{x}'-\boldsymbol{x}''|}\,\mathrm{d}^3x''-2\pi\int\frac{\rho_0(\boldsymbol{x}'',t)v_k(\boldsymbol{x}'',t)(x_k-x_k'')(x_j-x_j'')}{|\boldsymbol{x}-\boldsymbol{x}''|^3}\,\mathrm{d}^3x''\\
&=2\pi[V_j(\boldsymbol{x},t)-W_j(\boldsymbol{x},t)]
\end{aligned}
\tag{39.8.6}
$$

(b)

$$
\begin{aligned}
&\int\frac{\partial U(\boldsymbol{x}',t)}{\partial x_j'}\frac{\partial U(\boldsymbol{x}',t)}{\partial x_j'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\\
&=-\int U(\boldsymbol{x}',t)\frac{\partial}{\partial x_j'}\left(\frac{\partial U(\boldsymbol{x}',t)}{\partial x_j'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\mathrm{d}^3x'\\
&=-\int\left[U(\boldsymbol{x}',t)\frac{\partial^2U(\boldsymbol{x}',t)}{\partial x_j'\partial x_j'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}+U(\boldsymbol{x}',t)\frac{\partial U(\boldsymbol{x}',t)}{\partial x_j'}\frac{\partial}{\partial x_j'}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right]\mathrm{d}^3x'\\
&=-\int\left[U(\boldsymbol{x}',t)(-4\pi\rho_0(\boldsymbol{x}',t))\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}+\frac{1}{2}\frac{\partial U^2(\boldsymbol{x}',t)}{\partial x_j'}\frac{\partial}{\partial x_j'}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right]\mathrm{d}^3x'\\
&=\int\left[\frac{4\pi\rho_0(\boldsymbol{x}',t)U(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}+\frac{1}{2}U^2(\boldsymbol{x}',t)\frac{\partial^2}{\partial x_j'\partial x_j'}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right]\mathrm{d}^3x'\\
&=\int\left[\frac{4\pi\rho_0(\boldsymbol{x}',t)U(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}-2\pi U^2(\boldsymbol{x}',t)\delta(\boldsymbol{x}-\boldsymbol{x}')\right]\mathrm{d}^3x'\\
&=-2\pi[U(\boldsymbol{x},t)]^2+4\pi\int\frac{\rho_0(\boldsymbol{x}',t)U(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,.
\end{aligned}
\tag{39.8.7}
$$

39.9 REMOVAL OF Σ TERM FROM g_{00}

Show that the coordinate transformation (39.25) removes the Σ term from the metric coefficient g_{00} of equation (39.24), as claimed in the text.

Solution:

After coordinate transformation, we have

$$g_{\alpha'\beta'}(x_{\rm new}) = g_{\alpha\beta}(x_{\rm old})\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}. \tag{39.9.1}$$

From (39.25), we have

$$x^0_{\rm new} = x^0_{\rm old} - \frac{1}{2}\Sigma\int\frac{\rho_0(\boldsymbol{x}',t)v_j(\boldsymbol{x}',t)(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x'\,. \tag{39.9.2}$$

And so

$$\frac{\partial x^0_{\rm old}}{\partial x^0_{\rm new}} = \left(1-\frac{1}{2}\Sigma\int\frac{(\rho_0 v_j)_{,t}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x'\right)^{-1}. \tag{39.9.3}$$

Notice that

$$\begin{aligned}\int\frac{(\rho_0 v_j)_{,t}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' &= -\int\frac{(t_{ji}+t_{\hat{j}\hat{i}}+\rho_0 v_j v_i)_{,i}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x'\\ &= \int(t_{ji}+t_{\hat{j}\hat{i}}+\rho_0 v_j v_i)\frac{\partial}{\partial x'_i}\left(\frac{(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right){\rm d}^3x' \sim \mathcal{O}(\epsilon^4).\end{aligned} \tag{39.9.4}$$

To the order of $\mathcal{O}(\epsilon^4)$, we have

$$\begin{aligned}g^{\rm new}_{00}(x_{\rm new}) &= g^{\rm old}_{00}(x_{\rm old})\frac{\partial x^0_{\rm old}}{\partial x^0_{\rm new}}\frac{\partial x^0_{\rm old}}{\partial x^0_{\rm new}}\\ &= g_{00}(x_{\rm new}) + g_{00,0}(x_{\rm new})(t_{\rm old}-t_{\rm new}) + g_{00}(x_{\rm new})\Sigma\int\frac{(\rho_0 v_j)_{,t}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' + \mathcal{O}(\epsilon^6)\\ &= g_{00} + U_{,t}\Sigma\int\frac{\rho_0 v_j(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' - \Sigma\int\frac{(\rho_0 v_j)_{,t}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' + \mathcal{O}(\epsilon^6)\end{aligned} \tag{39.9.5}$$

Notice that

$$U_{,t}\Sigma\int\frac{\rho_0 v_j(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' \sim U\frac{\partial}{\partial t}\left(\Sigma\int\frac{\rho_0 v_j(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x'\right) \sim \mathcal{O}(\epsilon^6) \tag{39.9.6}$$

Thus we have

$$g^{\rm new}_{00}(x_{\rm new}) = g_{00} - \Sigma\int\frac{(\rho_0 v_j)_{,t}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' + \mathcal{O}(\epsilon^6). \tag{39.9.7}$$

Using

$$\frac{\partial\rho_0 v_j}{\partial t} + \frac{\partial t_{\hat{j}\hat{i}}+\rho_0 v_i v_j}{\partial x^i} - \rho_0\frac{\partial U}{\partial x^j} = 0, \tag{39.9.8}$$

we have

$$\begin{aligned}g^{\rm new}_{00} &= g^{\rm old}_{00} - \Sigma\int\frac{\rho_0 U_{,j}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' + \Sigma\int\frac{(t_{\hat{j}\hat{k}}+\rho_0 v_k v_j)_{,k}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' + \mathcal{O}(\epsilon^6)\\ &= g^{\rm old}_{00} - \Sigma\int\frac{\rho_0 U_{,j}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x' - \Sigma\int(t_{\hat{j}\hat{k}}+\rho_0 v_j v_k)\frac{\partial}{\partial x'_k}\left(\frac{(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right){\rm d}^3x' + \mathcal{O}(\epsilon^6)\\ &= g^{\rm old}_{00} - \Sigma\int\frac{\rho_0 U_{,j}(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,{\rm d}^3x'\\ &+ \Sigma\int(t_{\hat{j}\hat{k}}+\rho_0 v_j v_k)\left[\frac{\delta_{jk}}{|\boldsymbol{x}-\boldsymbol{x}'|} - \frac{(x_j-x'_j)(x_k-x'_k)}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right]{\rm d}^3x' + \mathcal{O}(\epsilon^6)\end{aligned} \tag{39.9.9}$$

Notice that the last term can be absorbed into (39.23g,h,i), and

$$-\Sigma\int\frac{\rho_0(\boldsymbol{x}',t)U_{,j}(\boldsymbol{x}',t)(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'=\Sigma\int\mathrm{d}^3x'\,\frac{\rho_0(\boldsymbol{x}',t)(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\int\mathrm{d}^3x''\,\frac{\rho_0(\boldsymbol{x}'',t)(x'_j-x''_j)}{|\boldsymbol{x}'-\boldsymbol{x}''|^3}$$
$$=\Sigma\int\int\frac{\rho_0(\boldsymbol{x}',t)\rho_0(\boldsymbol{x}'',t)[(\boldsymbol{x}-\boldsymbol{x}')\cdot(\boldsymbol{x}'-\boldsymbol{x}'')]}{|\boldsymbol{x}-\boldsymbol{x}'||\boldsymbol{x}'-\boldsymbol{x}''|^3}\,\mathrm{d}^3x'\,\mathrm{d}^3x''\,. \tag{39.9.10}$$

We conclude that the coordinate transformation (39.25) removes the Σ term from the metric coefficient g_{00} of equation (39.24).

39.10 VERIFICATION OF FORMS OF POST-NEWTONIAN CORRECTIONS

Verify the claims in the text immediately preceding equations (39.23a,b,c,f).

Solution:

The elements we can use to construct post-Newtonian correction of metric is

$$\rho_0,\quad \rho_0\Pi\sim\mathcal{O}\big(\rho_0\epsilon^2\big),\quad t_{\hat{i}\hat{j}}\sim\mathcal{O}\big(\rho_0\epsilon^2\big),\quad p=\sim\mathcal{O}\big(\rho_0\epsilon^2\big),\quad \boldsymbol{v}\sim\mathcal{O}(\epsilon) \tag{39.10.1}$$

and

$$X\equiv\int\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}\sim\mathcal{O}\big(\epsilon^2\rho_0^{-1}\big)\sim\mathcal{O}\left(\frac{1}{r}\right),\quad \mathbf{Y}\equiv\int\mathrm{d}^3x'\,\frac{(\boldsymbol{x}-\boldsymbol{x}')(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\sim\mathcal{O}\big(\epsilon^2\rho_0^{-1}\big)\sim\mathcal{O}\left(\frac{1}{r}\right). \tag{39.10.2}$$

Claims preceding equations (29.23a)

The corrections to the spatial components are $\mathcal{O}\big(\epsilon^2\big)$. The only choice is the combination of X or $\mathbf{Y}$ with ρ_0. So we have equation (39.23a).

Claims preceding equations (29.23b)

For coordinate transformation

$$x^i_{\rm new}=x^i_{\rm old}+\Gamma\frac{\partial\chi}{\partial x^i}\quad\text{where}\quad\chi(\boldsymbol{x},t)=-\int\rho_0(\boldsymbol{x}',t)|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x'\,. \tag{39.10.3}$$

We have

$$\frac{\partial x^i_{\rm old}}{\partial x^j_{\rm new}}=\delta_{ij}-\Gamma\frac{\partial^2\chi}{\partial x^i\partial x^j}+\mathcal{O}\big(\epsilon^4\big)=\delta_{ij}+\Gamma\int\rho_0(\boldsymbol{x}',t)\left(\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{x}'|}-\frac{(x_i-x'_i)(x_j-x'_j)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\mathrm{d}^3x'+\mathcal{O}\big(\epsilon^4\big)$$
$$=\delta_{ij}+\Gamma(U\delta_{ij}-U_{ij})+\mathcal{O}\big(\epsilon^4\big). \tag{39.10.4}$$

It follows that

$$g^{\rm new}_{ij}(\boldsymbol{x}^{\rm new})=g^{\rm old}_{kl}(\boldsymbol{x}^{\rm old})\frac{\partial x^k_{\rm old}}{\partial x^j_{\rm new}}\frac{\partial x^l_{\rm old}}{\partial x^j_{\rm new}}=g^{\rm old}_{ij}(\boldsymbol{x}^{\rm old})+2\Gamma(U\delta_{ij}-U_{ij})+\mathcal{O}\big(\epsilon^4\big). \tag{39.10.5}$$

Notice that

$$g^{\rm old}_{ij}(\boldsymbol{x}^{\rm old})-g^{\rm old}_{ij}(\boldsymbol{x}^{\rm new})\sim g^{\rm old}_{ij,k}(x^k_{\rm old}-x^i_{\rm new})\sim U_{,i}\frac{\partial\chi}{\partial x^j}\sim\mathcal{O}\big(\epsilon^4\big). \tag{39.10.6}$$

So we have

$$g^{\rm new}_{ij}(\boldsymbol{x}^{\rm new})=g^{\rm old}_{ij}(\boldsymbol{x}^{\rm new})+2\Gamma(U\delta_{ij}-U_{ij})+\mathcal{O}\big(\epsilon^4\big). \tag{39.10.7}$$

The term $2\Gamma U_{ij}$ in $g_{ij}^{\rm old}$ will be eliminated.

Claims preceding equations (29.23c)

The corrections to the time-spatial components are $\mathcal{O}(\epsilon^3)$. The only choice is the combination of X or $\mathbf{Y}$ with $\rho_0 \boldsymbol{v}$. So we have equation (39.23cd).

Claims preceding equations (29.23f)

The corrections to the 00 components are $\mathcal{O}(\epsilon^4)$. The possible choice can be the combination of $\boldsymbol{v}^2$, $\rho\Pi$, p, U with $\rho_0 X$ or the combination of $v_i v_j$, $t_{\hat{i}\hat{j}}$ with Y_{ij}. One more possible chose is U^2.

39.11 TRANSFORMATION TO MOVING FRAME

Show that the change of coordinates (39.29) changes the PPN metric coefficients from the form (39.23) to the form (39.32).

Solution:

From the definition of $\chi(\boldsymbol{x}, t)$, we have

$$\begin{aligned}
\frac{\partial^2 \chi}{\partial x^j \partial t} &= -\int \frac{\partial \rho_0(\boldsymbol{x}', t)}{\partial t} \frac{x_j - x_j'}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' = \int \frac{\partial \rho_0 v^i}{\partial x'^i} \frac{x_j - x_j'}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' + \text{fractional errors of } \mathcal{O}(\epsilon^2) \\
&= -\int \rho_0 v^i \frac{\partial}{\partial x'^i} \left(\frac{x_j - x_j'}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) \mathrm{d}^3 x' + \text{fractional errors of } \mathcal{O}(\epsilon^2) \\
&= \int \rho_0 v^i \left[\frac{\delta_{ij}}{|\boldsymbol{x} - \boldsymbol{x}'|} - \frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{x} - \boldsymbol{x}'|} \right] \mathrm{d}^3 x' + \text{fractional errors of } \mathcal{O}(\epsilon^2) \\
&= V_j - W_j + \mathcal{O}(\epsilon^5). \qquad (39.11.1)
\end{aligned}$$

Similarly, we also have

$$\frac{\partial^2 \chi}{\partial x^j \partial x^k} = -U \delta_{jk} + U_{jk}. \qquad (39.11.2)$$

From equation (39.23), we have

$$\begin{aligned}
\frac{\partial t_{\rm old}}{\partial t_{\rm new}} &= 1 + \frac{1}{2}\boldsymbol{w}^2 + \frac{3}{8}\boldsymbol{w}^4 + \left(\frac{1}{2}\Delta_2 + \zeta - 1 \right) w^j (V_j - W_j) + \mathcal{O}(\epsilon^6) \\
\frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} &= (1 + \frac{1}{2}\boldsymbol{w}^2) w_j + \left(\frac{1}{2}\Delta_2 + \zeta - 1 \right) (U_{jk} w^k - U w_j) + \mathcal{O}(\epsilon^5) \\
\frac{\partial x^j_{\rm old}}{\partial t_{\rm new}} &= (1 + \frac{1}{2}\boldsymbol{w}^2) w^j + \mathcal{O}(\epsilon^5) \\
\frac{\partial x^j_{\rm old}}{\partial x^k_{\rm new}} &= \delta^j_k + \frac{1}{2} w^j w_k + \mathcal{O}(\epsilon^4). \qquad (39.11.3)
\end{aligned}$$

The transformation of the metric is

$$g^{\rm new}_{\alpha\beta}(x_{\rm new}) = g^{\rm old}_{\mu\nu}(x_{\rm old}) \frac{\partial x^\mu_{\rm old}}{\partial x^\alpha_{\rm new}} \frac{\partial x^\nu_{\rm old}}{\partial x^\beta_{\rm new}}. \qquad (39.11.4)$$

Keep firmly in mind the fact that the potentials U, V_j, W_j, $\mathcal{A}$ and $\mathcal{D}$ are not scalar fields. Each coordinate

system possesses its own potentials. We have

$$
\begin{aligned}
U_{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \left[U_{\rm new} - w^j(V_j^{\rm new} - W_j^{\rm new}) + \frac{1}{2}w^j w^k \chi_{,jk}\right]_{\boldsymbol{x}_{\rm new},t_{\rm new}} + \mathcal{O}\big(\epsilon^6\big)\\
V_j^{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \big(V_j^{\rm new} + w_i U_{\rm new}\big)_{\boldsymbol{x}_{\rm new},t_{\rm new}} + \mathcal{O}\big(\epsilon^5\big)\\
W_j^{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \big(W_j^{\rm new} + w^i U_{ij}^{\rm new}\big)_{\boldsymbol{x}_{\rm new},t_{\rm new}} + \mathcal{O}\big(\epsilon^5\big)\\
\Phi_{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \big(\Phi_{\rm new} + 2\beta_1 w^i V_i^{\rm new} + \beta_1 \boldsymbol{w}^2 U_{\rm new}\big)_{\boldsymbol{x}_{\rm new},t_{\rm new}} + \mathcal{O}\big(\epsilon^6\big)\\
\mathcal{A}_{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \big(\mathcal{A}_{\rm new} + 2w^i W_i^{\rm new} + w^j w^k U_{jk}^{\rm new}\big)_{\boldsymbol{x}_{\rm new},t_{\rm new}} + \mathcal{O}\big(\epsilon^6\big)\\
\mathcal{D}_{\rm old}(\boldsymbol{x}_{\rm old},t_{\rm old}) &= \mathcal{D}_{\rm new}(\boldsymbol{x}_{\rm new},t_{\rm new}) + \mathcal{O}\big(\epsilon^6\big)
\end{aligned}
\tag{39.11.5}
$$

The metric in new coordinates will be

$$
\begin{aligned}
g_{00}^{\rm new}(x_{\rm new}) &= g_{00}^{\rm old}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial t_{\rm new}}\frac{\partial t_{\rm old}}{\partial t_{\rm new}} + 2g_{0j}^{\rm old}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial t_{\rm new}}\frac{\partial x^j_{\rm old}}{\partial t_{\rm new}} + g_{jk}^{\rm old}(x_{\rm old})\frac{\partial x^j_{\rm old}}{\partial t_{\rm new}}\frac{\partial x^k_{\rm old}}{\partial t_{\rm new}}\\
&= (-1 + 2U_{\rm old} - 2\beta U_{\rm old}^2 + 4\Psi_{\rm old} - \zeta\mathcal{A}_{\rm old} - \eta\mathcal{D}_{\rm old}) \times \left[1 + \boldsymbol{w}^2 + \boldsymbol{w}^4 + 2\left(\frac{1}{2}\Delta_2 + \zeta - 1\right)w^j(V_j^{\rm new} - W_j^{\rm new})\right]\\
&+ 2\left(-\frac{7}{2}\Delta_1 V_j^{\rm old} - \frac{1}{2}\Delta_2 W_j^{\rm old}\right)w^j + \delta_{jk}(1 + 2\gamma U_{\rm old})(1 + \boldsymbol{w}^2)w^j w^k + \mathcal{O}\big(\epsilon^6\big)\\
&= -\left[1 + \boldsymbol{w}^2 + \boldsymbol{w}^4 + 2\left(\frac{1}{2}\Delta_2 + \zeta - 1\right)w^j(V_j^{\rm new} - W_j^{\rm new})\right] + 2U_{\rm new}(1 + \boldsymbol{w}^2)\\
&+ 2\left[-w^j(V_j^{\rm new} - W_j^{\rm new}) + \frac{1}{2}w^j w^k \chi_{,jk}\right] - 2\beta^2 U_{\rm new}^2 + 4\left(\Phi_{\rm new} + 2\beta_1 w^i V_i^{\rm new} + \beta_1 \boldsymbol{w}^2 U_{\rm new}\right)\\
&- \zeta\left(\mathcal{A}_{\rm new} + 2w^i W_i^{\rm new} + w^j w^k U_{jk}^{\rm new}\right) - \eta\mathcal{D}_{\rm new} + 2\left(-\frac{7}{2}\Delta_1 V_j^{\rm new} - \frac{1}{2}\Delta_2 W_j^{\rm new}\right)w^j\\
&+ 2\left(-\frac{7}{2}\Delta_1 w_j U_{\rm new} - \frac{1}{2}\Delta_2 w^k U_{kj}^{\rm new}\right)w^j + (1 + \boldsymbol{w}^2)\boldsymbol{w}^2 + 2\gamma U_{\rm new}\boldsymbol{w}^2 + \mathcal{O}\big(\epsilon^6\big)\\
&= (-1 + 2U_{\rm new} - 2\beta U_{\rm new}^2 + 4\Psi_{\rm new} - \zeta\mathcal{A}_{\rm new} - \eta\mathcal{D}_{\rm new}) + (2 - 2\zeta + 8\beta_1 - 7\Delta_1 - \Delta_2)w^j V_j^{\rm new}\\
&+ (1 + 2\gamma + 4\beta_1 - 7\Delta_2)\boldsymbol{w}^2 U_{\rm new} + (1 - \zeta - \Delta_2)U_{jk}^{\rm new} w^j w^k + \mathcal{O}\big(\epsilon^6\big).
\end{aligned}
\tag{39.11.6}
$$

$$
\begin{aligned}
g_{0j}^{\rm new}(x_{\rm new}) &= g_{00}^{\rm old}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial t_{\rm new}}\frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} + g_{0k}^{\rm old}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial t_{\rm new}}\frac{\partial x^k_{\rm old}}{\partial x^j_{\rm new}} + g_{0k}^{\rm old}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial x^j_{\rm new}}\frac{\partial x^k_{\rm old}}{\partial t_{\rm new}} + g_{kl}^{\rm old}(x_{\rm old})\frac{\partial x^k_{\rm old}}{\partial t_{\rm new}}\frac{\partial x^l_{\rm old}}{\partial x^j_{\rm new}}\\
&= (-1 + 2U_{\rm old})\left(1 + \frac{1}{2}\boldsymbol{w}^2\right)\left[(1 + \frac{1}{2}\boldsymbol{w}^2)w_j + \left(\frac{1}{2}\Delta_2 + \zeta - 1\right)(U_{jk}^{\rm new} w^k - U_{\rm new} w_j)\right]\\
&+ \left(-\frac{7}{2}\Delta_1 V_k^{\rm old} - \frac{1}{2}\Delta_2 W_k^{\rm old}\right)\delta_j^k + \delta_{kl}(1 + 2\gamma U_{\rm old})(1 + \frac{1}{2}\boldsymbol{w}^2)w^k\left(\delta_j^l + \frac{1}{2}w^l w_j\right) + \mathcal{O}\big(\epsilon^5\big)\\
&= -\left[(1 + \boldsymbol{w}^2)w_j + \left(\frac{1}{2}\Delta_2 + \zeta - 1\right)(U_{jk}^{\rm new} w^k - U_{\rm new} w_j)\right] + 2U_{\rm new} w_j + \left(-\frac{7}{2}\Delta_1 V_j^{\rm new} - \frac{1}{2}\Delta_2 W_j^{\rm new}\right)\\
&+ \left(-\frac{7}{2}\Delta_1 w_j U_{\rm new} - \frac{1}{2}\Delta_2 w^k U_{kj}^{\rm new}\right) + w_k\left(\delta_j^k + \frac{1}{2}w^k w_j\right) + 2\gamma U_{\rm new} w_j + \frac{1}{2}\boldsymbol{w}^2 w_j + \mathcal{O}\big(\epsilon^5\big)\\
&= \left(-\frac{7}{2}\Delta_1 V_j^{\rm new} - \frac{1}{2}\Delta_2 W_j^{\rm new}\right) + (1 - \zeta - \Delta_2)w^k U_{kj}^{\rm new} + \left(1 + 2\gamma + \zeta + \frac{1}{2}\Delta_2 - \frac{7}{2}\Delta_1\right)w_j U_{\rm new} + \mathcal{O}\big(\epsilon^5\big).
\end{aligned}
\tag{39.11.7}
$$

$$
\begin{aligned}
g^{\rm new}_{jk}(x_{\rm new}) &= g^{\rm old}_{00}(x_{\rm old})\frac{\partial t_{\rm old}}{\partial x^j_{\rm new}}\frac{\partial t_{\rm old}}{\partial x^k_{\rm new}} + g^{\rm old}_{il}(x_{\rm old})\frac{\partial x^i_{\rm old}}{\partial x^j_{\rm new}}\frac{\partial x^l_{\rm old}}{\partial x^k_{\rm new}} + \mathcal{O}(\epsilon^4) \\
&= -w^j w^k + \delta_{il}(1+2\gamma U_{\rm old})\left(\delta^i_j + \frac{1}{2}w^i w_j\right)\left(\delta^l_k + \frac{1}{2}w^l w_k\right) + \mathcal{O}(\epsilon^4) \\
&= \delta_{jk}(1+2\gamma U_{\rm new}) + \mathcal{O}(\epsilon^4). \qquad (39.11.8)
\end{aligned}
$$

Now we obtain the components for the metric in the new PPN coordinates.

39.12 THE TRANSFORMATION BETWEEN COMOVING FRAME AND PPN FRAME

Carry out the details of the derivation of the transformation matrix (39.41); and in the process calculate the correction of $\mathcal{O}(\epsilon^4)$ to $A^{\hat{0}}{}_{\hat{0}}$.

Solution:

Firstly, we have the

$$
\boldsymbol{\omega}^{\tilde{\alpha}} = A^{\tilde{\alpha}}{}_{\hat{\beta}}\boldsymbol{\omega}^{\hat{\beta}}. \qquad (39.12.1)
$$

The transformation matrix is

$$
A^{\tilde{\alpha}}{}_{\hat{\beta}} = \begin{pmatrix} \tilde{\gamma} & \tilde{\gamma}v_{\tilde{i}} \\ \tilde{\gamma}v_{\tilde{i}} & \delta_{ij} + (\tilde{\gamma}-1)n_{\tilde{i}}n_{\tilde{i}} \end{pmatrix}, \qquad (39.12.2)
$$

where $v_{\tilde{i}} = v_i(1+U+\gamma U) + \mathcal{O}(\epsilon^5)$, $\tilde{\gamma} = (1-v^2_{\tilde{i}})^{-1/2}$ and $n_{\tilde{i}} = v_{\tilde{i}}/|\tilde{v}|$. From equation (39.38), we have

$$
\boldsymbol{d}x^{\alpha} = A^{\alpha}{}_{\tilde{\beta}}\boldsymbol{\omega}^{\tilde{\beta}}, \qquad (39.12.3)
$$

where

$$
A^{\alpha}{}_{\hat{\beta}} = \begin{pmatrix} 1+U+\mathcal{O}(\epsilon^4) & -\frac{7}{2}\Delta_1 V_j - \frac{1}{2}\Delta_2 W_j - \left(\frac{1}{2}\alpha_1 - \alpha_2\right)w_j U - \alpha_2 w_k U_{kj} + \mathcal{O}(\epsilon^5) \\ \mathcal{O}(\epsilon^5) & (1-\gamma U)\delta_{jk} + \mathcal{O}(\epsilon^4) \end{pmatrix} \qquad (39.12.4)
$$

So for $\boldsymbol{d}x^{\alpha} = A^{\alpha}{}_{\hat{\beta}}\boldsymbol{\omega}^{\hat{\beta}}$, we have

$$
A^0{}_{\hat{0}} = A^0{}_{\tilde{\beta}}A^{\tilde{\beta}}{}_{\hat{0}} = (1+U)\tilde{\gamma} + \mathcal{O}(\epsilon^4) = 1 + U + \frac{1}{2}v^2 + \mathcal{O}(\epsilon^4). \qquad (39.12.5)
$$

$$
\begin{aligned}
A^0{}_{\hat{i}} &= A^0{}_{\tilde{\beta}}A^{\tilde{\beta}}{}_{\hat{i}} = (1+U)\tilde{\gamma}v_{\tilde{i}} + A^0{}_{\tilde{j}}A^{\tilde{j}}{}_{\hat{i}} + \mathcal{O}(\epsilon^5) \\
&= \left(1 + \frac{1}{2}v^2 + \gamma U + 2U\right)v_i - \frac{7}{2}\Delta_1 V_i - \frac{1}{2}\Delta_2 W_i - \left(\frac{1}{2}\alpha_1 - \alpha_2\right)w_i U - \alpha_2 w_j U_{ji} + \mathcal{O}(\epsilon^5) \qquad (39.12.6)
\end{aligned}
$$

$$
A^i{}_{\hat{0}} = A^i{}_{\tilde{\beta}}A^{\tilde{\beta}}{}_{\hat{0}} = A^i{}_{\tilde{j}}A^{\tilde{j}}{}_{\hat{0}} + \mathcal{O}(\epsilon^5) = (1-\gamma U)\tilde{\gamma}v_{\tilde{i}} + \mathcal{O}(\epsilon^5) = \left(1 + \frac{1}{2}v^2 + U\right) + \mathcal{O}(\epsilon^5) \qquad (39.12.7)
$$

$$
A^i{}_{\hat{j}} = A^i{}_{\tilde{\beta}}A^{\tilde{\beta}}{}_{\hat{j}} = (1-\gamma U)\delta_{ik}\left[\delta_{kj} + (\tilde{\gamma}-1)n_{\tilde{k}}n_{\tilde{j}}\right] + \mathcal{O}(\epsilon^4) = (1-\gamma U)\,\delta_{ij} + \frac{1}{2}v_i v_j + \mathcal{O}(\epsilon^4) \qquad (39.12.8)
$$

The calculation of the correction of $\mathcal{O}(\epsilon^4)$ to $A^{\hat{0}}{}_{\hat{0}}$ is similar but more tedious. I do not perform the calculation here.

39.13 EQUATIONS OF MOTION

Carry out the details of the derivation of the equations of motion (39.44), (39.46), and (39.47). As part of the derivation, calculate the following values of the Christoffel symbols in the PPN coordinate frame:

$$
\begin{aligned}
&\Gamma^0{}_{00} = -U_{,t} + \mathcal{O}\big(U_{,j}\epsilon^3\big), \quad \Gamma^0{}_{0j} = -U_{,j} + \mathcal{O}\big(U_{,j}\epsilon^2\big),\\
&\Gamma^0{}_{jk} = \gamma U_{,t}\delta_{jk} + \frac{7}{2}\Delta_1 V_{(j,k)} + \frac{1}{2}\Delta_2 W_{(j,k)} + \left(\frac{1}{2}\alpha_1 - \alpha_2\right) w_{(j}U_{,k)} + \alpha_2 w_i U_{i(j,k)} + \mathcal{O}\big(U_{,j}\epsilon^3\big)\\
&\Gamma^j{}_{00} = -U_{,j} + \left[(\beta+\gamma)U^2 - 2\Psi + \frac{1}{2}\zeta\mathcal{A} + \frac{1}{2}\eta\mathcal{D} + \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3)\boldsymbol{w}^2 U + \frac{1}{2}(\alpha_1 - 2\alpha_3)w_i V_i + \frac{1}{2}\alpha_2 w_i w_k U_{ik}\right]_{,j}\\
&\qquad - \frac{7}{2}\Delta_1 V_{j,t} - \frac{1}{2}\Delta_2 W_{j,t} + \left(\alpha_2 - \frac{1}{2}\alpha_1\right) w_j U_{,t} - \alpha_2 w_i U_{ij,t} + \mathcal{O}\big(U_{,j}\epsilon^4\big)\\
&\Gamma^j{}_{0k} = \gamma U_{,t}\delta_{jk} - \left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right) V_{[j,k]} - \frac{1}{2}\alpha_1 w_{[j}U_{,k]} + \mathcal{O}\big(U_{,j}\epsilon^3\big)\\
&\Gamma^j{}_{kl} = -\gamma(U_{,j}\delta_{kl} - 2U_{,(k}\delta_{l)j}) + \mathcal{O}\big(U_{,j}\epsilon^2\big).
\end{aligned}
\tag{39.13.1}
$$

Here square brackets on tensor indices denote antisymmetrization, and round brackets denote symmetrization. As part of the derivation, it may be useful to prove and use the relations

$$\chi(t,\boldsymbol{x}) = \int \rho_0(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x', \tag{39.13.2a}$$

$$\chi_{,jk} = -\delta_{jk}U + U_{jk}, \tag{39.13.2b}$$

$$\chi_{,it} = V_i - W_i + \mathcal{O}\big(\epsilon^5\big), \tag{39.13.2c}$$

$$W_{[k,j]} = V_{[k,j]}. \tag{39.13.3}$$

Here χ is the function originally defined in equation (39.29c).

Solution:

The metric is

$$
\begin{aligned}
&g_{00} = -1 + 2U - 2\beta U^2 + 4\Psi - \zeta\mathcal{A} - \eta\mathcal{D} + (\alpha_2 + \alpha_3 - \alpha_1)\boldsymbol{w}^2 U + (2\alpha_3 - \alpha_1)w^j V_j - \alpha_2 U_{jk}w^j w^k + \mathcal{O}\big(\epsilon^6\big),\\
&g_{0j} = -\frac{7}{2}\Delta_1 V_j - \frac{1}{2}\Delta_2 W_j - \alpha_2 w^k U_{kj} + \left(\alpha_2 - \frac{1}{2}\alpha_1\right) w_j U + \mathcal{O}\big(\epsilon^5\big),\\
&g_{jk} = \delta_{jk}(1 + 2\gamma U) + \mathcal{O}\big(\epsilon^4\big).
\end{aligned}
\tag{39.13.4}
$$

The inverse of the metric is

$$
\begin{aligned}
&g^{00} = -1 - 2U - 4U^2 + 2\beta U^2 - 4\Psi + \zeta\mathcal{A} + \eta\mathcal{D} - (\alpha_2 + \alpha_3 - \alpha_1)\boldsymbol{w}^2 U - (2\alpha_3 - \alpha_1)w^j V_j - \alpha_2 U_{jk}w^j w^k + \mathcal{O}\big(\epsilon^6\big),\\
&g^{0j} = -\frac{7}{2}\Delta_1 V_j - \frac{1}{2}\Delta_2 W_j - \alpha_2 w^k U_{kj} + \left(\alpha_2 - \frac{1}{2}\alpha_1\right) w_j U + \mathcal{O}\big(\epsilon^5\big),\\
&g^{jk} = \delta^{jk}(1 - 2\gamma U) + \mathcal{O}\big(\epsilon^4\big).
\end{aligned}
\tag{39.13.5}
$$

Christoffel symbols of the first kind are

$$
\begin{aligned}
&\Gamma_{000} = \frac{1}{2}g_{00,0} \sim \mathcal{O}(U_{,j}\epsilon), \quad \Gamma_{00j} = \frac{1}{2}g_{00,j} \sim \mathcal{O}(U_{,j}), \quad \Gamma_{0jk} = g_{0(j,k)} - \frac{1}{2}g_{jk,0} \sim \mathcal{O}(U_{,j}\epsilon)\\
&\Gamma_{j00} = g_{j0,0} - \frac{1}{2}g_{00,j} \sim \mathcal{O}(U_{,j}), \quad \Gamma_{j0k} = g_{0[j,k]} + \frac{1}{2}g_{jk,0} \sim \mathcal{O}(U_{,j}\epsilon), \quad \Gamma_{jkl} = \frac{1}{2}(g_{jk,l} + g_{jl,k} - g_{kl,j}) \sim \mathcal{O}(U_{,j})
\end{aligned}
\tag{39.13.6}
$$

Christoffel symbols of the second kind are

$$\begin{aligned}
\Gamma^0{}_{00} &= g^{00}\Gamma_{000} + g^{0i}\Gamma_{i00} = -\Gamma_{000} + \mathcal{O}\big(U_{,j}\epsilon^3\big) = -\frac{1}{2}g_{00,0} + \mathcal{O}\big(U_{,j}\epsilon^3\big),\\
\Gamma^0{}_{0j} &= g^{00}\Gamma_{00j} + g^{0i}\Gamma_{i0j} = -\Gamma_{00j} + \mathcal{O}\big(U_{,j}\epsilon^2\big) = -\frac{1}{2}g_{00,j} + \mathcal{O}\big(U_{,j}\epsilon^2\big),\\
\Gamma^0{}_{jk} &= g^{00}\Gamma_{0jk} + g^{0i}\Gamma_{ijk} = -\Gamma_{0jk} + \mathcal{O}\big(U_{,j}\epsilon^3\big) = \frac{1}{2}g_{jk,0} - g_{0(j,k)},\\
\Gamma^j{}_{00} &= g^{j0}\Gamma_{000} + g^{ji}\Gamma_{i00} = (1-2\gamma U)\Gamma_{j00} + \mathcal{O}\big(U_{,j}\epsilon^4\big) = g_{j0,0} - \frac{1}{2}(1-2\gamma U)g_{00,j} + \mathcal{O}\big(U_{,j}\epsilon^4\big),\\
\Gamma^j{}_{0k} &= g^{j0}\Gamma_{00k} + g^{ji}\Gamma_{i0k} = \Gamma_{j0k} + \mathcal{O}\big(U_{,j}\epsilon^3\big) = g_{0[j,k]} + \frac{1}{2}g_{jk,0} + \mathcal{O}\big(U_{,j}\epsilon^3\big),\\
\Gamma^j{}_{kl} &= g^{j0}\Gamma_{0kl} + g^{ji}\Gamma_{ikl} = \Gamma_{jkl} + \mathcal{O}\big(U_{,j}\epsilon^2\big) = \frac{1}{2}(g_{jk,l} + g_{jl,k} - g_{kl,j}) + \mathcal{O}\big(U_{,j}\epsilon^2\big).
\end{aligned} \tag{39.13.7}$$

Substituting equation 39.13.4 into 39.13.7, we can derive the equation 39.13.1 easily.

The only trick lies in the simplification of $g_{0[j,k]}$. We need to derive equation 39.13.2 and 39.13.3 first. We have already derived the equation 39.13.2 in Exercise 39.11. As for 39.13.3, we have

$$V_{j,k} = -\int \rho_0(t,\boldsymbol{x}')v_j(t,\boldsymbol{x}')\frac{x_k - x'_k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}^3x' \tag{39.13.8}$$

and

$$\begin{aligned}
W_{j,k} &= \int \rho_0(t,\boldsymbol{x}')v_i(t,\boldsymbol{x}')\left[\frac{\delta_{ik}(x_j - x'_j) + \delta_{jk}(x_i - x'_i)}{|\boldsymbol{x}-\boldsymbol{x}'|^3} - 3\frac{(x_i - x'_i)(x_j - x'_j)(x_k - x'_k)}{|\boldsymbol{x}-\boldsymbol{x}'|^5}\right]\mathrm{d}^3x'\\
&= -V_{k,j} + \int \rho_0(t,\boldsymbol{x}')v_i(t,\boldsymbol{x}')\frac{(x_i - x'_i)}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[\delta_{jk} - 3\frac{(x_j - x'_j)(x_k - x'_k)}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right]\mathrm{d}^3x'\,.
\end{aligned} \tag{39.13.9}$$

It follows that $W_{[k,j]} = V_{[k,j]}$. Now we have

$$\begin{aligned}
g_{0[j,k]} &= -\frac{7}{2}\Delta_1 V_{[j,k]} - \frac{1}{2}\Delta_2 W_{[j,k]} - \alpha_2 w^i U_{i[j,k]} + \left(\alpha_2 - \frac{1}{2}\alpha_1\right)w_{[j}U_{,k]} + \mathcal{O}\big(U_{,j}\epsilon^3\big)\\
&= -\left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_{[j,k]} - \frac{1}{2}\alpha_1 w_{[j}U_{,k]} - \alpha_2 w^i(U_{i[j,k]} - \delta_{i[j}U_{,k]}) + \mathcal{O}\big(U_{,j}\epsilon^3\big)\\
&= -\left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_{[j,k]} - \frac{1}{2}\alpha_1 w_{[j}U_{,k]} - \alpha_2 w^i\chi_{,i[jk]} + \mathcal{O}\big(U_{,j}\epsilon^3\big)\\
&= -\left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_{[j,k]} - \frac{1}{2}\alpha_1 w_{[j}U_{,k]} + \mathcal{O}\big(U_{,j}\epsilon^3\big)
\end{aligned} \tag{39.13.10}$$

And then we can get the expression for $\Gamma^j{}_{0k}$.

Using the law of conservation of baryon number, we have

$$(\rho_0 u^\alpha)_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\rho_0 u^\alpha)_{,\alpha} = 0. \tag{39.13.11}$$

Since

$$\rho_0 u^0\sqrt{-g} = \rho_0\left(1 + \frac{1}{2}v^2 + U\right)\sqrt{(1-2U)(1+2\gamma U)^3} + \mathcal{O}\big(\rho_0\epsilon^4\big) = \rho_0\left(1 + \frac{1}{2}v^2 + 3\gamma U\right) + \mathcal{O}\big(\rho_0\epsilon^4\big), \tag{39.13.12}$$

we would define

$$\rho^* \equiv \rho_0\left(1 + \frac{1}{2}v^2 + 3\gamma U\right), \tag{39.13.13}$$

and so we have

$$\sqrt{-g}\rho_0 u^0 = \rho^* + \mathcal{O}\big(\rho_0\epsilon^4\big), \quad \sqrt{-g}\rho_0 u^i = \sqrt{-g}\rho_0 u^0 v^i = \rho^* v^i + \mathcal{O}\big(\rho_0\epsilon^5\big) \tag{39.13.14}$$

The law of conservation of baryon number would be

$$\rho^*_{,t} + (\rho^* v^i)_{,i} = \mathcal{O}\left(\rho_{0,j}\epsilon^5\right). \tag{39.13.15}$$

The stress-energy tensor in the PPN coordinate frame is

$$\begin{aligned}
T^{00} &= \rho_0(1+\Pi+v^2+2U) + \mathcal{O}\left(\rho_0\epsilon^4\right),\\
T^{0j} &= \rho_0(1+\Pi+v^2+2U)v^j + t_{\hat{j}\hat{m}}v_m + \mathcal{O}\left(\rho_0\epsilon^5\right),\\
T^{jk} &= t_{\hat{j}\hat{k}}(1-2\gamma U) + \rho_0(1+\Pi+v^2+2U)v_j v_k + \frac{1}{2}(v_j t_{\hat{k}\hat{m}}v_m + v_k t_{\hat{j}\hat{m}}v_m) + \mathcal{O}\left(\rho_0\epsilon^6\right).
\end{aligned} \tag{39.13.16}$$

The law of conservation of local energy-momentum gives

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{\alpha\mu} = 0. \tag{39.13.17}$$

Notice that

$$\Gamma^{\beta}{}_{\mu\beta} = (\ln\sqrt{-g})_{,\mu} = \frac{1}{2}[\ln(1-2U)(1+2\gamma U)^3]_{,\mu} + \mathcal{O}\left(U_{,\mu}\epsilon^2\right) = (3\gamma-1)U_{,\mu} + \mathcal{O}\left(U_{,\mu}\epsilon^2\right). \tag{39.13.18}$$

For $\alpha = 0$, we have

$$T^{0\beta}{}_{;\beta} = T^{00}{}_{,0} + T^{0j}{}_{,j} + \Gamma^{0}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{0\mu} = 0. \tag{39.13.19}$$

Since

$$\begin{aligned}
\Gamma^{0}{}_{\mu\beta}T^{\mu\beta} &= \Gamma^{0}{}_{00}T^{00} + 2\Gamma^{0}{}_{0j}T^{0j} + \Gamma^{0}{}_{jk}T^{jk}\\
&= -U_{,t}\rho_0 - 2U_{,j}\rho_0 v^j + \mathcal{O}\left(\rho_{0,j}\epsilon^5\right)
\end{aligned} \tag{39.13.20}$$

and

$$\begin{aligned}
\Gamma^{\beta}{}_{\mu\beta}T^{0\mu} &= (3\gamma-1)U_{,\mu}T^{0\mu} + \mathcal{O}\left(\rho_{0,j}\epsilon^5\right)\\
&= (3\gamma-1)U_{,t}\rho_0 + (3\gamma-1)U_{,j}\rho_0 v^j + \mathcal{O}\left(\rho_{0,j}\epsilon^5\right),
\end{aligned} \tag{39.13.21}$$

equation 39.13.19 becomes

$$[\rho_0(1+\Pi+v^2+2U)]_{,t} + [\rho_0(1+\Pi+v^2+2U)v^j + t_{\hat{j}\hat{m}}v_m]_{,j} + (3\gamma-2)U_{,t}\rho_0 + (3\gamma-3)U_{,j}\rho_0 v^j = \mathcal{O}\left(\rho_{0,j}\epsilon^5\right). \tag{39.13.22}$$

Using

$$\rho_{0,t} + (\rho_0 v^j)_{,j} = \mathcal{O}\left(\rho_{0,j}\epsilon^3\right), \tag{39.13.23}$$

we can get

$$\begin{aligned}
\mathcal{O}\left(\rho_{0,j}\epsilon^5\right) &= \rho_{0,t} + (\rho_0 v^j)_{,j} + \rho_0(\Pi+v^2+2U)_{,t} + \rho_0 v^j(\Pi+v^2+2U)_{,j} + (t_{\hat{j}\hat{m}}v_m)_{,j} + (3\gamma-2)U_{,t}\rho_0 + (3\gamma-3)U_{,j}\rho_0 v^j\\
&= \left(1+\frac{1}{2}v^2+3\gamma U\right)[\rho_{0,t} + (\rho_0 v^j)_{,j}] + \rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + \rho_0(v^2+3\gamma U)_{,t} + \rho_0 v^j(v^2+3\gamma U-U)_{,j} + (t_{\hat{j}\hat{m}}v_m)_{,j}.
\end{aligned} \tag{39.13.24}$$

Using

$$\frac{1}{2}\rho_0\frac{\mathrm{d}v^2}{\mathrm{d}t} = \rho_0 U_{,j}v^j - t_{\hat{j}\hat{k},\hat{k}}v^j + \mathcal{O}\left(\rho_{0,j}\epsilon^5\right), \tag{39.13.25}$$

we can get

$$\begin{aligned}
\mathcal{O}\left(\rho_{0,j}\epsilon^5\right) &= \left(1+\frac{1}{2}v^2+3\gamma U\right)[\rho_{0,t} + (\rho_0 v^j)_{,j}] + \rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + \rho_0(v^2+3\gamma U)_{,t} + \rho_0 v^j(v^2+3\gamma U-U)_{,j}\\
&\quad + t_{\hat{j}\hat{m}}v_{m,j} + \rho_0 U_{,j}v^j - \frac{1}{2}\rho_0\frac{\mathrm{d}v^2}{\mathrm{d}t}\\
&= \left(1+\frac{1}{2}v^2+3\gamma U\right)[\rho_{0,t} + (\rho_0 v^j)_{,j}] + \rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + \rho_0\left(\frac{1}{2}v^2+3\gamma U\right)_{,t} + \rho_0 v^j\left(\frac{1}{2}v^2+3\gamma U\right)_{,j} + t_{\hat{j}\hat{m}}v_{m,j}\\
&= \rho^*_{,t} + (\rho^* v^i)_{,i} + \rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + t_{\hat{j}\hat{m}}v_{m,j}\\
&= \rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + t_{\hat{j}\hat{k}}v_{j,k}.
\end{aligned} \tag{39.13.26}$$

For $\alpha = j$, we have

$$T^{j\beta}{}_{;\beta} = T^{j0}{}_{,0} + T^{jk}{}_{,k} + \Gamma^{j}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{j\mu} = 0. \tag{39.13.27}$$

Using equation 39.13.22, we have

$$\begin{aligned}
T^{j0}{}_{,0} + T^{jk}{}_{,k} &= [\rho_0(1+\Pi+v^2+2U)v^j + t_{\hat{j}\hat{m}}v_m]_{,t} \\
&\quad + [t_{\hat{j}\hat{k}}(1-2\gamma U) + \rho_0(1+\Pi+v^2+2U)v_jv_k + \frac{1}{2}(v_jt_{\hat{k}\hat{m}}v_m + v_kt_{\hat{j}\hat{m}}v_m)]_{,k} + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big) \\
&= \rho_0(1+\Pi+v^2+2U)\frac{\mathrm{d}v^j}{\mathrm{d}t} - v^j[(t_{\hat{k}\hat{m}}v_m)_{,k} + (3\gamma-2)U_{,t}\rho_0 + (3\gamma-3)U_{,k}\rho_0v^k] \\
&\quad + [t_{\hat{j}\hat{k}}(1-2\gamma U) + \frac{1}{2}(v_jt_{\hat{k}\hat{m}}v_m + v_kt_{\hat{j}\hat{m}}v_m)]_{,k} + (t_{\hat{j}\hat{k}}v_k)_{,t} + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big).
\end{aligned} \tag{39.13.28}$$

Using

$$t_{\hat{j}\hat{k}}v_{k,t} + t_{\hat{j}\hat{k}}v_mv_{k,m} = t_{\hat{j}\hat{k}}U_{,k} - \frac{t_{\hat{j}\hat{k}}t_{\hat{k}\hat{m},m}}{\rho_0} + \mathcal{O}\big(\rho_{0,j}\epsilon^5\big), \tag{39.13.29}$$

we have

$$\begin{aligned}
T^{j0}{}_{,0} + T^{jk}{}_{,k} &= \rho_0(1+\Pi+v^2+2U)\frac{\mathrm{d}v^j}{\mathrm{d}t} + [t_{\hat{j}\hat{k}}(1-2\gamma U)]_{,k} - v^j[(3\gamma-2)U_{,t}\rho_0 + (3\gamma-3)U_{,k}\rho_0v^k] + t_{\hat{j}\hat{k}}U_{,k} \\
&\quad + t_{\hat{j}\hat{k},t}v_k + \frac{1}{2}(v_{j,k}t_{\hat{k}\hat{m}}v_m - t_{\hat{j}\hat{m}}v_{m,k}v_k) + \frac{1}{2}[v_m(t_{\hat{m}\hat{j}}v_k)_{,k} - v_j(t_{\hat{k}\hat{l}}v_k)_{,l}] - \frac{t_{\hat{j}\hat{k}}t_{\hat{k}\hat{m},m}}{\rho^*} + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big)
\end{aligned} \tag{39.13.30}$$

Also, we have

$$\begin{aligned}
\Gamma^{j}{}_{00}T^{00} &= -\rho_0(1+\Pi+v^2+2U)U_{,j} - \rho_0\left[2\Psi - \frac{1}{2}\zeta\mathcal{A} - \frac{1}{2}\eta\mathcal{D} - \frac{1}{2}\alpha_2w_iw_kU_{ik}\right]_{,j} + \frac{1}{2}\rho_0(\alpha_1-2\alpha_3)w_kV_{k,j} \\
&\quad - \rho_0U_{,j}\left[-2(\beta+\gamma)U + \frac{1}{2}(\alpha_2+\alpha_3-\alpha_1)\boldsymbol{w}^2\right] \\
&\quad - \frac{7}{2}\rho_0\Delta_1V_{j,t} - \frac{1}{2}\rho_0\Delta_2W_{j,t} + \left(\alpha_2 - \frac{1}{2}\alpha_1\right)\rho_0w_jU_{,t} - \alpha_2\rho_0w_iU_{ij,t} + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big)
\end{aligned} \tag{39.13.31a}$$

$$2\Gamma^{j}{}_{0k}T^{0k} = 2\gamma\rho_0U_{,t}v_j - \left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)(V_{j,k} - V_{k,j})\rho_0v_k - \frac{1}{2}\alpha_1w_jU_{,k}\rho_0v_k + \frac{1}{2}\alpha_1\rho_0w_kv_kU_{,j} + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big) \tag{39.13.31b}$$

$$\Gamma^{j}{}_{kl}T^{kl} = -\gamma U_{,j}(3p + \rho_0v^2) + 2\gamma U_{,k}(t_{\hat{j}\hat{k}} + \rho_0v_jv_k) + \mathcal{O}\big(\rho_{0,j}\epsilon^6\big). \tag{39.13.31c}$$

Adding equations 39.13.31 together and using the fact that

$$U_{ij,t} - \delta_{ij}U_{,t} = \chi_{,ijt} = \chi_{,itj} = (V_i - W_i)_{,j} + \mathcal{O}\big(\epsilon^5\big), \tag{39.13.32}$$

we have

$$\begin{aligned}
\Gamma^{j}{}_{\mu\beta}T^{\mu\beta} &= -\rho_0(1+\Pi+v^2+2U)U_{,j} - \rho_0\left[2\Psi - \frac{1}{2}\zeta\mathcal{A} - \frac{1}{2}\eta\mathcal{D} - \frac{1}{2}\alpha_2w_iw_kU_{ik} + \alpha_2w_i(V_i - W_i)\right]_{,j} \\
&\quad + \frac{1}{2}\rho_0(\alpha_1-2\alpha_3)w_kV_{k,j} - \rho_0U_{,j}\left[\gamma v^2 - \frac{1}{2}\alpha_1v_iw_i + \frac{1}{2}(\alpha_2+\alpha_3-\alpha_1)\boldsymbol{w}^2 - 2(\beta+\gamma)U + 3\gamma p/\rho_0\right] \\
&\quad - \frac{7}{2}\rho_0\Delta_1V_{j,t} - \frac{1}{2}\rho_0\Delta_2W_{j,t} - \frac{1}{2}\alpha_1\rho_0w_jU_{,t} + 2\gamma\rho_0U_{,t}v_j - \left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_{j,k}\rho_0v_k - \frac{1}{2}\alpha_1w_jU_{,k}\rho_0v_k \\
&\quad + 2\gamma U_{,k}(t_{\hat{j}\hat{k}} + \rho_0v_jv_k) + \left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_{k,j}\rho_0v_k\mathcal{O}\big(\rho_{0,j}\epsilon^6\big).
\end{aligned} \tag{39.13.33}$$

Using $\mathrm{d}/\mathrm{d}t = \partial/\partial t + v^k\,\partial/\partial x^k$, we have

$$\begin{aligned}\Gamma^j{}_{\mu\beta}T^{\mu\beta} =& -\rho_0(1+\Pi+v^2+2U)U_{,j} - \rho^*\left[2\Psi - \frac{1}{2}\zeta\mathcal{A} - \frac{1}{2}\eta\mathcal{D} - \frac{1}{2}\alpha_2 w_i w_k U_{ik} + \alpha_2 w_i(V_i - W_i)\right]_{,j}\\ &- \rho^* U_{,j}\left[\gamma v^2 - \frac{1}{2}\alpha_1 v_i w_i + \frac{1}{2}(\alpha_2+\alpha_3-\alpha_1)\boldsymbol{w}^2 - (2\beta-2)U + 3\gamma p/\rho^*\right] + (2\gamma+2)\rho_0 U U_{,j}\\ &+ \rho^*\frac{\mathrm{d}}{\mathrm{d}t}\left[-\left(\frac{7}{2}\Delta_1 + \frac{1}{2}\Delta_2\right)V_j - \frac{1}{2}\alpha_1 w_j U\right] + \frac{1}{2}\Delta_2\rho^*(V_j - W_j)_{,t} + 2\gamma\rho^* v_j(U_{,t} + v_k U_{,k})\\ &+ \frac{1}{2}\rho^*[(7\Delta_1+\Delta_2)v_k + (\alpha_1 - 2\alpha_3)w_k]V_{k,j} + 2\gamma U_{,k}t_{\hat{j}\hat{k}} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right). \end{aligned} \tag{39.13.34}$$

Finally, we have

$$\Gamma^\beta{}_{\mu\beta}T^{j\mu} = (3\gamma-1)U_{,t}\rho_0 v^j + (3\gamma-1)U_{,k}\left(t_{\hat{j}\hat{k}} + \rho_0 v_j v_k\right) \tag{39.13.35}$$

Now focus on the expression

$$\begin{aligned}C \equiv& \rho_0(1+\Pi+v^2+2U)\frac{\mathrm{d}v^j}{\mathrm{d}t} + [t_{\hat{j}\hat{k}}(1-2\gamma U)]_{,k} - v^j[(3\gamma-2)U_{,t}\rho_0 + (3\gamma-3)U_{,k}\rho_0 v^k] + t_{\hat{j}\hat{k}}U_{,k}\\ &- \rho_0(1+\Pi+v^2+2U)U_{,j} + (2\gamma+2)\rho_0 U U_{,j} + 2\gamma\rho^* v_j(U_{,t} + v_k U_{,k}) + 2\gamma U_{,k}t_{\hat{j}\hat{k}}\\ &+ (3\gamma-1)U_{,t}\rho_0 v^j + (3\gamma-1)U_{,k}\left(t_{\hat{j}\hat{k}} + \rho_0 v_j v_k\right) + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right). \end{aligned} \tag{39.13.36}$$

We have

$$\begin{aligned}C =& \rho_0(1+\Pi+v^2+2U)\frac{\mathrm{d}v^j}{\mathrm{d}t} - \rho_0(1+\Pi+v^2-2\gamma U)U_{,j} + (1-2\gamma U)t_{\hat{j}\hat{k},k}\\ &+ 3\gamma t_{\hat{j}\hat{k}}U_{,k} + (2\gamma+1)\rho_0 v^j U_{,t} + (2\gamma+2)\rho_0 v^j v^k U_{,k} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right). \end{aligned} \tag{39.13.37}$$

As

$$\begin{aligned}&(2\gamma+1)\rho_0 v^j U_{,t} + (2\gamma+2)\rho_0 v^j v^k U_{,k} = (2\gamma+2)\rho_0 v^j\frac{\mathrm{d}U}{\mathrm{d}t} - \rho_0 v^j v^k U_{,k}\\ &= (2\gamma+2)\rho^*\frac{\mathrm{d}Uv^j}{\mathrm{d}t} - (2\gamma+2)\rho_0 U\frac{\mathrm{d}v^j}{\mathrm{d}t} - \rho^* v^j U_{,t} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right), \end{aligned} \tag{39.13.38}$$

we have

$$\begin{aligned}&C = \rho_0(1+\Pi+v^2-2\gamma U)\left(\frac{\mathrm{d}v^j}{\mathrm{d}t} - U_{,j}\right) + (1-2\gamma U)t_{\hat{j}\hat{k},k} + 3\gamma t_{\hat{j}\hat{k}}U_{,k} - \rho^* v^j U_{,t} + (2\gamma+2)\rho^*\frac{\mathrm{d}Uv^j}{\mathrm{d}t} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right)\\ &= \rho^*\left(\frac{\mathrm{d}v^j}{\mathrm{d}t} - U_{,j}\right) + \left(\frac{1}{2}v^2 + \Pi - 5\gamma U\right)\left(\rho_0\frac{\mathrm{d}v^j}{\mathrm{d}t} - \rho_0 U_{,j} + t_{\hat{j}\hat{k},k}\right) - \left(\frac{1}{2}v^2 + \Pi - 5\gamma U\right)t_{\hat{j}\hat{k},k}\\ &+ (1-2\gamma U)t_{\hat{j}\hat{k},k} + 3\gamma t_{\hat{j}\hat{k}}U_{,k} - \rho^* v^j U_{,t} + (2\gamma+2)\rho^*\frac{\mathrm{d}Uv^j}{\mathrm{d}t} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right)\\ &= \rho^*\left(\frac{\mathrm{d}v^j}{\mathrm{d}t} - U_{,j}\right) + [t_{\hat{j}\hat{k}}(1+3\gamma)U]_{,k} - \left(\frac{1}{2}v^2 + \Pi\right)t_{\hat{j}\hat{k},k} - \rho^* v^j U_{,t} + (2\gamma+2)\rho^*\frac{\mathrm{d}Uv^j}{\mathrm{d}t} + \mathcal{O}\left(\rho_{0,j}\epsilon^6\right) \end{aligned} \tag{39.13.39}$$

Now bring equations 39.13.30, 39.13.34, 39.13.35, 39.13.36 and 39.13.39 together, we have the post-

Newtonian Euler equation

$$\rho^*\left(\frac{\mathrm{d}v^j}{\mathrm{d}t}-U_{,j}\right)+[t_{\hat{j}\hat{k}}(1+3\gamma)U]_{,k}-\left(\frac{1}{2}v^2+\Pi\right)t_{\hat{j}\hat{k},k}-\rho^*v^jU_{,t}+t_{\hat{j}\hat{k},t}v_k-\frac{t_{\hat{j}\hat{k}}t_{\hat{k}\hat{m},m}}{\rho^*}$$
$$+\rho^*\frac{\mathrm{d}}{\mathrm{d}t}\left[(2\gamma+2)Uv^j-\left(\frac{7}{2}\Delta_1+\frac{1}{2}\Delta_2\right)V_j-\frac{1}{2}\alpha_1w_jU\right]+\frac{1}{2}\Delta_2\rho^*(V_j-W_j)_{,t}$$
$$-\rho^*\left[2\Psi-\frac{1}{2}\zeta\mathcal{A}-\frac{1}{2}\eta\mathcal{D}-\frac{1}{2}\alpha_2w_iw_kU_{ik}+\alpha_2w_i(V_i-W_i)\right]_{,j}+\frac{1}{2}\rho^*[(7\Delta_1+\Delta_2)v_k+(\alpha_1-2\alpha_3)w_k]V_{k,j}$$
$$-\rho^*U_{,j}\left[\gamma v^2-\frac{1}{2}\alpha_1v_iw_i+\frac{1}{2}(\alpha_2+\alpha_3-\alpha_1)\boldsymbol{w}^2-(2\beta-2)U+3\gamma p/\rho^*\right]$$
$$+\frac{1}{2}(v_{j,k}t_{\hat{k}\hat{m}}v_m-t_{\hat{j}\hat{m}}v_{m,k}v_k)+\frac{1}{2}[v_m(t_{\hat{m}\hat{j}}v_k)_{,k}-v_j(t_{\hat{k}\hat{l}}v_k)_{,l}]=\mathcal{O}\big(\rho_{0,j}\epsilon^6\big). \tag{39.13.40}$$

39.14 POST-NEWTONIAN APPROXIMATION TO GENERAL RELATIVITY

Perform a post-Newtonian expansion of Einstein's field equations, thereby obtaining the values cited in Box 39.2 for the PPN parameters of general relativity. The calculations might best follow the approach of Chandrasekhar (1965a): Set $g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}$ and assume

$$h_{00}=\mathcal{O}\big(\epsilon^2\big)+\mathcal{O}\big(\epsilon^4\big),\quad h_{0j}=\mathcal{O}\big(\epsilon^3\big),\quad h_{jk}=\mathcal{O}\big(\epsilon^2\big). \tag{39.14.1}$$

Choose the space and time coordinates so that the four "gauge conditions"

$$h_{jk,k}-\frac{1}{2}h_{,j}=\mathcal{O}\left(\frac{\epsilon^4}{R_\odot}\right),\quad h_{0k,k}-\frac{1}{2}h_{kk,0}=\mathcal{O}\left(\frac{\epsilon^5}{R_\odot}\right)\quad\text{with}\quad h=h_{\alpha\beta}\eta^{\alpha\beta}=-h_{00}+h_{ll} \tag{39.14.2}$$

are satisfied.

(a) Show that the spatial gauge conditions are the post-Newtonian approximations to those (35.1a) used in the study of weak gravitational waves, but that the temporal gauge condition is not.

(b) Use these gauge conditions and the post-Newtonian limit in equations (8.24) and (8.47) to obtain for the Ricci tensor, accurate to linearized order,

$$R_{00}=-\frac{1}{2}h_{00,mm}+\mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right),\quad R_{jk}=-\frac{1}{2}h_{jk,mm}+\mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right),\quad R_{0j}=-\frac{1}{2}h_{0j,mm}-\frac{1}{4}h_{00,0j}+\mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right). \tag{39.14.3}$$

(c) Combine these with the Newtonian form (39.13) of the stress-energy tensor, and with equation (39.27), to obtain the following metric coefficients, accurate to linearized order:

$$h_{00}=2U+k_{00}+\mathcal{O}\big(\epsilon^6\big),\quad h_{0j}=-\frac{7}{2}V_j-\frac{1}{2}W_j+\mathcal{O}\big(\epsilon^5\big),\quad h_{jk}=2U\delta_{jk}+\mathcal{O}\big(\epsilon^4\big). \tag{39.14.4}$$

Here U, V_j, and W_j are to be regarded as defined by equations (39.34a,b,c). By comparing these metric coefficients with equations 39.13.4, discover that

$$\gamma=1,\quad \Delta_1=1,\quad \Delta_2=1 \tag{39.14.5}$$

for general relativity.

(d) With this knowledge of the metric in linearized order, one can carry out the analysis of section 39.10 to obtain the post-Newtonian corrections to the stress-energy tensor.

(e) Calculate, similarly, the post-Newtonian corrections to the Ricci tensor component R_{00}, using $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, using $h_{\alpha\beta}$ as given in equations (39.54), and using the gauge conditions (39.52). The answer should be

$$R_{00} = \left(-U - \frac{1}{2}k_{00} - U^2\right)_{,mm} + 4UU_{,mm} + \mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right). \tag{39.14.6}$$

(f) Evaluate the Einstein equation $R_{00} = 8\pi(T_{00} - g_{00}T/2)$, accurate to post-Newtonian order, and solve it to obtain the post-Newtonian metric correction

$$k_{00} = -2U^2 + 4\Psi, \tag{39.14.7}$$

where Ψ is given by equation (39.43d) with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$. By comparing with equations (39.32c) and (39.34d), discover that

$$\beta = \beta_1 = \beta_2 = \beta_3 = \beta_4 = 1, \quad \zeta = \eta = 0 \tag{39.14.8}$$

for general relativity.

(g) Knowing the full post-Newtonian metric, and the full post-Newtonian stress-energy tensor, one can carry out the calculations of section 39.11 to obtain the post-Newtonian equations of motion for the matter [equations (39.44), (39.46), and (39.47)].

Solution:

(a) Gauge conditions used in the study of weak gravitational waves is

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{39.14.9}$$

The temporal component is

$$\bar{h}^{00}{}_{,0} + \bar{h}^{0k}{}_{,k} = \left(h_{00} + \frac{1}{2}h\right)_{,0} - h_{0k,k} = \frac{1}{2}h_{00,0} + \frac{1}{2}h_{kk,0} - h_{0k,k} = 0. \tag{39.14.10}$$

Since $h_{00,0} \sim \mathcal{O}(\epsilon^3/R_\odot)$, temporal component is different from those in 39.14.1.

The spatial components are

$$\bar{h}^{j0}{}_{,0} + \bar{h}^{jk}{}_{,k} = -h_{j0,0} + \left(h_{jk} - \frac{1}{2}\delta_{jk}h\right)_{,k} = h_{jk,k} - \frac{1}{2}h_{,j} - h_{j0,0} = 0. \tag{39.14.11}$$

Since $h_{j0,0} \sim \mathcal{O}(\epsilon^4/R_\odot)$, spatial components are the same as those in 39.14.1.

(b) To calculate Ricci tensor to linearized order, we are free to use η to lower and raise indices. We have

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \mathcal{O}(\Gamma^2) = \frac{1}{2}(h_\mu{}^\alpha{}_{,\nu\alpha} + h_\nu{}^\alpha{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^\alpha - h_{,\mu\nu}) + \mathcal{O}(\Gamma^2). \tag{39.14.12}$$

So

$$R_{00} = \frac{1}{2}(2h_0{}^\alpha{}_{,0\alpha} - h_{00,\alpha}{}^\alpha - h_{,00}) + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) = -\frac{1}{2}h_{00,mm} + +\mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \tag{39.14.13}$$

$$\begin{aligned} R_{0j} &= \frac{1}{2}(h_0{}^\alpha{}_{,j\alpha} + h_j{}^\alpha{}_{,0\alpha} - h_{0j,\alpha}{}^\alpha - h_{,0j}) + \mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right) \\ &= \frac{1}{2}(-h_{00,j0} + h_{0k,jk} + h_{jk,0k} - h_{0j,kk} - h_{,0j}) + \mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right) \\ &= \frac{1}{2}(-h_{00,j0} + \frac{1}{2}h_{kk,0j} - h_{0j,kk} - \frac{1}{2}h_{,0j}) + \mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right) \\ &= -\frac{1}{2}h_{0j,mm} - \frac{1}{4}h_{00,0j} + \mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right). \end{aligned} \tag{39.14.14}$$

$$
\begin{aligned}
R_{jk} &= \frac{1}{2}(h_j{}^{\alpha}{}_{,k\alpha} + h_k{}^{\alpha}{}_{,j\alpha} - h_{jk,\alpha}{}^{\alpha} - h_{,jk}) + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \\
&= \frac{1}{2}(h_{jm,km} + h_{km,jm} - h_{jk,mm} - h_{,jk}) + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \\
&= \frac{1}{2}(\frac{1}{2}h_{,jk} + \frac{1}{2}h_{,kj} - h_{jk,mm} - h_{,jk}) + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \\
&= -\frac{1}{2}h_{jk,mm} + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right). \qquad (39.14.15)
\end{aligned}
$$

Notice that

$$
R = \eta_{\alpha\beta}R^{\alpha\beta} + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) = \frac{1}{2}h_{00,mm} - \frac{1}{2}h_{kk,mm} + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right). \qquad (39.14.16)
$$

(c)The Newtonian form (39.13) of the stress-energy tensor is

$$
T^{00} = \rho_0 + \mathcal{O}\big(\rho_0\epsilon^2\big), \quad T^{0i} = \rho_0 v_i + \mathcal{O}\big(\rho_0\epsilon^3\big), \quad T^{jk} = t_{\hat{j}\hat{k}} + \rho_0 v_j v_k + \mathcal{O}\big(\rho_0\epsilon^4\big). \qquad (39.14.17)
$$

Since

$$
\rho_0 \sim \mathcal{O}\left(\frac{\epsilon^2}{R_\odot^2}\right), \qquad (39.14.18)
$$

so we have

$$
-\frac{1}{4}h_{00,mm} - \frac{1}{4}h_{kk,mm} = 8\pi\rho_0 + \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \qquad (39.14.19)
$$

$$
-\frac{1}{2}h_{jk,mm} - \frac{1}{4}\delta_{jk}h_{00,mm} + \frac{1}{4}\delta_{jk}h_{ll,mm} = \mathcal{O}\left(\frac{\epsilon^4}{R_\odot^2}\right) \qquad (39.14.20)
$$

$$
-\frac{1}{2}h_{0j,mm} - \frac{1}{4}h_{00,0j} = -8\pi\rho_0 v_i + \mathcal{O}\left(\frac{\epsilon^5}{R_\odot^2}\right). \qquad (39.14.21)
$$

It is easy to find that

$$
h_{00} = 2U + \mathcal{O}\big(\epsilon^4\big), \quad h_{jk} = 2U\delta_{jk} + \mathcal{O}\big(\epsilon^4\big). \qquad (39.14.22)
$$

It follows that

$$
h_{0j,mm} = 16\pi\rho_0 v_i - U_{,0j}. \qquad (39.14.23)
$$

The solution is

$$
h_{0j}(\boldsymbol{x},t) = -4\int \frac{\rho_0(\boldsymbol{x}',t)v_i(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' + \frac{1}{4\pi}\int \frac{\partial^2 U(\boldsymbol{x}',t)}{\partial x_j'\partial t}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} + \mathcal{O}\big(\epsilon^5\big). \qquad (39.14.24)
$$

Using equation 39.8.1, we have

$$
h_{0j}(\boldsymbol{x},t) = -4V_i(\boldsymbol{x},t) + \frac{1}{2}[V_j(\boldsymbol{x},t) - W_j(\boldsymbol{x},t)] + \mathcal{O}\big(\epsilon^5\big) = -\frac{7}{2}V_i(\boldsymbol{x},t) - \frac{1}{2}W_j(\boldsymbol{x},t) + \mathcal{O}\big(\epsilon^5\big). \qquad (39.14.25)
$$

Now we conclude that $\gamma = \Delta_1 = \Delta_2 = 1$ in general relativity.

(d) Substitute $\gamma = \Delta_1 = \Delta_2 = 1$ into equation (39.42). We obtain the stress-energy tensor to PPN order in general relativity.

$$
\begin{aligned}
T^{00} &= \rho_0(1 + \Pi + v^2 + 2U) + \mathcal{O}\big(\rho_0\epsilon^4\big), \\
T^{0j} &= \rho_0(1 + \Pi + v^2 + 2U)v^j + t_{\hat{j}\hat{m}}v_m + \mathcal{O}\big(\rho_0\epsilon^5\big), \\
T^{jk} &= t_{\hat{j}\hat{k}}(1 - 2U) + \rho_0(1 + \Pi + v^2 + 2U)v_j v_k + \frac{1}{2}(v_j t_{\hat{k}\hat{m}}v_m + v_k t_{\hat{j}\hat{m}}v_m) + \mathcal{O}\big(\rho_0\epsilon^6\big). \qquad (39.14.26)
\end{aligned}
$$

(e)Suppose the unknown post-Newtonian correction to h_{00} is k_{00}, i.e.,

$$h_{00} = 2U + k_{00} + \mathcal{O}(\epsilon^6). \tag{39.14.27}$$

Using equations 39.13.1 and 39.13.7, we have

$$\begin{aligned} R_{00} &= R^{\alpha}{}_{0\alpha 0} = R^{k}{}_{0k0} = \Gamma^{k}{}_{00,k} - \Gamma^{k}{}_{0k,0} + \Gamma^{k}{}_{\mu k}\Gamma^{\mu}{}_{00} - \Gamma^{k}{}_{\mu 0}\Gamma^{\mu}{}_{0k} \\ &= \Gamma^{k}{}_{00,k} - \Gamma^{k}{}_{0k,0} + \Gamma^{k}{}_{0k}\Gamma^{0}{}_{00} + \Gamma^{k}{}_{lk}\Gamma^{l}{}_{00} - \Gamma^{k}{}_{00}\Gamma^{0}{}_{0k} - \Gamma^{k}{}_{l0}\Gamma^{l}{}_{0k} \\ &= \left[h_{k0,0} - \frac{1}{2}(1-2U)h_{00,k}\right]_{,k} - 3U_{,tt} - 3U_{,l}^2 - U_{,k}^2 + \mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right). \end{aligned} \tag{39.14.28}$$

Using the gauge condition, we have

$$h_{k0,0k} = \frac{1}{2}h_{kk,00} + \mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right) = 3U_{,tt} + +\mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right) \tag{39.14.29}$$

So we obtain

$$\begin{aligned} R_{00} &= -\frac{1}{2}k_{00,kk} - U_{,kk} - 2U_{,k}^2 + 2UU_{,kk} + \mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right) \\ &= \left(-U - \frac{1}{2}k_{00} - U^2\right)_{,mm} + 4UU_{,mm} + \mathcal{O}\left(\frac{\epsilon^6}{R_\odot^2}\right). \end{aligned} \tag{39.14.30}$$

(f)Firstly, we have

$$\begin{aligned} T_{00} &= g_{00}^2 T^{00} + 2g_{00}g_{0i}T^{i0} + g_{0i}g_{0j}T^{ij} = (-1+2U)^2 T^{00} + \mathcal{O}(\rho_0\epsilon^4) \\ &= \rho_0(1 + \Pi + v^2 - 2U) + \mathcal{O}(\rho_0\epsilon^4) \end{aligned} \tag{39.14.31}$$

and

$$\begin{aligned} T &= g_{00}T^{00} + 2g_{0i}T^{0i} + g_{ij}T^{ij} = (-1+2U)T^{00} + \delta_{ij}T^{ij} + \mathcal{O}(\rho_0\epsilon^6) \\ &= -\rho_0(1 + \Pi + v^2) + 3p + \rho_0 v^2 + \mathcal{O}(\rho_0\epsilon^6). \end{aligned} \tag{39.14.32}$$

So we obtain the equation

$$\begin{aligned} R_{00} &= 8\pi\left(T_{00} - \frac{1}{2}g_{00}T\right) = 4\pi[\rho_0(1 + \Pi + 2v^2 - 2U) + 3p] + \mathcal{O}(\rho_0\epsilon^6) \\ &= \left(-U - \frac{1}{2}k_{00} - U^2\right)_{,mm} + 4UU_{,mm} + \mathcal{O}(\rho_0\epsilon^6), \end{aligned} \tag{39.14.33}$$

i.e.,

$$\left(-U - \frac{1}{2}k_{00} - U^2\right)_{,mm} = 4\pi[\rho_0(1 + \Pi + 2v^2 + 2U) + 3p] + \mathcal{O}(\rho_0\epsilon^6) \tag{39.14.34}$$

We have the solution

$$U + \frac{1}{2}k_{00} + U^2 = U + 2\int \rho_0\left(v^2 + U + \frac{1}{2}\Pi + \frac{3p}{2\rho_0}\right)\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}^3x'\,. \tag{39.14.35}$$

We can conclude that

$$\beta = \beta_1 = \beta_2 = \beta_3 = \beta_4 = 1, \quad \zeta = \eta = 0. \tag{39.14.36}$$

(g) The post-Newtonian equations of motion for the matter is

$$\rho^*_{,t} + (\rho^* v^i)_{,i} = \mathcal{O}(\rho_{0,j}\epsilon^5) \quad \text{where} \quad \rho^* \equiv \rho_0\left(1 + \frac{1}{2}v^2 + 3U\right) \tag{39.14.37}$$

$$\rho_0\frac{\mathrm{d}\Pi}{\mathrm{d}t} + t_{\hat{j}\hat{k}}v_{j,k} = \mathcal{O}(\rho_{0,j}\epsilon^5) \tag{39.14.38}$$

and

$$\rho^*\left(\frac{\mathrm{d}v^j}{\mathrm{d}t}-U_{,j}\right)+[t_{\hat{j}\hat{k}}(1+3U)]_{,k}-\left(\frac{1}{2}v^2+\Pi\right)t_{\hat{j}\hat{k},k}-\rho^*v^jU_{,t}+t_{\hat{j}\hat{k},t}v_k-\frac{t_{\hat{j}\hat{k}}t_{\hat{k}\hat{m},m}}{\rho^*}$$
$$+4\rho^*\frac{\mathrm{d}}{\mathrm{d}t}\big(Uv^j-V_j\big)+\frac{1}{2}\rho^*(V_j-W_j)_{,t}-2\rho^*\Psi_{,j}+4\rho^*v_kV_{k,j}-\rho^*U_{,j}(v^2+3p/\rho_0)$$
$$+\frac{1}{2}(v_{j,k}t_{\hat{k}\hat{m}}v_m-t_{\hat{j}\hat{m}}v_{m,k}v_k)+\frac{1}{2}[v_m(t_{\hat{m}\hat{j}}v_k)_{,k}-v_j(t_{\hat{k}\hat{l}}v_k)_{,l}]=\mathcal{O}\big(\rho_{0,j}\epsilon^6\big). \tag{39.14.39}$$

39.15 MANY-BODY SYSTEM IN POST-NEWTONIAN LIMIT OF GENERAL RELATIVITY

Consider, in the post-Newtonian limit of general relativity, a system made up of many gravitationally interacting bodies with separations large compared to their sizes (example: the solar system). Idealize each body to be spherically symmetric, to be free of internal motions, and to have isotropic internal stresses, $t_{\hat{j}\hat{k}}=p\delta_{jk}$. Let the world line of the center of body A, in some chosen PPN coordinate frame, be $\boldsymbol{x}_A(t)$; and let the (coordinate) velocity of the center of body A be

$$\boldsymbol{v}_A=\frac{\mathrm{d}\boldsymbol{x}_A}{\mathrm{d}t}. \tag{39.15.1}$$

The total mass-energy of body A as measured in its neighborhood (rest mass-energy plus internal energy plus self-gravitational energy) is given by

$$M_A=\int_{\mathcal{V}_A}=\left(1+\Pi-\frac{1}{2}U_{\rm self}\right)\mathrm{d}(\text{rest mass})+\mathcal{O}\big(M_A\epsilon^4\big), \tag{39.15.2}$$

where $U_{\rm self}$ is the body's own Newtonian potential (no contributions from other bodies), and $\mathcal{V}_A$ is the interior of the body.

(a) Show that, when written in the chosen PPN coordinate frame, this expression for M_A becomes

$$M_A=\int_{\mathcal{V}_A}\rho_0\left(1+\Pi+\frac{1}{2}v_A^2+3U-\frac{1}{2}U_{\rm self}\right)\mathrm{d}^3x+\mathcal{O}\big(M_A\epsilon^4\big). \tag{39.15.3}$$

Use equations 39.13.13, 39.13.15, and 39.13.26 to show that M_A is conserved as the bodies move about, $\mathrm{d}M_A/\mathrm{d}t=0$.

(b) Pick an event $(t,\boldsymbol{x})$ outside all the bodies, and at time t denote

$$\boldsymbol{r}_A\equiv\boldsymbol{x}_A-\boldsymbol{x},\quad \boldsymbol{r}_{AB}\equiv\boldsymbol{x}_A-\boldsymbol{x}_B,\quad r_A\equiv|\boldsymbol{r}_A|,\quad r_{AB}\equiv|\boldsymbol{r}_{AB}|. \tag{39.15.4}$$

Show that the general-relativistic, post-Newtonian metric 39.13.4 at the chosen event has the form

$$g_{jk}=\delta_{jk}\left(1+2\sum_A\frac{M_A}{r_A}\right)+\mathcal{O}\big(\epsilon^4\big),$$
$$g_{0j}=-\sum_A\frac{M_A}{r_A}\left[\frac{7}{2}v_{Aj}+\frac{1}{2}\frac{(\boldsymbol{v}_A\cdot\boldsymbol{r}_A)r_{Aj}}{r_A^2}\right]+\mathcal{O}\big(\epsilon^5\big),$$
$$g_{00}=-1+2\sum_A\frac{M_A}{r_A}-2\left(\sum_A\frac{M_A}{r_A}\right)^2+3\sum_A\frac{M_Av_A^2}{r_A}-2\sum_A\sum_{B\neq A}\frac{M_AM_B}{r_Ar_{AB}}+\mathcal{O}\big(\epsilon^6\big). \tag{39.15.5}$$

(c) Perform an infinitesimal coordinate transformation,

$$t_{\rm old}=t_{\rm new}-\frac{1}{2}\sum_A\frac{M_A(\boldsymbol{v}_A\cdot\boldsymbol{r}_A)}{r_A},\quad \boldsymbol{x}_{\rm old}=\boldsymbol{x}_{\rm new}, \tag{39.15.6}$$

to bring the metric 39.15.5 into the standard form originally devised by Einstein, Infeld, and Hoffman (1938), and by Eddington and Clark (1938):

$$
\begin{aligned}
g_{jk} &= \delta_{jk}\left(1 + 2\sum_A \frac{M_A}{r_A}\right) + \mathcal{O}(\epsilon^4),\\
g_{0j} &= -4\sum_A \frac{M_A}{r_A} v_{Aj} + \mathcal{O}(\epsilon^5),\\
g_{00} &= -1 + 2\sum_A \frac{M_A}{r_A} - 2\left(\sum_A \frac{M_A}{r_A}\right)^2 + 3\sum_A \frac{M_A v_A^2}{r_A} - 2\sum_A \sum_{B\neq A} \frac{M_A M_B}{r_A r_{AB}} - \frac{\partial^2 \chi}{\partial t^2} + \mathcal{O}(\epsilon^6),
\end{aligned}
\tag{39.15.7}
$$

where χ [equation (39.49a)] is given by

$$\chi = -\sum M_A r_A .$$

(d) The equations of motion for the bodies can be obtained in either of two ways: by performing a volume integral of the Euler equation 39.13.40 over the interior of each body; or by invoking the general arguments of section 20.6. The latter way is the easier. Use it to conclude that any chosen body K moves along a geodesic of the metric obtained by omitting the terms $A = K$ from the sums in 39.15.7. Show that the geodesic equation for body K reduces to

$$
\begin{aligned}
\frac{\mathrm{d}^2 \boldsymbol{x}_K}{\mathrm{d}t^2} \equiv \frac{\mathrm{d}\boldsymbol{v}_A}{\mathrm{d}t} = \sum_{A\neq K} \boldsymbol{r}_{AK} \frac{M_A}{r_{AK}^3} \Bigg[& 1 - 4\sum_{B\neq K} \frac{M_B}{r_{BK}} - \sum_{C\neq A} \frac{M_C}{r_{CA}} \left(1 - \frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\\
& + \boldsymbol{v}_K^2 + 2\boldsymbol{v}_A^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_K - \frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2 \Bigg]\\
& - \sum_{A\neq K} (\boldsymbol{v}_A - \boldsymbol{v}_K) \frac{M_A \boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_K)}{r_{AK}^3} + \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A} \boldsymbol{r}_{CA} \frac{M_A M_C}{r_{AK} r_{CA}^3}.
\end{aligned}
\tag{39.15.8}
$$

Equations 39.15.7 and 39.15.8 are called the Einstein-Infeld-Hoffman ("EIH") equations for the geometry and evolution of a many-body system. They are used widely in analyses of planetary orbits in the solar system.

Solution:

(a) The rest mass energy in volume d^3x is

$$\mathrm{d}(\text{rest mass}) = \rho_0 u^0 \sqrt{-g}\,\mathrm{d}^3x = \rho^*\left[1 + \mathcal{O}(\epsilon^4)\right]\mathrm{d}^3x = \rho_0\left[1 + \frac{1}{2}v^2 + 3U + \mathcal{O}(\epsilon^4)\right]\mathrm{d}^3x\,. \tag{39.15.9}$$

The total mass energy of the star is

$$
\begin{aligned}
M_A &= \int_{\mathcal{V}_A} \rho_0 \left(1 + \Pi - \frac{1}{2}U_{\text{self}}\right)\left[1 + \frac{1}{2}v^2 + 3U + \mathcal{O}(\epsilon^4)\right]\mathrm{d}^3x + \mathcal{O}(M_A\epsilon^4)\\
&= \int_{\mathcal{V}_A} \rho_0 \left(1 + \Pi + \frac{1}{2}v_A^2 + 3U - \frac{1}{2}U_{\text{self}}\right)\mathrm{d}^3x + \mathcal{O}(M_A\epsilon^4).
\end{aligned}
\tag{39.15.10}
$$

I have not figured out how to prove M_A is conserved as the bodies move about. Any help is appreciated.

(b) Define $m_A \equiv \int_{\mathcal{V}_A} \rho_0 \,\mathrm{d}^3x$. For an event $(t, \boldsymbol{x})$ outside all the bodies, we have

$$
\begin{aligned}
U(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{m_A}{r_A} = \sum_A \frac{M_A}{r_A} + \mathcal{O}(\epsilon^4) \\
\Psi_1(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)\boldsymbol{v}^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{M_A v_A^2}{r_A} + \mathcal{O}(\epsilon^6) \\
\Psi_2(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)U}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{\int_{\mathcal{V}_A} \rho_0(\boldsymbol{x}',t)U_{\mathrm{self}}\,\mathrm{d}^3x'}{r_A} + \sum_A \sum_{B\neq A} \frac{M_A M_B}{r_A r_{AB}} + \mathcal{O}(\epsilon^6) \\
\Psi_3(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)\Pi}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{\int_{\mathcal{V}_A} \rho_0(\boldsymbol{x}',t)\Pi\,\mathrm{d}^3x'}{r_A} + \mathcal{O}(\epsilon^6) \\
\Psi_4(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{p(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{\int_{\mathcal{V}_A} p(\boldsymbol{x}',t)\,\mathrm{d}^3x'}{r_A} + \mathcal{O}(\epsilon^6) \\
V_j(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)\boldsymbol{v}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \sum_A \frac{M_A v_A^j}{r_A} + \mathcal{O}(\epsilon^5) \\
W_j(\boldsymbol{x},t) &= \sum_A \int_{\mathcal{V}_A} \frac{\rho_0(\boldsymbol{x}',t)[\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{x}')(x_j-x_j')]}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}^3x' = \sum_A \frac{M_A(\boldsymbol{v}_A\cdot\boldsymbol{r}_A)r_A^j}{r_A} + \mathcal{O}(\epsilon^5).
\end{aligned}
\tag{39.15.11}
$$

From the Newtonian virial theorem 39.6.6, applied to body A by itself in its own rest frame, we can conclude that

$$
-\int \frac{1}{2}\rho_0 U_{\mathrm{self}}\,\mathrm{d}^3x + 3\int p\,\mathrm{d}^3x = \mathcal{O}(M_A\epsilon^4). \tag{39.15.12}
$$

In General relativity, we have

$$
g_{00} = -1 + 2U - 2U^2 + 4\Psi_1 + 4\Psi_2 + 2\Psi_3 + 6\Psi_4 + \mathcal{O}(\epsilon^6) \tag{39.15.13}
$$

We also have

$$
\sum_A \frac{M_A}{r_A} = U + \Psi_3 + \frac{1}{2}\Psi_1 + 3\Psi_2 - 3\Psi_4 + \mathcal{O}(\epsilon^6). \tag{39.15.14}
$$

So we can get

$$
\begin{aligned}
g_{00} &= -1 + 2\sum_A \frac{M_A}{r_A} - 2U^2 + 3\Psi_1 - 2\Psi_2 + 12\Psi_4 + \mathcal{O}(\epsilon^6) \\
&= -1 + 2\sum_A \frac{M_A}{r_A} - 2\left(\sum_A \frac{M_A}{r_A}\right)^2 + 3\sum_A \frac{M_A v_A^2}{r_A} - 2\sum_A\sum_{B\neq A} \frac{M_A M_B}{r_A r_{AB}} \\
&\quad - 2\sum_A \frac{\int_{\mathcal{V}_A} \rho_0(\boldsymbol{x}',t)U_{\mathrm{self}}\,\mathrm{d}^3x'}{r_A} + 12\sum_A \frac{\int_{\mathcal{V}_A} p(\boldsymbol{x}',t)\,\mathrm{d}^3x'}{r_A} + \mathcal{O}(\epsilon^6) \\
&= -1 + 2\sum_A \frac{M_A}{r_A} - 2\left(\sum_A \frac{M_A}{r_A}\right)^2 + 3\sum_A \frac{M_A v_A^2}{r_A} - 2\sum_A\sum_{B\neq A} \frac{M_A M_B}{r_A r_{AB}} + \mathcal{O}(\epsilon^6);
\end{aligned}
\tag{39.15.15a}
$$

$$
g_{0j} = -\frac{7}{2}V_j - \frac{1}{2}W_j + \mathcal{O}(\epsilon^5) = -\sum_A \frac{M_A}{r_A}\left[\frac{7}{2}v_{Aj} + \frac{1}{2}\frac{(\boldsymbol{v}_A\cdot\boldsymbol{r}_A)r_{Aj}}{r_A^2}\right] + \mathcal{O}(\epsilon^5); \tag{39.15.15b}
$$

$$
g_{jk} = \delta_{jk}(1+2U) + \mathcal{O}(\epsilon^4) = \delta_{jk}\left(1 + 2\sum_A \frac{M_A}{r_A}\right) + \mathcal{O}(\epsilon^4). \tag{39.15.15c}
$$

(c) Firstly, we have

$$
\frac{\partial^2\chi}{\partial t\partial x^j} = V_j - W_j + \mathcal{O}(\epsilon^5), \quad t_{\mathrm{old}} = t_{\mathrm{new}} + \frac{1}{2}\frac{\partial\chi}{\partial t}. \tag{39.15.16}
$$

The transformation matrix is

$$\frac{\partial t_{\rm old}}{\partial t_{\rm new}} = 1 + \frac{1}{2}\frac{\partial^2 \chi}{\partial t^2}, \quad \frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} = \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x^j} = \frac{1}{2}(V_j - W_j) + \mathcal{O}\big(\epsilon^5\big), \quad \frac{\partial x^j_{\rm old}}{\partial t_{\rm new}} = 0, \quad \frac{\partial x^j_{\rm old}}{\partial x^k_{\rm new}} = \delta^j_k. \tag{39.15.17}$$

Thus,

$$\begin{aligned}
&g^{\rm new}_{00}(\boldsymbol{x}_{\rm new}, t_{\rm new}) = g^{\rm old}_{00}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \left(\frac{\partial t_{\rm old}}{\partial t_{\rm new}}\right)^2 = g^{\rm old}_{00}(\boldsymbol{x}_{\rm new}, t_{\rm old}) \left(1 + \frac{\partial^2 \chi}{\partial t^2}\right) + \mathcal{O}\big(\epsilon^8\big) \\
&= g^{\rm old}_{00}(\boldsymbol{x}_{\rm new}, t_{\rm new}) + \frac{1}{2} g^{\rm old}_{00,0}(\boldsymbol{x}_{\rm new}, t_{\rm new}) \frac{\partial \chi}{\partial t} - \frac{\partial^2 \chi}{\partial t^2} + \mathcal{O}\big(\epsilon^6\big) = g^{\rm old}_{00}(\boldsymbol{x}_{\rm new}, t_{\rm new}) - \frac{\partial^2 \chi}{\partial t^2} + \mathcal{O}\big(\epsilon^6\big) \\
&g^{\rm new}_{0j}(\boldsymbol{x}_{\rm new}, t_{\rm new}) = g^{\rm old}_{00}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \frac{\partial t_{\rm old}}{\partial t_{\rm new}} \frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} + g^{\rm old}_{0k}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \frac{\partial t_{\rm old}}{\partial t_{\rm new}} \frac{\partial x^k_{\rm old}}{\partial x^j_{\rm new}} \\
&= -\frac{1}{2}(V_j - W_j) + g^{\rm old}_{0k}(\boldsymbol{x}_{\rm new}, t_{\rm new}) + \mathcal{O}\big(\epsilon^5\big) \\
&g_{jk}(\boldsymbol{x}_{\rm new}, t_{\rm new}) = g^{\rm old}_{00}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} \frac{\partial t_{\rm old}}{\partial x^k_{\rm new}} + g_{0k}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \frac{\partial t_{\rm old}}{\partial x^j_{\rm new}} + g_{j0}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \frac{\partial t_{\rm old}}{\partial x^k_{\rm new}} + g^{\rm old}_{jk}(\boldsymbol{x}_{\rm old}, t_{\rm old}) \\
&= g^{\rm old}_{jk}(\boldsymbol{x}_{\rm new}, t_{\rm new}) + \mathcal{O}\big(\epsilon^4\big).
\end{aligned} \tag{39.15.18}$$

So we derive the equation 39.15.7.

(d) The geodesic equation is

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = 0. \tag{39.15.19}$$

So we have

$$\frac{\mathrm{d}v^j}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{u^j}{u^0}\right) = \frac{1}{(u^0)^2}\frac{\mathrm{d}u^j}{\mathrm{d}\tau} - \frac{u^j}{(u^0)^3}\frac{\mathrm{d}u^0}{\mathrm{d}\tau} = -(\Gamma^j{}_{00} + 2\Gamma^j{}_{0k} v^k + \Gamma^j{}_{kl} v^k v^l) + v^j(\Gamma^0{}_{00} + 2\Gamma^0{}_{0k} v^k + \Gamma^0{}_{kl} v^k v^l) \tag{39.15.20}$$

Using equations 39.13.7, 39.15.7, and omitting the terms $A = K$ from the sums in 39.15.7, we have

$$\Gamma^0{}_{00} = \sum_{A\neq K} \frac{M_A(\boldsymbol{r}_{AK} \cdot \boldsymbol{v}_A)}{r^3_{AK}} + \mathcal{O}\left(\frac{\epsilon^5}{R}\right) \tag{39.15.21a}$$

$$\Gamma^0{}_{0j} = -\sum_{A\neq K} \frac{M_A r^j_{AK}}{r^3_{AK}} + \mathcal{O}\left(\frac{\epsilon^4}{R}\right) \tag{39.15.21b}$$

$$\Gamma^0{}_{jk} = -\sum_{A\neq K} \frac{M_A(\boldsymbol{r}_{AK} \cdot \boldsymbol{v}_A)}{r^3_{AK}} \delta_{jk} + 2\sum_{A\neq K} \frac{M_A(v^j_A r^k_{AK} + v^k_A r^j_{AK})}{r^3_{AK}} + \mathcal{O}\left(\frac{\epsilon^5}{R}\right) \tag{39.15.21c}$$

$$\Gamma^j{}_{00} = -\sum_{A\neq K} \frac{M_A r^j_{AK}}{r^3_{AK}} \left[1 - 4\sum_{B\neq K} \frac{M_B}{r_{BK}} + \frac{3}{2} v^2_A - \sum_{C\neq A} \frac{M_C}{r_{AC}}\right] + \frac{1}{2}\chi_{,ttj} + g_{j0,0} + \mathcal{O}\left(\frac{\epsilon^6}{R}\right) \tag{39.15.21d}$$

$$\Gamma^j{}_{0k} = -\sum_{A\neq K} \frac{M_A(\boldsymbol{r}_{AK} \cdot \boldsymbol{v}_A)}{r^3_{AK}} \delta_{jk} - 2\sum_{A\neq K} \frac{M_A(v^j_A r^k_{AK} - v^k_A r^j_{AK})}{r^3_{AK}} + \mathcal{O}\left(\frac{\epsilon^5}{R}\right) \tag{39.15.21e}$$

$$\Gamma^j{}_{kl} = \sum_{A\neq K} \frac{M_A}{r^3_{AK}}(r^k_{AK}\delta_{jl} + r^l_{AK}\delta_{kj} - r^j_{AK}\delta_{kl}) + \mathcal{O}\left(\frac{\epsilon^4}{R}\right) \tag{39.15.21f}$$

Using

$$\frac{\mathrm{d}\boldsymbol{v}_A}{\mathrm{d}t} = \sum_{C\neq A} \frac{M_C \boldsymbol{r}_{CA}}{r^3_{CA}} + \mathcal{O}\left(\frac{\epsilon^4}{R}\right), \tag{39.15.22}$$

we find that

$$\begin{aligned}\frac{1}{2}\chi_{,ttj}+g_{j0,0}&=-\frac{7}{2}\frac{\partial}{\partial t}\left(\sum_{A\neq K}\frac{M_A}{r_{AK}}v_A^j\right)-\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{A\neq K}\frac{M_A(\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK})r_{AK}^j}{r_{AK}^3}\right)+\mathcal{O}\left(\frac{\epsilon^6}{R}\right)\\&=-\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\frac{M_AM_Cr_{CA}^j}{r_{AK}r_{CA}^3}+\frac{7}{2}\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_A)}{r_{AK}^3}v_A^j\\&\quad-\frac{1}{2}\sum_{A\neq K}\frac{M_Ar_{AK}^j}{r_A^3}\sum_{C\neq A}\left(\frac{M_C}{r_{CA}^3}\boldsymbol{r}_{CA}\cdot\boldsymbol{r}_{AK}\right)-\frac{1}{2}\sum_{A\neq K}\frac{M_A(\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK})v_A^j}{r_{AK}^3}\\&\quad-\frac{1}{2}\sum_{A\neq K}\frac{M_Av_A^2}{r_{AK}^j}r_{AK}^3+\frac{3}{2}\sum_{A\neq K}\frac{M_A(\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK})^2r_{AK}^j}{r_{AK}^5}+\mathcal{O}\left(\frac{\epsilon^4}{R}\right)\end{aligned}\tag{39.15.23}$$

We then have

$$\begin{aligned}\Gamma^j{}_{00}&=-\sum_{A\neq K}\frac{M_Ar_{AK}^j}{r_{AK}^3}\left[1-4\sum_{B\neq K}\frac{M_B}{r_{BK}}+2v_A^2-\sum_{C\neq A}\frac{M_C}{r_{AC}}\left(1-\frac{\boldsymbol{r}_{CA}\cdot\boldsymbol{r}_{AK}}{2r_{CA}^2}\right)-\frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\right]\\&\quad+3\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_A)}{r_{AK}^3}v_A^j-\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\frac{M_AM_Cr_{CA}^j}{r_{AK}r_{CA}^3}+\mathcal{O}\left(\frac{\epsilon^6}{R}\right)\end{aligned}\tag{39.15.24}$$

We can also figure out that

$$\begin{aligned}&-2\Gamma^j{}_{0k}v_K^k+\Gamma^0{}_{00}v_K^j\\&=2\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_A)}{r_{AK}^3}v_K^j+4\sum_{A\neq K}\frac{M_A[v_A^j(\boldsymbol{v}_K\cdot\boldsymbol{r}_{AK})-(v_A\cdot v_K)r_{AK}^j]}{r_{AK}^3}+\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_A)v_K^j}{r_{AK}^3}+\mathcal{O}\left(\frac{\epsilon^6}{R}\right)\\&=3\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_A)}{r_{AK}^3}v_K^j+4\sum_{A\neq K}\frac{M_A(\boldsymbol{v}_K\cdot\boldsymbol{r}_{AK})}{r_{AK}^3}v_A^j-4\sum_{A\neq K}\frac{M_Ar_{AK}^j}{r_{AK}^3}v_A\cdot v_K+\mathcal{O}\left(\frac{\epsilon^6}{R}\right)\end{aligned}\tag{39.15.25}$$

and

$$\begin{aligned}&-\Gamma^j{}_{kl}v_K^kv_K^l+2\Gamma^0{}_{0k}v_K^jv_K^k\\&=\sum_{A\neq K}\frac{M_A}{r_{AK}^3}[-2(r_{AK}\cdot\boldsymbol{v}_K)v_K^j+r_{AK}^j\boldsymbol{v}_K^2]-2\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_K)}{r_{AK}^3}v_K^j+\mathcal{O}\left(\frac{\epsilon^6}{R}\right)\\&=\sum_{A\neq K}\frac{M_Ar_{AK}^j}{r_{AK}^3}\boldsymbol{v}_K^2-4\sum_{A\neq K}\frac{M_A(\boldsymbol{r}_{AK}\cdot\boldsymbol{v}_K)}{r_{AK}^3}v_K^j+\mathcal{O}\left(\frac{\epsilon^6}{R}\right).\end{aligned}\tag{39.15.26}$$

We also notice that $\Gamma^0{}_{kl}v^kv^lv^j\sim\mathcal{O}(\epsilon^6/R)$. Putting equations 39.15.20, 39.15.24, 39.15.25 and 39.15.26 together, we can get the equation 39.15.8.

Chapter 40

SOLAR-SYSTEM EXPERIMENTS

40.1 PPN METRIC FOR IDEALIZED SUN

Show that for an isolated, static, spherical sun at rest at the origin of the PPN coordinate system, the PPN metric (39.32) reduces to expressions (40.3), (40.3'). As part of the reduction, show that the sun's total mass-energy is given by

$$M_\odot = \int_0^{R_\odot} \rho_0(1 + 2\beta_2 U + \beta_3 \Pi + 3\beta_4 p/\rho_0) 4\pi r^2 \, \mathrm{d}r \, . \tag{40.1.1}$$

Solution:

For an isolated, static, spherical sun at rest at the origin of the PPN coordinate system, we have

$$\begin{aligned} U(\boldsymbol{x}, t) &= \int \frac{\rho_0(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' = \frac{1}{r} \int_0^{R_\odot} \rho_0 4\pi r'^2 \, \mathrm{d}r' \\ \Psi(\boldsymbol{x}, t) &= \int \frac{\rho_0(\boldsymbol{x}', t)\psi(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' = \frac{1}{r} \int_0^{R_\odot} \rho_0 \left(\beta_2 + \frac{1}{2}\beta_3 \Pi + \frac{3p}{2\rho_0} \right) 4\pi r'^2 \, \mathrm{d}r' \\ V_i &= W_i = \mathcal{A} = \mathcal{D} = 0 \end{aligned} \tag{40.1.2}$$

The PPN metric will be

$$g_{00} = -1 + 2U - 2\beta U^2 + 4\Psi + \mathcal{O}(\epsilon^6), \quad g_{0j} = \mathcal{O}(\epsilon^5), \quad g_{jk} = (1 + 2\gamma U)\delta_{jk} + \mathcal{O}(\epsilon^4). \tag{40.1.3}$$

If we define $M_\odot$ as 40.1.1, it will be easy to find that

$$\frac{2M_\odot}{r} = 2U + 4\Psi = 2U + \mathcal{O}(\epsilon^4). \tag{40.1.4}$$

So we have

$$g_{00} = -1 + \frac{2M_\odot}{r} - \frac{2M_\odot^2}{r^2} + \mathcal{O}(\epsilon^5), \quad g_{0j} = \mathcal{O}(\epsilon^5), \quad g_{jk} = \left(1 + \frac{2\gamma M_\odot}{r} \right) \delta_{jk} + \mathcal{O}(\epsilon^4). \tag{40.1.5}$$

In order to calculate the metric for a theory with a preferred "universal rest frame", we have to know

$$U_{jk}(\boldsymbol{x}, t) = \int \frac{\rho_0(\boldsymbol{x}', t)(x_j - x'_j)(x_k - x'_k)}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, \mathrm{d}^3 x' \tag{40.1.6}$$

Recall that

$$\frac{\partial^2 \chi}{\partial x^j \partial x^k} = U_{ij} - U\delta_{jk} \tag{40.1.7}$$

where

$$\chi(\boldsymbol{x}, t) = -\int \rho_0(\boldsymbol{x}', t)|\boldsymbol{x} - \boldsymbol{x}'|\, \mathrm{d}^3x' \,. \tag{40.1.8}$$

Notice that

$$\begin{aligned}\int \rho_0(\boldsymbol{x}', t)|\boldsymbol{x} - \boldsymbol{x}'|\, \mathrm{d}^3x' &= \int_0^{R_\odot} \rho_0(r') r'^2\, \mathrm{d}r' \int_{-1}^{1} \sqrt{r^2 + r'^2 - 2rr'\cos\theta}\, \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\phi \\ &= \int_0^{R_\odot} \rho_0(r') 4\pi r'^2 \left(r + \frac{r'^2}{3r}\right) \mathrm{d}r' \\ &= M_\odot r + \frac{I_\odot}{3r} + \mathcal{O}\big(\epsilon^4 r^2\big),\end{aligned} \tag{40.1.9}$$

where

$$I_\odot = \int \rho(\boldsymbol{x}', t) r'^2\, \mathrm{d}^3x' \,. \tag{40.1.10}$$

So we have

$$U_{jk} = \delta_{jk}\frac{M_\odot}{r} - \left(M_\odot r + \frac{I_\odot}{3r}\right)_{,jk} + \mathcal{O}\big(\epsilon^4\big) = \frac{M_\odot}{r^3} x_j x_k - \frac{I_\odot}{r^5}\left(x_j x_k - \frac{1}{3} r^2 \delta_{jk}\right) + \mathcal{O}\big(\epsilon^4\big). \tag{40.1.11}$$

Now we have

$$\begin{aligned} g_{00} &= -1 + \frac{2M_\odot}{r} - \frac{2M_\odot^2}{r^2} + (\alpha_2 + \alpha_3 - \alpha_1) w^2 \frac{M_\odot}{r} - \alpha_2 w_j w_k \left[\frac{M_\odot}{r^3} x_j x_k - \frac{I_\odot}{r^5}\left(x_j x_k - \frac{1}{3} r^2 \delta_{jk}\right)\right] + \mathcal{O}\big(\epsilon^6\big); \\ g_{0j} &= \left(\alpha_2 - \frac{1}{2}\alpha_1\right) w_j \frac{M_\odot}{r} - \alpha_2 w_k \left[\frac{M_\odot}{r^3} x_j x_k - \frac{I_\odot}{r^5}\left(x_j x_k - \frac{1}{3} r^2 \delta_{jk}\right)\right] + \mathcal{O}\big(\epsilon^5\big); \\ g_{jk} &= \left(1 + \frac{2\gamma M_\odot}{r}\right)\delta_{jk} + \mathcal{O}\big(\epsilon^4\big). \end{aligned} \tag{40.1.12}$$

40.2 TRAJECTORY OF LIGHT RAY IN SUN'S GRAVITATIONAL FIELD

Derive equation (40.6) for the path of a light ray in isotropic coordinates (40.3) in the sun's "equatorial plane". Use one or more of three alternative approaches: (1) direct integration of the geodesic equation (the hardest approach!); (2) computation based on the three integrals of the motion

$$\boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{k}\cdot\frac{\partial}{\partial t} = k_0, \quad \boldsymbol{k}\cdot\frac{\partial}{\partial \phi} = k_\phi = -bk_0, \quad \boldsymbol{k} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} = \text{ tangent vector to geodesic} \tag{40.2.1}$$

(see section 25.2 and 25.3); (3) computation based on the Hamilton-Jacobi method (Box 25.4), which for photons (zero rest mass) reduces to the "eikonal method" of geometric optics (see section 22.5).

Solution:

I would use Hamilton-Jacobi method to derive the trajectory. The metric of the spacetime is

$$\mathrm{d}s^2 = -\left[1 - \frac{2M_\odot}{r} + \mathcal{O}\big(\epsilon^4\big)\right]\mathrm{d}t^2 + \left[1 + \frac{2\gamma M_\odot}{r} + \mathcal{O}\big(\epsilon^4\big)\right]\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right]. \tag{40.2.2}$$

For massless photons, we have

$$g^{\alpha\beta} p_\alpha p_\beta = 0 \tag{40.2.3}$$

Suppose the photon is in plane $\theta = \pi/2$. Hamilton-Jacobi equation would be

$$-\left(1+\frac{2M_\odot}{r}\right)\left(\frac{\partial S}{\partial t}\right)^2+\left(1-\frac{2\gamma M_\odot}{r}\right)\left[\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right]=0. \tag{40.2.4}$$

Assume

$$S=-Et+L\phi+S_r(r). \tag{40.2.5}$$

We can obtain the solution

$$S=-Et+L\phi\pm\int^r\sqrt{\left(1+\frac{2(\gamma+1)M_\odot}{r}\right)E^2-\frac{L^2}{r^2}}\,\mathrm{d}r\,. \tag{40.2.6}$$

The trajectory is given by

$$\frac{\partial S}{\partial L}=\phi_0. \tag{40.2.7}$$

We can obtain

$$\phi=\phi_0\mp\int^r\frac{L}{r^2\sqrt{\left(1+\frac{2(\gamma+1)M_\odot}{r}\right)E^2-\frac{L^2}{r^2}}}\,\mathrm{d}r \tag{40.2.8}$$

Far from the sun, the ray trajectory is $\phi = b/r$. So we have $L = bE$ and

$$\begin{aligned}\phi&=\int_r^\infty\frac{b}{r^2\sqrt{\left(1+\frac{2(\gamma+1)M_\odot}{r}\right)-\frac{b^2}{r^2}}}\,\mathrm{d}r=\int_0^{b/r}\frac{1}{\sqrt{-t^2+\frac{2(\gamma+1)M_\odot}{b}t+1}}\,\mathrm{d}t\\&=\int_0^{b/r}\left(\frac{1}{\sqrt{-t^2+1}}-\frac{(\gamma+1)M_\odot}{b}\frac{t}{(1-t^2)^{3/2}}\right)\mathrm{d}t+\mathcal{O}\left(\frac{M_\odot^2}{b^2}\right)\\&=\arcsin\left(\frac{b}{r}\right)+\frac{(\gamma+1)M_\odot}{b}\left[1-\frac{1}{\sqrt{1-b^2/r^2}}\right]+\mathcal{O}\left(\frac{M_\odot^2}{b^2}\right)\end{aligned} \tag{40.2.9}$$

It can be simplified into

$$\frac{b}{r}=\sin\phi-\frac{(\gamma+1)M_\odot}{b}(\cos\phi-1)+\mathcal{O}\left(\frac{M_\odot^2}{b^2}\right). \tag{40.2.10}$$

40.3 FERMAT'S PRINCIPLE

Prove Fermat's principle for a static gravitational field.

Solution:

The geodesic equation is

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}+\Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma=0. \tag{40.3.1}$$

Using t as the parameter of the geodesic, we have

$$\frac{\mathrm{d}v^j}{\mathrm{d}t}=\frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{u^j}{u^0}\right)=\frac{1}{(u^0)^2}\frac{\mathrm{d}u^j}{\mathrm{d}\tau}-\frac{u^j}{(u^0)^3}\frac{\mathrm{d}u^0}{\mathrm{d}\tau}=-(\Gamma^j{}_{00}+2\Gamma^j{}_{0k}v^k+\Gamma^j{}_{kl}v^kv^l)+v^j(\Gamma^0{}_{00}+2\Gamma^0{}_{0k}v^k+\Gamma^0{}_{kl}v^kv^l) \tag{40.3.2}$$

Using the fact that $g_{0j} = 0$ and $g_{\alpha\beta,0} = 0$, we can get

$$\frac{\mathrm{d}v^j}{\mathrm{d}t}+\Gamma^j{}_{00}+\Gamma^j{}_{kl}v^kv^l-2\Gamma^0{}_{0k}v^kv^j=0. \tag{40.3.3}$$

Lowering the indices, we have

$$g_{jk}\frac{\mathrm{d}v^k}{\mathrm{d}t}+\Gamma_{j00}+\Gamma_{jkl}v^kv^l-2\frac{g_{jk}}{g_{00}}\Gamma_{00l}v^lv^k=0. \tag{40.3.4}$$

For photons, we also have

$$(g_{00}+g_{kl}v^kv^l)u^0u^0=0. \tag{40.3.5}$$

Now the geodesic equation becomes

$$g_{jk}\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2}+\left(\Gamma_{jkl}-\Gamma_{j00}\frac{g_{kl}}{g_{00}}-2\frac{g_{jk}}{g_{00}}\Gamma_{00l}\right)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^l}{\mathrm{d}t} \tag{40.3.6}$$

Define $\gamma_{jk}=g_{jk}/g_{00}$. We have

$$g_{jk,l}=g_{00}\gamma_{jk,l}+\gamma_{jk}g_{00,l} \tag{40.3.7}$$

So

$$2\Gamma_{jkl}=g_{jk,l}+g_{jl,k}-g_{kl,j}=g_{00}(\gamma_{jk,l}+\gamma_{jl,k}-\gamma_{kl,j})+\gamma_{jk}g_{00,l}+\gamma_{jl}g_{00,l}-\gamma_{kl}g_{00,j}. \tag{40.3.8}$$

Notice that

$$2\Gamma_{j00}=-g_{00,j},\quad 2\Gamma_{00l}=g_{00,l}. \tag{40.3.9}$$

The geodesic equation will become

$$\gamma_{jk}\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2}+\frac{1}{2}\left(\gamma_{jk,l}+\gamma_{jl,k}-\gamma_{kl,j}\right)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^l}{\mathrm{d}t}=0. \tag{40.3.10}$$

Then notice that this is a geodesic equation with affine parameter t in a three dimensional manifold with metric γ_{jk}. The familiar extremum principle for this geodesic is

$$\delta\int_{a^j}^{b^j}(\gamma_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k)^{1/2}=\delta\int_{a^j}^{b^j}\mathrm{d}t=0, \tag{40.3.11}$$

which is precisely Fermat's principle.

40.4 DERIVATION OF PERIHELION SHIFT IN PPN FORMALISM

Derive equation (40.17) for the shape of any bound orbit of a test particle moving in the equatorial plane of the PPN gravitational field (40.3). Keep only "first-order" corrections beyond Newtonian theory (first order in powers of $M_\odot/r$).

Solution:

Hamilton-Jacobi equation, referred to a test body of unit mass, is

$$-1=g^{\alpha\beta}\tilde{S}_{,\alpha}\tilde{S}_{,\beta}=-\left[1+\frac{2M_\odot}{r}+(4-2\beta)\left(\frac{M_\odot}{r}\right)^2\right]\left(\frac{\partial\tilde{S}}{\partial t}\right)^2+\left[1-2\gamma\frac{M_\odot}{r}\right]\left[\left(\frac{\partial\tilde{S}}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\tilde{S}}{\partial\phi}\right)^2\right]. \tag{40.4.1}$$

Assume $\tilde{S}=-\tilde{E}t+\tilde{L}\phi+\tilde{S}_r$. We have

$$\left[1+\frac{2M_\odot}{r}+(4-2\beta)\left(\frac{M_\odot}{r}\right)^2\right]\tilde{E}^2-1=\left[1-2\gamma\frac{M_\odot}{r}\right]\left[\left(\frac{\partial\tilde{S}_r}{\partial r}\right)^2+\frac{\tilde{L}^2}{r^2}\right]. \tag{40.4.2}$$

Notice that

$$\tilde{E}^2-1\sim\mathcal{O}\left(\frac{M_\odot}{r}\right),\quad \frac{\tilde{L}^2}{r^2}\sim\mathcal{O}\left(\frac{M_\odot}{r}\right). \tag{40.4.3}$$

We have

$$S_r = \pm \int^r \left\{ (\tilde{E}^2 - 1) + \frac{2M_\odot}{r}\left[\tilde{E}^2 + \gamma(\tilde{E}^2 - 1)\right] + \frac{M_\odot^2}{r^2}(4 - 2\beta + 4\gamma) - \frac{\tilde{L}^2}{r^2} \right\}^{1/2} \mathrm{d}r \tag{40.4.4}$$

The shape of the orbit is determined by the "condition of constructive interference", $\partial\tilde{S}\Big/\partial\tilde{L} = 0$:

$$\phi = \pm \int^r \left\{ -\frac{1-\tilde{E}^2}{\tilde{L}^2} + \frac{2M_\odot}{r\tilde{L}^2}\left[1 - (1+\gamma)(1-\tilde{E}^2)\right] - \frac{1}{r^2}\left[1 - 2\left(\frac{M_\odot}{\tilde{L}}\right)^2 (2 - \beta + 2\gamma)\right] \right\}^{-1/2} \mathrm{d}\left(\frac{1}{r}\right) \tag{40.4.5}$$

Define

$$A \equiv 1 - 2\left(\frac{M_\odot}{\tilde{L}}\right)^2 (2 - \beta + 2\gamma), \quad B \equiv \frac{M_\odot}{\tilde{L}^2}\left[1 - (1+\gamma)(1-\tilde{E}^2)\right], \quad C \equiv \frac{1-\tilde{E}^2}{\tilde{L}^2} \tag{40.4.6}$$

For the case $\tilde{E}^2 < 1$, we have the solution

$$\phi = A^{-1/2} \arcsin\left(\frac{A/r - B}{\sqrt{B^2 - AC}}\right). \tag{40.4.7}$$

By redefine the position of $\phi = 0$, we can get

$$r = \frac{A/B}{1 + \sqrt{1 - AC/B^2}\cos\left(A^{1/2}\phi\right)}. \tag{40.4.8}$$

Define

$$\begin{aligned} 1 - e^2 \equiv \frac{AC}{B^2} &= \left(\frac{\tilde{L}}{M_\odot}\right)^2 \left(1 - \tilde{E}^2\right)\left[1 + 2(1+\gamma)\left(1 - \tilde{E}^2\right) - 2\left(\frac{M_\odot}{\tilde{L}}\right)^2 (2 - \beta + 2\gamma)\right] + \mathcal{O}\left(\frac{M_\odot^2}{r^2}\right) \\ a \equiv \frac{B}{C} &= \frac{M_\odot}{1 - \tilde{E}^2}\left[1 - (1+\gamma)\left(1 - \tilde{E}^2\right)\right] \\ \delta\phi_0 \equiv 2\pi(1 - A^{1/2}) &= 2\pi\left(\frac{M_\odot}{\tilde{L}}\right)^2 (2 - \beta + 2\gamma) + \mathcal{O}\left(\frac{M_\odot^2}{r^2}\right). \end{aligned} \tag{40.4.9}$$

The solution can be rewritten as

$$r = \frac{a(1-e^2)}{1 + e\cos\left[(1 - \delta\phi_0/2\pi)\,\phi\right]}. \tag{40.4.10}$$

The perihelion shifts is

$$\delta\phi_0 = 2\pi\left(\frac{M_\odot}{\tilde{L}}\right)^2 (2 - \beta + 2\gamma) + \mathcal{O}\left(\frac{M_\odot^2}{r^2}\right) = \frac{2 - \beta + 2\gamma}{3}\frac{6\pi M_\odot}{a(1-e^2)} + \mathcal{O}\left(\frac{M_\odot^2}{r^2}\right) \tag{40.4.11}$$

40.5 PERIHELION SHIFT FOR OBLATE SUN

(a) The Newtonian potential for an oblate sun has the form

$$U = \frac{M_\odot}{r}\left(1 - J_2\frac{R^2}{r^2}\frac{3\cos^2\theta - 1}{2}\right), \tag{40.5.1}$$

where J_2 is the "quadrupole-moment parameter". One knows that $J_2 \lesssim 3 \times 10^{-5}$. Show that if an oblate sun is at rest at the origin of the PPN coordinate system, the metric of the surrounding spacetime [equations (39.32)] can be put into the form

$$\begin{aligned} \mathrm{d}s^2 = &-\left[1 - 2\frac{M_\odot}{r} + 2J_2\frac{M_\odot R^2}{r^3}\frac{3\cos^2\theta - 1}{2} + 2\beta\left(\frac{M_\odot}{r}\right)^2\right]\mathrm{d}t^2 + \left[1 + \frac{2\gamma M_\odot}{r}\right]\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \\ &+ \text{ corrections of post-post-Newtonian magnitude.} \end{aligned} \tag{40.5.2}$$

(b) Let a test particle move in a bound orbit in the equatorial plane. Use Hamilton-Jacobi theory to show that its orbit is a precessing ellipse [equation (40.17)] with a precession per orbit given by

$$\delta\phi_0 = \frac{2-\beta+2\gamma}{3}\frac{6\pi M_\odot}{a(1-e^2)} + J_2\frac{3\pi R^2}{a^2(1-e^2)^2}. \tag{40.5.3}$$

For the significance of this result, see Box 40.3.

Solution:

The equation 40.5.2 (40.25) in the textbook is incorrect.

(a) Notice that

$$g_{00} = -1 + 2U + 2\beta U^2 = -1 + 2\frac{M_\odot}{r} - 2J_2\frac{M_\odot R^2}{r^3}\frac{3\cos^2\theta - 1}{2} + 2\beta\frac{M_\odot^2}{r^2} + \mathcal{O}(\epsilon^6). \tag{40.5.4}$$

So we have the metric 40.5.2.

(b) Hamilton-Jacobi equation, referred to a test body of unit mass at plane $\theta = \pi/2$, is

$$-1 = g^{\alpha\beta}\tilde{S}_{,\alpha}\tilde{S}_{,\beta} = -\left[1 + \frac{2M_\odot}{r} + (4-2\beta)\left(\frac{M_\odot}{r}\right)^2 + J_2\frac{M_\odot R^2}{r^3}\right]\left(\frac{\partial\tilde{S}}{\partial t}\right)^2 + \left[1 - 2\gamma\frac{M_\odot}{r}\right]\left[\left(\frac{\partial\tilde{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\tilde{S}}{\partial\phi}\right)^2\right]. \tag{40.5.5}$$

Assume $\tilde{S} = -\tilde{E}t + \tilde{L}\phi + \tilde{S}_r$. We have

$$\left[1 + \frac{2M_\odot}{r} + (4-2\beta)\left(\frac{M_\odot}{r}\right)^2 + J_2\frac{M_\odot R^2}{r^3}\right]\tilde{E}^2 - 1 = \left[1 - 2\gamma\frac{M_\odot}{r}\right]\left[\left(\frac{\partial\tilde{S}_r}{\partial r}\right)^2 + \frac{\tilde{L}^2}{r^2}\right]. \tag{40.5.6}$$

The solution for S_r is

$$S_r = \pm\int^r\left\{(\tilde{E}^2-1) + \frac{2M_\odot}{r}\left[\tilde{E}^2 + \gamma(\tilde{E}^2-1)\right] + \frac{M_\odot^2}{r^2}(4-2\beta+4\gamma) - \frac{\tilde{L}^2}{r^2} + J_2\frac{M_\odot R^2}{r^3}\right\}^{1/2}\mathrm{d}r \tag{40.5.7}$$

The shape of the orbit is determined by the "condition of constructive interference", $\partial\tilde{S}\big/\partial\tilde{L} = 0$:

$$\phi = \pm\int^r\left\{-\frac{1-\tilde{E}^2}{\tilde{L}^2} + \frac{2M_\odot}{r\tilde{L}^2}\left[1-(1+\gamma)(1-\tilde{E}^2)\right] - \frac{1}{r^2}\left[1 - 2\left(\frac{M_\odot}{\tilde{L}}\right)^2(2-\beta+2\gamma)\right] + J_2\frac{M_\odot R^2}{r^3\tilde{L}^2}\right\}^{-1/2}\mathrm{d}\left(\frac{1}{r}\right) \tag{40.5.8}$$

Define $x = \tilde{L}^2/M_\odot r$. We have

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + \left[1 - 2\left(\frac{M_\odot}{\tilde{L}}\right)^2(2-\beta+2\gamma)\right]x^2 - 2\left[1-(1+\gamma)(1-\tilde{E}^2)\right]x - \frac{J_2R^2M_\odot^2}{\tilde{L}^4}x^3 + \frac{(1-\tilde{E}^2)\tilde{L}^2}{M_\odot^2} = 0. \tag{40.5.9}$$

We take the derivative of with respect to ϕ to get

$$\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2} + (1-2\epsilon_1)\,x - (1-\epsilon_0) - \epsilon_2x^2 = 0, \tag{40.5.10}$$

where

$$\epsilon_0 = (1+\gamma)(1-\tilde{E}^2), \quad \epsilon_1 = \left(\frac{M_\odot}{\tilde{L}}\right)^2(2-\beta+2\gamma), \quad \epsilon_2 = \frac{3J_2R^2M_\odot^2}{2\tilde{L}^4}. \tag{40.5.11}$$

Suppose $x = x_0 + x_1 + \mathcal{O}(\epsilon^2)$, where $x_0 = 1 + e\cos\phi$. So we have

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + x_1 = \frac{\epsilon_2e^2}{2}\cos 2\phi + 2(\epsilon_1+\epsilon_2)e\cos\phi + \left(1+\frac{1}{2}e^2\right)\epsilon_2 + 2\epsilon_1 - \epsilon_0. \tag{40.5.12}$$

The solution is

$$x_1 = -\frac{\epsilon_2 e^2}{6}\cos 2\phi + (\epsilon_1+\epsilon_2)e\phi\sin\phi + \left(1+\frac{1}{2}e^2\right)\epsilon_2 + 2\epsilon_1 - \epsilon_0 \tag{40.5.13}$$

The three terms here have different characters. The first oscillates around zero, while the third is simply a constant displacement. The important effect is thus contained in the second term, which accumulates over successive orbits. We therefore combine this term with the zeroth-order solution to write

$$x = 1 + e\cos\phi + (\epsilon_1+\epsilon_2)e\phi\sin\phi \tag{40.5.14}$$

This is not a full solution, even to the perturbed equation, but it encapsulates the part that we care about. In particular, this expression for x can be conveniently rewritten as the equation for an ellipse with an angular period that is not quite 2π:

$$1 + e\cos\left[(1-\frac{\delta\phi_0}{2\pi})\phi\right] \tag{40.5.15}$$

where

$$\delta\phi_0 = 2\pi(\epsilon_1+\epsilon_2) = 2\pi(2-\beta+2\gamma)\left(\frac{M_\odot}{\tilde{L}}\right)^2 + \frac{3\pi J_2 R^2 M_\odot^2}{\tilde{L}^4} = \frac{2-\beta+2\gamma}{3}\frac{6\pi M_\odot}{a(1-e^2)} + J_2\frac{3\pi R^2}{a^2(1-e^2)^2} + \mathcal{O}(\epsilon^2). \tag{40.5.16}$$

40.6 PRECESSIONAL ANGULAR VELOCITY

Derive equations (40.33) for the precession of a gyroscope in the post-Newtonian limit. Base the derivation on equations (40.29)-(40.32).

Solution:

$$\begin{aligned}
\frac{\mathrm{d}S_{\hat{j}}}{\mathrm{d}\tau} &= \boldsymbol{S}\cdot\boldsymbol{\nabla_u}\boldsymbol{e}_{\hat{j}} = g_{\alpha\beta}S^\alpha\left(\frac{\mathrm{d}\boldsymbol{e}^\beta_{\hat{j}}}{\mathrm{d}\tau} + \Gamma^\beta{}_{\mu\nu}\boldsymbol{e}^\mu_{\hat{j}}u^\nu\right)\\
&= -v_k S_{\hat{k}}\left(\frac{\mathrm{d}v_j}{\mathrm{d}\tau} + \Gamma^0{}_{\mu\nu}\boldsymbol{e}^\mu_{\hat{j}}u^\nu\right) + \delta_{ab}S_{\hat{a}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\gamma U\delta_{jb} + \frac{1}{2}v_j v_b\right) + \Gamma^b{}_{\mu\nu}\boldsymbol{e}^\mu_{\hat{j}}u^\nu\right) + \mathcal{O}\big(\epsilon^3 U_{,j}S_k\big)\\
&= -v_k S_{\hat{k}}\left(a_j + U_{,j} + \Gamma^0{}_{l0}\boldsymbol{e}^l_{\hat{j}}u^0\right) + S_{\hat{b}}\left(-\gamma U_{,\alpha}u^\alpha\delta_{jb} + \frac{1}{2}(U_{,j}+a_j)v_b + \frac{1}{2}(U_{,b}+a_b)v_j + \Gamma^b{}_{\mu\nu}\boldsymbol{e}^\mu_{\hat{j}}u^\nu\right) + \mathcal{O}\big(\epsilon^3 U_{,j}S_k\big)\\
&= -v_k S_{\hat{k}}a_j - \gamma S_{\hat{j}}(U_{,t}+U_{,k}v_k) + \frac{1}{2}(U_{,j}+a_j)S_{\hat{b}}v_b + \frac{1}{2}(U_{,b}+a_b)S_{\hat{b}}v_j + S_{\hat{b}}(-U_{,b}v_j + \Gamma^b{}_{j0} + \Gamma^b{}_{jc}v_c) + \mathcal{O}\big(\epsilon^3 U_{,j}S_k\big)\\
&= \frac{1}{2}(U_{,j}-a_j)S_{\hat{k}}v_k - \gamma S_{\hat{j}}(U_{,t}+U_{,k}v_k) + \frac{1}{2}(-U_{,k}+a_k)S_{\hat{k}}v_j + S_{\hat{j}}\gamma U_{,t} + g_{0[b,j]}S_{\hat{b}} + \gamma(\delta_{bj}U_{,c} + 2\delta_{c[b}U_{,j]})v_c S_{\hat{b}} + \mathcal{O}\big(\epsilon^3 U_{,}\\
&= \frac{1}{2}(U_{,j}-a_j)S_{\hat{k}}v_k + \frac{1}{2}(-U_{,k}+a_k)S_{\hat{k}}v_j + g_{0[b,j]}S_{\hat{b}} + 2\gamma v_{[b}U_{,j]}S_{\hat{b}} + \mathcal{O}\big(\epsilon^3 U_{,j}S_k\big)\\
&= S_{\hat{k}}\left[(2\gamma+1)v_{[k}U_{,j]} - v_{[k}a_{j]} + g_{0[k,j]}\right] + \mathcal{O}\big(\epsilon^3 U_{,j}S_k\big).
\end{aligned} \tag{40.6.1}$$

It is straightforward to bring the equation above into three-dimensional vector form, which is equation (40.33) in the textbook.

40.7 OFF-DIAGONAL TERMS IN METRIC ABOUT THE EARTH

Idealize the Earth as an isolated, rigidly rotating sphere with angular momentum J. Use equations (39.34b,c) and (39.27) to show that (in three-dimensional vector notation)

$$\boldsymbol{V} \equiv V_j \boldsymbol{e}_{\hat{j}} = \boldsymbol{W} \equiv W_j \boldsymbol{e}_{\hat{j}} = \frac{1}{2}\boldsymbol{J} \times \boldsymbol{r}/r^3 \tag{40.7.1}$$

outside the Earth, in the Earth's PPN rest frame. From this, infer equation (40.36).

Solution:

For an isolated, rigidly rotating sphere, we have

$$U(\boldsymbol{x},t) = \int \frac{\rho_0(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \frac{M}{r} \quad \text{where} \quad M = \int \rho_0(\boldsymbol{x}',t)\,\mathrm{d}^3x', \quad r = |\boldsymbol{x}|. \tag{40.7.2}$$

Equations (29.27)

$$\int \frac{\partial^2 U(\boldsymbol{x}',t)}{\partial x_j' \partial t}\frac{\mathrm{d}^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} = 2\pi[V_j(\boldsymbol{x},t) - W_j(\boldsymbol{x},t)] + \mathcal{O}\big(\epsilon^5\big) \tag{40.7.3}$$

then implies that

$$W_j = V_j + \mathcal{O}\big(\epsilon^5\big). \tag{40.7.4}$$

We also have

$$\boldsymbol{V} = \int \frac{\rho_0(\boldsymbol{x}',t)\boldsymbol{v}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = \boldsymbol{\Omega} \times \int \frac{\rho_0(\boldsymbol{x}')\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,. \tag{40.7.5}$$

And

$$\begin{aligned}
\int \frac{\rho_0(\boldsymbol{x}')z'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' &= \sum_{l,m}\frac{4\pi Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\int \mathrm{d}r'\,\rho(r')r'^{l+3}\int Y_{lm}(\theta',\phi')\cos\theta'\,\mathrm{d}\Omega' \\
&= \sum_{l,m}\frac{4\pi Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\int \mathrm{d}r'\,\rho(r')r'^{l+3}\int Y_{lm}(\theta',\phi')2\sqrt{\frac{\pi}{3}}Y_{10}(\theta',\phi')\,\mathrm{d}\Omega' \\
&= \frac{Y_{10}(\theta,\phi)}{3r^2}\int \mathrm{d}r'\,\rho(r')r'^4\int 2\sqrt{\frac{\pi}{3}}\,\mathrm{d}\Omega' \\
&= \frac{z}{3r^3}\int \rho(\boldsymbol{x}',t)r'^2\,\mathrm{d}^3x'\,.
\end{aligned} \tag{40.7.6}$$

So generally, we have

$$\boldsymbol{V} = \boldsymbol{\Omega} \times \frac{\boldsymbol{r}}{3r^3}\int \rho(\boldsymbol{x}',t)r'^2\,\mathrm{d}^3x'\,. \tag{40.7.7}$$

If we notice that the inertial tensor for the earth is

$$I_{ij} = \int \rho(\boldsymbol{x}',t)(r'^2\delta_{ij} - x_i'x_j')\,\mathrm{d}^3x' = \frac{2}{3}\delta_{ij}\int \rho(\boldsymbol{x}',t)r'^2\,\mathrm{d}^3x'\,. \tag{40.7.8}$$

The angular momentum of the earth is

$$J_i = I_{ij}\Omega_j = \frac{2}{3}\Omega_i\int \rho(\boldsymbol{x}',t)r'^2\,\mathrm{d}^3x' \tag{40.7.9}$$

Now we obtain that

$$\boldsymbol{V} = \frac{1}{2}\boldsymbol{J} \times \frac{\boldsymbol{r}}{r^3}. \tag{40.7.10}$$

40.8 SPIN-CURVATURE COUPLING

Consider a spinning body (e.g., the Earth or a gyroscope or an electron) moving through curved spacetime. Tidal gravitational forces produced by the curvature of spacetime act on the elementary pieces of the spinning body. These forces should depend not only on the positions of the pieces relative to the center of the object, but also on their relative velocities. Moreover, the spin of the body,

$$\boldsymbol{S} \equiv \int (\rho \boldsymbol{r} \times \boldsymbol{v})\, \mathrm{d}(\text{volume}) \tag{40.8.1}$$

is a measure of the relative positions and velocities of its pieces. Therefore one expects the spin to couple to the tidal gravitational forces-i.e., to the curvature of spacetime-producing deviations from geodesic motion. Careful solution of the PPN equations of Chapter 39 for general relativity reveals [Papapetrou (1951), Pirani (1956)] that such coupling occurs and causes a deviation of the worldline from the course that it would otherwise take; thus,

$$m\frac{\mathrm{D}u^\alpha}{\mathrm{d}\tau} = -S_\mu u_\nu \frac{\mathrm{D}^2 u_\beta}{\mathrm{d}\tau^2}\epsilon^{\alpha\mu\nu\beta} + \frac{1}{2}\epsilon^{\lambda\mu\rho\tau} R^{\alpha\nu}{}_{\lambda\mu} u_\nu S_\rho u_\tau. \tag{40.8.2}$$

Evaluate, in order of magnitude, the effects of the supplementary term on planetary orbits in the solar system.

Solution:

The magnitude of $\epsilon^{\lambda\mu\rho\tau} R^{\alpha\nu}{}_{\lambda\mu} u_\nu S_\rho u_\tau$ is about

$$\epsilon^{\lambda\mu\rho\tau} R^{\alpha\nu}{}_{\lambda\mu} u_\nu S_\rho u_\tau \sim \epsilon^{jkl0} R^{i0}{}_{jk} S_l \sim \frac{M_\odot v}{R^3} \times M_\odot R v \sim \frac{M_\odot^3}{R^3} \sim (2\times 10^{-6})^3 \sim 10^{-17}. \tag{40.8.3}$$

So the effects of the supplementary term on planetary orbits in the solar system is too small to be detected.

40.9 CAVENDISH CONSTANT FOR IDEALIZED SUN

Idealize the sun as a static sphere of perfect fluid at rest at the origin of the PPN coordinates. Then its external gravitational field has the form 40.1.5, with $M_\odot$ given by 40.1.1. Consequently, a test body of mass m, located far away at radius r, is accelerated by a gravitational force

$$\text{Force} = -\frac{mM_\odot}{r^2} \tag{40.9.1}$$

(a) Calculate the mass of the sun, M, in the sense of the amount of energy required to construct it by adding one spherical shell of matter on top of another, working from the inside outward.

(b) Use the virial theorem to rewrite equation 40.1.1 in the form

$$M_\odot = \int_0^{R_\odot} \rho_0 \left[1 + \beta_3 \Pi + \left(2\beta_2 + \frac{1}{2}\beta_4\right) U\right] 4\pi r^2\, \mathrm{d}r\,. \tag{40.9.2}$$

(c) Combine the above equations with the definition

$$\text{Force} = -\frac{G_\mathrm{C} mM}{r^2} \tag{40.9.3}$$

of the Cavendish constant for r far outside the sun, to obtain

$$\begin{aligned} G_\mathrm{C} &= \frac{\text{mass of sun as defined by its effect in bending world line of a faraway test particle}}{\text{mass-energy as defined by applying law of conservation of energy}} \\ &= 1 + \int \frac{\rho_0}{M_0}\left[(\beta_3 - 1)\Pi + \frac{1}{2}(4\beta_2 + \beta_4 - 6\gamma + 1)U\right] 4\pi r^2\, \mathrm{d}r \end{aligned} \tag{40.9.4}$$

Unless $\beta_3 = 1$, and $4\beta_2 + \beta_4 - 6\gamma + 1 = 0$ (as they are, of course, in Einstein's theory), G_C will depend on the sun's internal structure! Specialize equation 40.9.4 to conservative theories of gravity, and explain why the result is what one would expect from equation (40.43).

Solution:

(a) The rest mass and internal energy required to construct one spherical shell of matter with thickness $\mathrm{d}r$ is

$$\mathrm{d}E_1 = \rho u^0 \sqrt{-g} 4\pi r^2 \,\mathrm{d}r + \mathcal{O}\big(\rho_0 r^2 \,\mathrm{d}r\, \epsilon^4\big), \tag{40.9.5}$$

where

$$u^0\sqrt{-g} = 1 + \frac{1}{2}v^2 + 3\gamma U + \mathcal{O}\big(\epsilon^4\big) = 1 + 3\gamma U + \mathcal{O}\big(\epsilon^4\big), \quad \rho = \rho_0(1+\Pi). \tag{40.9.6}$$

So we have

$$\int \mathrm{d}E_1 = \int_0^{R_\odot} \rho_0(1 + \Pi + 3\gamma U) 4\pi r^2 \,\mathrm{d}r\,. \tag{40.9.7}$$

The work done to overcome gravity to add one spherical shell of matter on top of another is

$$\mathrm{d}E_2 = \rho_0 \tilde{U} 4\pi r^2 \,\mathrm{d}r + \mathcal{O}\big(\rho_0 r^2 \,\mathrm{d}r\, \epsilon^4\big), \tag{40.9.8}$$

where

$$\tilde{U}(r) = \frac{M(r)}{r} + \mathcal{O}\big(\epsilon^4\big) \tag{40.9.9}$$

After construction, we have

$$U(r) = \tilde{U}(R_\odot) + \int_r^{R_\odot} \frac{M(r')}{r'^2}\,\mathrm{d}r' + \mathcal{O}\big(\epsilon^4\big) \tag{40.9.10}$$

Notice that

$$\begin{aligned}\int \mathrm{d}E_2 &= \int_0^{R_\odot} \frac{M(r)}{r}\,\mathrm{d}M(r) + \mathcal{O}\big(M\epsilon^4\big) = \frac{M^2(r)}{r}\bigg|_0^{R_\odot} - \int_0^{R_\odot} M(r)\,\mathrm{d}\left(\frac{M(r)}{r}\right) + \mathcal{O}\big(M\epsilon^4\big) \\ &= \frac{M^2}{R_\odot} - \int \mathrm{d}E_2 + \int_0^{R_\odot} \frac{M^2(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big).\end{aligned} \tag{40.9.11}$$

We have

$$2\int \mathrm{d}E_2 = \frac{M^2}{R_\odot} + \int_0^{R_\odot} \frac{M^2(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big) \tag{40.9.12}$$

On the other hand, we have

$$\begin{aligned}\int_0^{R_\odot} \rho_0 U 4\pi r^2\,\mathrm{d}r &= \int_0^{R_\odot} \left[\tilde{U}(R_\odot) + \int_r^{R_\odot} \frac{M(r')}{r'^2}\,\mathrm{d}r'\right]\mathrm{d}M(r) + \mathcal{O}\big(M\epsilon^4\big) \\ &= \left[\tilde{U}(R_\odot) + \int_r^{R_\odot} \frac{M(r')}{r'^2}\,\mathrm{d}r'\right] M(r)\bigg|_0^{R_\odot} - \int_0^{R_\odot} M(r)\frac{-M(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big) \\ &= \frac{M^2}{R_\odot} + \int_0^{R_\odot} \frac{M^2(r)}{r^2}\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big).\end{aligned} \tag{40.9.13}$$

So we conclude that

$$\int \mathrm{d}E_2 = \frac{1}{2}\int_0^{R_\odot} \rho_0 U 4\pi r^2\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big). \tag{40.9.14}$$

The mass of the sun is

$$M \equiv \int (\mathrm{d}E_1 + \mathrm{d}E_2) = \int_0^{R_\odot} \rho_0 \left[1 + \Pi + \left(3\gamma - \frac{1}{2}\right)U\right] 4\pi r^2\,\mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big) \tag{40.9.15}$$

(b) From equation 40.1.1, we have

$$M_\odot = \int_0^{R_\odot} \rho_0(1 + 2\beta_2 U + \beta_3 \Pi + 3\beta_4 p/\rho_0) 4\pi r^2 \, \mathrm{d}r \, . \tag{40.9.16}$$

From the Newtonian virial theorem ,

$$\left\langle \int \rho_0 \boldsymbol{v}^2 \, \mathrm{d}^3x - \int \frac{1}{2}\rho_0 U \, \mathrm{d}^3x + 3\int p \, \mathrm{d}^3x \right\rangle_{\text{long time}} = \mathcal{O}\left(\int \rho_0 \epsilon^4 \, \mathrm{d}^3x \right). \tag{40.9.17}$$

applied to the sun by itself in its own rest frame, we can conclude that

$$-\int \frac{1}{2}\rho_0 U \, \mathrm{d}^3x + 3\int p \, \mathrm{d}^3x = \mathcal{O}\big(M\epsilon^4\big). \tag{40.9.18}$$

So we have

$$M_\odot = \int_0^{R_\odot} \rho_0 \left[1 + \beta_3 \Pi + \left(2\beta_2 + \frac{1}{2}\beta_4 \right) U \right] 4\pi r^2 \, \mathrm{d}r \, . \tag{40.9.19}$$

(c)

$$\begin{aligned} G_{\mathrm{C}} = \frac{M_\odot}{M} &= \frac{\int_0^{R_\odot} \rho_0 \left[1 + \beta_3 \Pi + \left(2\beta_2 + \frac{1}{2}\beta_4 \right) U \right] 4\pi r^2 \, \mathrm{d}r}{\int_0^{R_\odot} \rho_0 \left[1 + \Pi + \left(3\gamma - \frac{1}{2} \right) U \right] 4\pi r^2 \, \mathrm{d}r} + \mathcal{O}\big(\epsilon^4\big) \\ &= \frac{1 + \int_0^{R_\odot} \rho_0/M_0 \left[\beta_3 \Pi + \left(2\beta_2 + \frac{1}{2}\beta_4 \right) U \right] 4\pi r^2 \, \mathrm{d}r}{1 + \int_0^{R_\odot} \rho_0/M_0 \left[1 + \Pi + \left(3\gamma - \frac{1}{2} \right) U \right] 4\pi r^2 \, \mathrm{d}r} + \mathcal{O}\big(\epsilon^4\big) \\ &= 1 + \int \frac{\rho_0}{M_0} \left[(\beta_3 - 1)\Pi + \frac{1}{2}(4\beta_2 + \beta_4 - 6\gamma + 1) U \right] 4\pi r^2 \, \mathrm{d}r + \mathcal{O}\big(\epsilon^4\big), \end{aligned} \tag{40.9.20}$$

where $M_0 = \int \rho_0 4\pi r^2 \, \mathrm{d}r$. For a conservative theory, we have $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0$, i.e.,

$$\zeta = 0, \quad 2\beta + 2\beta_2 - 3\gamma - 1 = 0, \quad \beta_3 - 1 = 0, \quad \beta_4 - \gamma = 0, \quad 4\beta_1 - 2\gamma - 2 - \zeta = 0. \tag{40.9.21}$$

So

$$\frac{\Delta G_c}{G_c} \sim (4\beta_2 + \beta_4 - 6\gamma + 1)U = [4\beta_2 + \gamma - (2\beta + 2\beta_2 - 1) - 3\gamma + 1]U = -2(\beta + \gamma - \beta_2 - 1)U \tag{40.9.22}$$

40.10 CAVENDISH CONSTANT FOR ANY BODY

Extend the analysis of exercise 40.9 to a source that is arbitrarily stressed and has arbitrary shape and internal velocities (subject to the constraints $v^2 \ll 1, |t_{\hat{j}\hat{k}}|/\rho_0 \ll 1, U \ll 1, \Pi \ll 1$ of the post-Newtonian approximation). Assume that the body is at rest relative to the universal rest frame. Show that G_{C} depends on the internal structure of the source unless

$$2\beta_1 - \beta_4 = 1, \quad 4\beta_2 + \beta_4 - 6\gamma = -1, \quad \beta_3 = 1, \quad \zeta = 0, \quad \eta = 0. \tag{40.10.1}$$

Of course, these PPN constraints are all satisfied by Einstein's theory.

Solution:

The mass of the star, M, in the sense of the amount of energy required to construct it by adding one spherical shell of matter on top of another, working from the inside outward is

$$M = \int_0^{R_\odot} \rho_0 \left[1 + \Pi + v^2 + \left(3\gamma - \frac{1}{2} \right) U \right] 4\pi r^2 \, \mathrm{d}r + \mathcal{O}\big(M\epsilon^4\big) \tag{40.10.2}$$

In order to have Newtonian potential far from the star, g_{00} must be isotropic and so terms $\mathcal{A}$ and $\mathcal{D}$ must vanish, i.e.,

$$\zeta = \eta = 0. \tag{40.10.3}$$

The mass of sun as defined by its effect in bending world line of a faraway test particle is

$$M_\odot = \int_0^{R_\odot} \rho_0(1 + 2\beta_1 v^2 + 2\beta_2 U + \beta_3 \Pi + 3\beta_4 p/\rho_0) 4\pi r^2 \, \mathrm{d}r \, . \tag{40.10.4}$$

Using 40.9.17, we have

$$M_\odot = \int_0^{R_\odot} \rho_0 \left[1 + (2\beta_1 - \beta_4)v^2 + \left(2\beta_2 + \frac{1}{2}\beta_4\right) U + \beta_3 \Pi\right] 4\pi r^2 \, \mathrm{d}r \, . \tag{40.10.5}$$

If $M = M_\odot$, i.e.,

$$2\beta_1 - \beta_4 = 1, \quad 4\beta_2 + \beta_4 - 6\gamma = -1, \quad \beta_3 = 1, \tag{40.10.6}$$

G_C will be independent of the internal structure of the source.

Chapter 41

SPINORS

41.1 ELEMENTARY FEATURES OF THE ROTATION MATRIX

Write equation (41.3) in the form

$$R(\theta) = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{n}) \tag{41.1.1}$$

and establish the following properties:

(a) $(\boldsymbol{\sigma}\cdot\boldsymbol{n})^2 = 1 \equiv$ unit matrix;

(b) $\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = 0$ (tr means "trace", i.e., sum of diagonal elements);

(c) $[R, (\boldsymbol{\sigma}\cdot\boldsymbol{n})] \equiv R(\boldsymbol{\sigma}\cdot\boldsymbol{n}) - (\boldsymbol{\sigma}\cdot\boldsymbol{n})R = 0$;

(d)

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = -\frac{i}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{n})R. \tag{41.1.2}$$

Solution:

(a) Using $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, we have

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n})^2 = \sum_{i,j} n_i n_j \sigma_i \sigma_j = \sum_i n_i^2 \sigma_i^2 + \sum_{i<j} \{\sigma_i, \sigma_j\} n_i n_j = \sum_i n_i^2 = 1. \tag{41.1.3}$$

(b) Since $\mathrm{tr}\,\sigma_i = 0$ for $i = x, y, z$, we have

$$\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = \sum_i n_i \,\mathrm{tr}\,\sigma_i = 0. \tag{41.1.4}$$

(c)

$$[R, (\boldsymbol{\sigma}\cdot\boldsymbol{n})] = \left[\cos\left(\frac{\theta}{2}\right), (\boldsymbol{\sigma}\cdot\boldsymbol{n})\right] - i\sin\left(\frac{\theta}{2}\right)[(\boldsymbol{\sigma}\cdot\boldsymbol{n}), (\boldsymbol{\sigma}\cdot\boldsymbol{n})] = 0. \tag{41.1.5}$$

(d)

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right) - \frac{i}{2}\cos\left(\frac{\theta}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{n})^2 - \frac{i}{2}\cos\left(\frac{\theta}{2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = -\frac{i}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{n})R. \tag{41.1.6}$$

41.2 ROTATION MATRIX HAS UNIT DETERMINANT

Recall from exercise 5.5 that for any matrix M one has

$$\mathrm{d}[\ln(\det M)] = \mathrm{tr}\big(M^{-1}\,\mathrm{d}M\big) \tag{41.2.1}$$

and use this to show that $\det R$ in 41.1.2 is constant, and therefore equal to $(\det R)_{\theta=0} = 1$.

Solution:

Using equations 41.1.2, 41.1.4 and 41.2.1

$$\frac{\mathrm{d}\ln(\det R)}{\mathrm{d}\theta} = \mathrm{tr}\left(R^{-1}\frac{\mathrm{d}R}{\mathrm{d}\theta}\right) = -\frac{i}{2}\,\mathrm{tr}\big(R^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{n})R\big) = -\frac{i}{2}\,\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = 0 \tag{41.2.2}$$

So $\det R$ is a constant, i.e.,

$$\det R = (\det R)_{\theta=0} = 1. \tag{41.2.3}$$

41.3 MORE PROPERTIES OF THE ROTATION MATRIX

Show that for $X = \boldsymbol{x}\cdot\boldsymbol{\sigma}$ one has the commutation relation

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{n}), X] = 2i(\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{\sigma}. \tag{41.3.1}$$

Use this to obtain, from equation (41.19) in the form $X = RX_0R^*$ [where X_0 is constant, while $R(\theta)$ is given by equation (41.17)], the formula

$$\frac{\mathrm{d}\boldsymbol{x}\cdot\boldsymbol{\sigma}}{\mathrm{d}\theta} = (\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{\sigma}. \tag{41.3.2}$$

Why is this equivalent to the standard definition

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{\omega}\times\boldsymbol{x} \tag{41.3.3}$$

for the angular velocity? Reverse the argument to show that equation (41.7') correctly defines the rotation $R(t)$ resulting from a time-dependent angular velocity $\boldsymbol{\omega}(t)$, even though the simple solution $R = \exp\big[-\frac{1}{2}it(\boldsymbol{\sigma}\cdot\boldsymbol{\omega})\big]$ of this equation can no longer be written when $\boldsymbol{\omega}$ is not constant.

Solution:

Using the fact that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, we have

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{n}), X] = n_ix_j[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}n_ix_j\sigma_k = 2i(\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{\sigma}. \tag{41.3.4}$$

Since

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = -\frac{i}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{n})R, \quad \frac{\mathrm{d}R^*}{\mathrm{d}\theta} = \frac{i}{2}R^*(\boldsymbol{\sigma}\cdot\boldsymbol{n}), \tag{41.3.5}$$

we have

$$\frac{\mathrm{d}\boldsymbol{x}\cdot\boldsymbol{\sigma}}{\mathrm{d}\theta} = \frac{\mathrm{d}X}{\mathrm{d}\theta} = \frac{\mathrm{d}R}{\mathrm{d}\theta}X_0R^* + RX_0\frac{\mathrm{d}R^*}{\mathrm{d}\theta} = -\frac{i}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{n})RX_0R^* + \frac{i}{2}RX_0R^*(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = -\frac{i}{2}[(\boldsymbol{\sigma}\cdot\boldsymbol{n}), X] = (\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{\sigma}. \tag{41.3.6}$$

Using the fact that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, we have

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\theta} = \frac{1}{2}\frac{\mathrm{d}\{\boldsymbol{x}\cdot\boldsymbol{\sigma}, \boldsymbol{\sigma}\}}{\mathrm{d}\theta} = \frac{1}{2}\{(\boldsymbol{n}\times\boldsymbol{x})\cdot\boldsymbol{\sigma}, \boldsymbol{\sigma}\} = \boldsymbol{n}\times\boldsymbol{x}, \tag{41.3.7}$$

and so

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\boldsymbol{n} \times \boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}. \tag{41.3.8}$$

Reverse the argument, we can derive the equation 41.3.5 from 41.3.8, and so equation (41.7') correctly defines the rotation $R(t)$.

Chapter 43

SUPERSPACE ARENA FOR THE DYNAMICS OF GEOMETRY

43.1 THE ACTION PRINCIPLE FOR A FREE PARTICLE IN NON RELATIVISTIC MECHANICS

Taking as action principle $I = \int L\,\mathrm{d}t =$ extremum, with specified x', t' and x'', t'' at the two limits, and with $L = m(\mathrm{d}x/\mathrm{d}t)^2/2$, find (1) the extremizing history $x = x(t)$ and (2) the dynamical path length or action $S(x'', t''; x', t') = I_{\text{extremum}}$ in its dependence on the end points. Also (3) write down the Hamilton-Jacobi equation for this problem, and (4) verify that $S(x, t; x', t')$ satisfies this equation. Finally, imagining the Hamilton-Jacobi equation not to be known, (5) derive it from the already known properties of the function S itself.

Solution:

(1)The variation of I with respect to $\delta x(t)$ is

$$\delta I = \int_{t'}^{t''} m\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}\,\delta x}{\mathrm{d}t}\,\mathrm{d}t = m\frac{\mathrm{d}x}{\mathrm{d}t}\,\delta x\,\Big|_{t'}^{t''} - \int m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\delta x\,\mathrm{d}t = -\int m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\delta x\,\mathrm{d}t\,. \tag{43.1.1}$$

If I takes the extremum, we have $\delta I = 0$ and so

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0. \tag{43.1.2}$$

It follows that

$$x(t) = x' + \frac{x'' - x'}{t'' - t'}(t - t'). \tag{43.1.3}$$

(2) The action is

$$S(x'', t''; x', t') = I_{\text{extremum}} = \int_{t'}^{t''} \frac{1}{2}m\left(\frac{x'' - x'}{t'' - t'}\right)^2 \mathrm{d}t = \frac{m(x'' - x')^2}{2(t'' - t')}. \tag{43.1.4}$$

(3) The Hamiltonian of the particle is $H = p^2/2m$. So the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 = 0. \tag{43.1.5}$$

(4) Using

$$S(x, t; x', t') = \frac{m(x - x')^2}{2(t - t')}, \tag{43.1.6}$$

we have

$$\frac{\partial S}{\partial t} = -\frac{m(x-x')^2}{2(t-t')^2}, \quad \frac{\partial S}{\partial x} = \frac{m(x''-x')}{t''-t'}, \tag{43.1.7}$$

so $S(x,t;x',t')$ satisfy the Hamilton-Jacobi equation.

(5) From 43.1.7, it is straight to derive equation 43.1.5.

43.2 THE ACTION FOR THE HARMONIC OSCILLATOR

The kinetic energy is $m(\mathrm{d}x/\mathrm{d}t)^2/2$ and the potential energy is $m\omega^2x^2/2$. Carry through the analysis of parts (1), (2), (3), (4) of the preceding exercise. Verify that $\partial S/\partial x$ gives momentum and $-\partial S/\partial t$ gives energy.

Solution:

(1)The variation of I with respect to $\delta x(t)$ leads to Euler-Lagrangian equation,

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{43.2.1}$$

For

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2, \tag{43.2.2}$$

we have the equation of motion

$$\ddot{x} + \omega^2 x = 0. \tag{43.2.3}$$

The solution is

$$x(t) = \frac{x''\sin[\omega(t-t')] - x'\sin[\omega(t-t'')]}{\sin[\omega(t''-t')]} \tag{43.2.4}$$

(2) Using 43.2.2 and 43.2.4, we have

$$\begin{aligned} L(t) &= \frac{1}{2}m\omega^2\left\{\left\{\frac{x''\cos[\omega(t-t')] - x'\cos[\omega(t-t'')]}{\sin[\omega(t''-t')]}\right\}^2 - \left\{\frac{x''\sin[\omega(t-t')] - x'\sin[\omega(t-t'')]}{\sin[\omega(t''-t')]}\right\}^2\right\} \\ &= \frac{m\omega^2}{2\sin^2[\omega(t''-t')]}\left\{(x'')^2\cos[2\omega(t-t')] + (x')^2\cos[2\omega(t-t'')] - 2x'x''\cos[\omega(2t-t'-t'')]\right\} \end{aligned} \tag{43.2.5}$$

The action is

$$\begin{aligned} S(x'',t'';x',t') &= \int_{t'}^{t''} L(t)\,\mathrm{d}t \\ &= \frac{m\omega}{4\sin^2[\omega(t''-t')]}\left\{(x'')^2\sin[2\omega(t-t')] + (x')^2\sin[2\omega(t-t'')] - 2x'x''\sin[\omega(2t-t'-t'')]\right\}\Big|_{t'}^{t''} \\ &= \frac{m\omega}{2\sin[\omega(t''-t')]}\left\{[(x'')^2 + (x')^2]\cos[\omega(t''-t')] - 2x'x''\right\} \end{aligned} \tag{43.2.6}$$

(3) The Hamiltonian of the particle is $H = p^2/2m + m\omega^2x^2/2$. So the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{m\omega^2}{2}x^2 = 0. \tag{43.2.7}$$

(4) Using

$$S(x,t;x',t') = \frac{m\omega}{2}\frac{[x^2+(x')^2]\cos[\omega(t-t')] - 2x'x}{\sin[\omega(t-t')]} \tag{43.2.8}$$

we have

$$\frac{\partial S}{\partial t} = -\frac{m\omega^2}{2}\frac{[x^2 + (x')^2] - 2x'x\cos[\omega(t-t')]}{\sin^2[\omega(t-t')]}, \quad \frac{\partial S}{\partial x} = m\omega\frac{x\cos[\omega(t-t')] - x'}{\sin[\omega(t-t')]}. \tag{43.2.9}$$

Now it is easy to verify that 43.2.9 satisfies equation 43.2.7.

(5) The energy of the oscillator is

$$\begin{aligned} E(t) &= \frac{1}{2}m\omega^2\left\{\left\{\frac{x''\cos[\omega(t-t')] - x'\cos[\omega(t-t'')]}{\sin[\omega(t''-t')]}\right\}^2 + \left\{\frac{x''\sin[\omega(t-t')] - x'\sin[\omega(t-t'')]}{\sin[\omega(t''-t')]}\right\}^2\right\} \\ &= \frac{m\omega^2}{2\sin^2[\omega(t''-t')]}\left\{(x'')^2 + (x')^2 - 2x'x''\cos[\omega(t''-t')]\right\} \end{aligned} \tag{43.2.10}$$

The momentum of the oscillator is

$$p(t) = m\omega\frac{x''\cos[\omega(t-t')] - x'\cos[\omega(t-t'')]}{\sin[\omega(t''-t')]} \tag{43.2.11}$$

For $t = t''$, we can verify that

$$-\frac{\partial S}{\partial t} = E, \quad \frac{\partial S}{\partial x} = p. \tag{43.2.12}$$

43.3 QUANTUM PROPAGATOR FOR HARMONIC OSCILLATOR

Show that the probability amplitude for a simple harmonic oscillator to transit from x', t' to x'' t'' is

$$\langle x'', t''; x', t'\rangle = \left(\frac{m\omega}{2\pi i\hbar\sin\omega(t''-t')}\right)^{1/2} \times \exp\frac{im\omega[(x''^2 + x'^2)\cos\omega(t''-t') - 2x'x'']}{2\hbar\sin\omega(t''-t')} \tag{43.3.1}$$

and that it reduces for the case of a free particle to

$$\langle x'', t''; x', t'\rangle = \left(\frac{m}{2\pi i\hbar(t''-t')}\right)^{1/2} \times \exp\frac{im(x''-x')^2}{2\hbar(t''-t')}. \tag{43.3.2}$$

Note that one can derive all the harmonic-oscillator wave functions from the solution by use of the formula

$$\langle x'', t''; x', t'\rangle = \sum_n u_n(x'')u_n^*(x')\exp iE_n(t-t'')/\hbar. \tag{43.3.3}$$

Solution:

The propagator of a harmonic oscillator in terms of path integral is

$$\begin{aligned} \langle x'', t''; x', t'\rangle &= \int \mathcal{D}[x(t)]e^{i/\hbar\int_{t'}^{t''}(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2)\mathrm{d}t} \\ &= \lim_{N\to\infty}\int\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}}\exp\left\{\frac{i}{\hbar}\sum_{j=0}^{N}\left[\frac{m(x_{j+1}-x_j)^2}{2\Delta t^2} - \frac{1}{2}m\omega^2x_{j+1}^2\right]\Delta t\right\}\mathrm{d}x_1\cdots\mathrm{d}x_N\,, \end{aligned} \tag{43.3.4}$$

where $x_0 = x'$, $x_{N+1} = x''$ and $\Delta t = (t''-t')/(N+1)$. Suppose $x_\mathrm{c}(t)$ is the classical path of the harmonic oscillator and $\delta x(t) \equiv x(t) - x_\mathrm{c}(t)$ is the deviation from the classical path. Substituting $x = x_\mathrm{c} + \delta x$ into 43.3.4, terms which is linear in δx can be dropped since

$$\left.\frac{\delta S}{\delta x}\right|_{x(t)=x_\mathrm{c}(t)} = 0. \tag{43.3.5}$$

Noticing that $\delta x_0 = \delta x_{N+1} = 0$, equation 43.3.4 can be simplified into

$$\begin{aligned}\langle x'', t''; x', t' \rangle &= e^{iS_c/\hbar} \lim_{N\to\infty} \int \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}} \exp\left\{ i \sum_{j=1,k=1}^{N} \delta x_j M_{jk} \delta x_k \right\} \mathrm{d}\delta x_1 \cdots \mathrm{d}\delta x_N \\ &= e^{iS_c/\hbar} \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}} \sqrt{\frac{\pi^N}{\det(-iM)}}, \end{aligned} \tag{43.3.6}$$

where

$$-iM = \frac{m}{2i\hbar\Delta t} \begin{pmatrix} 2-\omega^2\Delta t^2 & -1 & 0 & \cdots \\ -1 & 2-\omega^2\Delta t^2 & -1 & \cdots \\ 0 & -1 & 2-\omega^2\Delta t^2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{43.3.7}$$

and

$$S_c = \frac{m\omega}{2\sin\omega(t''-t')} \left\{ (x''^2 + x'^2)\cos\omega(t''-t') - 2x'x'' \right\}. \tag{43.3.8}$$

Using the fact that

$$\det(-iM_N) = \frac{m}{2i\hbar\Delta t}(2-\omega^2\Delta t^2)\det(-iM_{N-1}) - \left(\frac{m}{2i\hbar\Delta t}\right)^2 \det(-iM_{N-2}) \tag{43.3.9}$$

and

$$\det(-iM_1) = \frac{m}{2i\hbar\Delta t}(2-\omega^2\Delta t^2), \quad \det(-iM_1) = \left(\frac{m}{2i\hbar\Delta t}\right)^2 [(2-\omega^2\Delta t^2)^2 - 1], \tag{43.3.10}$$

we can derive that

$$\det(-iM_N) = \left(\frac{m}{2i\hbar\Delta t}\right)^N \frac{\sin(N+1)u}{\sin u} \quad \text{where} \quad \sin\frac{u}{2} = \frac{1}{2}\omega\Delta t \tag{43.3.11}$$

In the limit of $N \to \infty$, we have

$$\det(-iM_N) = \left(\frac{m}{2i\hbar\Delta t}\right)^N \frac{\sin\omega(t''-t')}{\omega\Delta t} \tag{43.3.12}$$

and so

$$\lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N+1}{2}} \sqrt{\frac{\pi^N}{\det(-iM)}} = \left(\frac{m\omega}{2\pi i\hbar\sin\omega(t''-t')}\right)^{1/2}. \tag{43.3.13}$$

Now we can obtain the probability amplitude for a simple harmonic oscillator to transit from x', t' to x'' t'' as equation 43.3.1.

43.4 QUANTUM PROPAGATOR FOR FREE ELECTROMAGNETIC FIELD

In flat spacetime, one is given on the spacelike hypersurface $t = t'$ the divergence-free magnetic field $B'(x, y, z)$ and on the spacelike hypersurface $t = t''$ the divergence-free magnetic field $B''(x, y, z)$. By Fourier analysis (reducing this problem to the preceding problem) or otherwise, find the probability amplitude to transit from B' at t' to B'' at t''.

Solution:

Choosing the Coulomb's gauge, we have

$$\boldsymbol{E} = -\dot{\boldsymbol{A}}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{43.4.1}$$

By Fourier analysis, we have

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{r=1,2} \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \tilde{A}_r(\boldsymbol{k}) \boldsymbol{\epsilon}_r(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{where} \quad \boldsymbol{k}\cdot\boldsymbol{\epsilon}_r(\boldsymbol{k}) = 0, \quad \boldsymbol{\epsilon}_r(\boldsymbol{k})\cdot\boldsymbol{\epsilon}_{r'}(\boldsymbol{k}) = \delta_{rr'} \tag{43.4.2}$$

Notice that $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ and $\boldsymbol{k}/k$ form a set of right-handed orthonormal basis. The Fourier component of electric and magnetic field is

$$\tilde{E} = -\dot{\tilde{A}}, \quad \tilde{B}_1 = -ik\tilde{A}_2, \quad \tilde{B}_2 = ik\tilde{A}_1 \tag{43.4.3}$$

The Hamiltonian of the free electromagnetic field is

$$H = \int \mathrm{d}^3 x \, \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{8\pi} = \sum_{r=1,2} \int \mathrm{d}^3 k \, \frac{|\tilde{E}_r|^2 + |\tilde{B}_r|^2}{8\pi} = \sum_{r=1,2} \int \mathrm{d}^3 k \, \frac{|\dot{\tilde{A}}_r|^2 + k^2 |\tilde{A}_r|^2}{8\pi}. \tag{43.4.4}$$

Since $\boldsymbol{A}$ is real, we have $\tilde{A}(r, \boldsymbol{k}) = \tilde{A}^*(r, -\boldsymbol{k})$. The Hamiltonian then becomes a collection of independent harmonic oscillators,

$$H = \sum_{s=R,I} \sum_{r=1,2} \int_{k_x>0} \mathrm{d}^3 k \, \frac{\dot{\tilde{A}}_{r,s}^2 + k^2 \tilde{A}_{r,s}^2}{4\pi}. \tag{43.4.5}$$

Suppose the length scale of the space is L. We can get

$$H = \frac{1}{2\pi} \left(\frac{2\pi\hbar}{L}\right)^3 \sum_{s=R,I} \sum_{r=1,2} \sum_{k_x>0} \frac{\dot{\tilde{A}}_{r,s}^2 + k^2 \tilde{A}_{r,s}^2}{2} \quad \text{where} \quad k_i = \frac{2\pi\hbar}{L} n_i. \tag{43.4.6}$$

Using 43.3.1, the quantum amplitude will be

$$\langle A'', t''; A', t' \rangle = \prod_{s=R,I} \prod_{r=1,2} \prod_{k_x>0} \left(\frac{Mk}{2\pi i\hbar \sin k(t''-t')} \right)^{1/2} \times \exp \frac{iMk[(\tilde{A}_{r,s}^{''2} + \tilde{A}_{r,s}^{'2}) \cos(t''-t') - 2\tilde{A}_{r,s}^{''} \tilde{A}_{r,s}^{'}]}{2\hbar \sin k(t''-t')}, \tag{43.4.7}$$

where

$$M = \frac{1}{2\pi} \left(\frac{2\pi\hbar}{L}\right)^3. \tag{43.4.8}$$

Notice that $\boldsymbol{A}$ is completely determined by $\boldsymbol{B}$ in Coulomb's gauge.

43.5 HAMILTON-JACOBI FORMULATION OF MAXWELL ELECTRODYNAMICS

Regard the four components A_μ of the electromagnetic 4-potential as the primary quantities; split them into a space part A_i and a scalar potential ϕ. (1) Derive from the action principle (in flat spacetime)

$$I = \frac{1}{8\pi} \int (E^2 - B^2) \, \mathrm{d}^4 x \,, \tag{43.5.1}$$

by splitting off an appropriate divergence, an expression qualitatively similar in character to (43.7). (2) Show that the appropriate quantity to be fixed on the initial and final spacelike hypersurface is not really A_i itself, but the magnetic field, defined by $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. (3) Derive the Hamilton-Jacobi equation for the dynamic phase or action $S(\boldsymbol{B}, \mathcal{S})$ in its dependence on the choice of hypersurface $\mathcal{S}$, and the choice of magnetic field $\boldsymbol{B}$ on this hypersurface,

$$-\frac{\delta S}{\delta \Omega} = \frac{1}{8\pi} \boldsymbol{B}^2 + \frac{(4\pi)^2}{8\pi} \left(\frac{\delta S}{\delta \boldsymbol{A}}\right)^2. \tag{43.5.2}$$

The quantity on the left is Tomonaga's "bubble time" derivative [Tomonaga (1946); see also Box 21.1].

Solution:

(1) The Lagrangian density for EM field is

$$\mathcal{L} = \frac{1}{8\pi}(E^2 - B^2) = \frac{(\dot{A}_i + \phi_{,i})^2 - (\epsilon_{ijk}A_{k,j})^2}{8\pi}. \tag{43.5.3}$$

The canonical momentum for A_i is

$$\pi^i_{\text{ture}} = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{\dot{A}_i + \phi_{,i}}{4\pi} = -\frac{E_i}{4\pi}. \tag{43.5.4}$$

Define $\pi^i \equiv 4\pi\pi^i_{\text{ture}} = -E_i$. We have

$$\begin{aligned} I &= \frac{1}{8\pi}\int \left[\pi_i^2 - (\epsilon_{ijk}A_{k,j})^2\right] \mathrm{d}^4x \\ &= \frac{1}{4\pi}\int \left\{\pi^i\dot{A}_i - \frac{1}{2}\left[\pi_i^2 + (\epsilon_{ijk}A_{k,j})^2\right] + \pi^i\phi_{,i}\right\} \mathrm{d}^4x \\ &= \frac{1}{4\pi}\int \left\{\pi^i\dot{A}_i - \frac{1}{2}\left[\pi_i^2 + (\epsilon_{ijk}A_{k,j})^2\right] - \pi^i_{,i}\phi\right\} \mathrm{d}^4x\,. \end{aligned} \tag{43.5.5}$$

(2)

$$\begin{aligned} \delta I &= \frac{1}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \left[-E_i(\delta\dot{A}_i + \delta\phi_{,i}) - B_i\epsilon_{ijk}\,\delta A_{k,j}\right] \\ &= \frac{1}{4\pi}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \left[(\dot{E}_k + \epsilon_{ijk}B_{i,j})\,\delta A_k + E_{i,i}\,\delta\phi\right] - \frac{1}{4\pi}\int \mathrm{d}^3x\, E_i\delta A_i\Big|_{t_1}^{t_2} \end{aligned} \tag{43.5.6}$$

If $\delta A_i = \chi_{,i}$, using the fact that $E_{i,i} = 0$, we can get

$$\int \mathrm{d}^3x\, E_i\delta A_i = \int \mathrm{d}^3x\, (E_i\chi)_{,i} = 0. \tag{43.5.7}$$

So a transformation $A_i \to A_i + \chi_{,i}$ on the space sheet t_1 and t_2 will not change the value of action. The appropriate quantity to be fixed on the initial and final spacelike hypersurface is not really A_i itself, but the magnetic field, defined by $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

(3) From equation 43.5.6, we have

$$\frac{\delta S}{\delta \boldsymbol{A}} = -\frac{E_i}{4\pi} = \frac{\pi^i}{4\pi} = \pi^i_{\text{ture}} \tag{43.5.8}$$

From equation 43.5.5, we can obtain the Hamiltonian density

$$\mathcal{H} = \frac{1}{8\pi}\left[\pi_i^2 + (\epsilon_{ijk}A_{k,j})^2\right] \tag{43.5.9}$$

Since $\delta S/\delta\Omega = -\mathcal{H}$, the Hamilton-Jacobi equation is

$$-\frac{\delta S}{\delta\Omega} = \frac{1}{8\pi}\boldsymbol{B}^2 + \frac{(4\pi)^2}{8\pi}\left(\frac{\delta S}{\delta \boldsymbol{A}}\right)^2. \tag{43.5.10}$$